

Almost Hermitian Manifold with Semi-Symmetric Recurrent Connection

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Abstract. The Riemannian manifold equipped with a semi-symmetric metric connection has been studied by O.C. Andonie¹, M.C. Chaki and A. Konar², U.C.De³ etc. while a special type of semi-symmetric metric connection on a Riemannian manifold has been studied by U.C. De and S.C. Biswas⁴. P.N. Pandey and S.K. Dubey⁵ discussed an almost Grayan manifold admitting a semi-symmetric metric connection in 2004. In the present paper, we have obtained certain results for an almost Hermite manifold equipped with a semi-symmetric recurrent connection. Apart from finding the necessary and sufficient condition for a covariant almost analytic vector field with respect to Riemannian connection D to be covariant almost analytic with respect to the semi-symmetric recurrent connection ∇ , we have also shown that the Nijenhuis tensor with respect to the connection ∇ on an almost Hermite manifold equipped with a semi-symmetric recurrent connection is equal to the Nijenhuis tensor with respect to the Riemannian connection. We have also discussed the geodesics on an almost Hermite manifold equipped with semi-symmetric recurrent connection ∇ .

Keywords: Semi-symmetric recurrent connection, almost Hermitian manifold, covariant almost analytic vector field, Nijenhuis tensor, geodesics.

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1. Introduction

An even dimensional differentiable manifold M_n is said to be an almost Hermite manifold with structure $\{F, g\}$ if

$$(1.1) \quad \overline{\overline{X}} + X = 0,$$

and

$$(1.2) \quad g(\overline{X}, \overline{Y}) = g(X, Y),$$

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where $\overline{X} = FX$, F is a tensor of type (1,1), g is the metric tensor and X, Y are arbitrary vector fields.

Let D be the Riemannian connection and ∇ be another connection on $\{F, g\}$ defined by

$$(1.3) \quad \nabla_X Y = D_X Y + a\omega(X)Y + b\omega(Y)X,$$

where $\omega(X) = g(X, \rho)$ and a, b are non-zero real or complex numbers such that $a \neq b$. The connection ∇ is said to be a semi-symmetric recurrent connection if the metric g of the manifold satisfies the following condition:

$$(1.4) \quad (\nabla_T g)(X, Y) = Tg(X, Y)$$

for arbitrary vector fields X and Y .

The torsion tensor S of the connection ∇ is given by

$$(1.5) \quad S(X, Y) = (a - b)\{\omega(X)Y - \omega(Y)X\}.$$

2. Some Theorems on an Almost Hermite Manifold with Connection ∇

Now we define a tensor $B(X, Y)$, such that

$$(2.1) \quad \nabla_X Y - D_X Y = B(X, Y).$$

Therefore equation (1.3) and (2.1) gives

$$(2.2) \quad B(X, Y) = a\omega(X)Y + b\omega(Y)X.$$

Let $A(X, Y)$ and $\text{'}A(X, Y)$ be the symmetric and skew-symmetric parts of the tensor $B(X, Y)$, therefore we have

$$(2.3) \quad \begin{cases} A(X, Y) = \frac{a+b}{2}\{\omega(X)Y + \omega(Y)X\}, \\ \text{'}A(X, Y) = \frac{a-b}{2}\{\omega(X)Y - \omega(Y)X\}, \end{cases}$$

Now from equation (1.5) and (2.3), we get

$$S(X, Y) = 2\text{'}A(X, Y),$$

and $A(X, Y) = 0$ if and only if $a = -b$ or $\omega(Y)X = -\omega(X)Y$.

Thus, we have:

Theorem 2.1. *An almost Hermite manifold equipped with a semi-symmetric recurrent connection ∇ admits the following:*

- (i) *The torsion tensor S of the connection ∇ is two times of the skew-symmetric part $\text{'}A(X, Y)$ of the tensor $B(X, Y)$, i.e. $S(X, Y) = 2\text{'}A(X, Y)$.*

- (ii) The symmetric part of the tensor $B(X, Y)$ vanishes if and only if $a = -b$ or $\omega(Y)X = -\omega(X)Y$.

The Nijenhuis tensor with respect to semi-symmetric recurrent connection ∇ is given by

$$(2.4) \quad \tilde{N}(X, Y) = (\nabla_{\bar{X}} F)Y - (\nabla_{\bar{Y}} F)X - \overline{(\nabla_X F)Y} + \overline{(\nabla_Y F)X}.$$

Operating F on Y in equation (1.3), we have

$$(2.5) \quad (\nabla_X F)Y = (D_X F)Y + \overline{D_X Y} - \overline{\nabla_X Y} + a\omega(X)\bar{Y} + b\omega(\bar{Y})X.$$

Using (1.3) in (2.5), we get

$$(2.6) \quad (\nabla_X F)Y = (D_X F)Y + b[\omega(\bar{Y})X - \omega(Y)\bar{X}]$$

Interchanging X and Y in (2.6), we get

$$(2.7) \quad (\nabla_Y F)X = (D_Y F)X + b[\omega(\bar{X})Y - \omega(X)\bar{Y}]$$

Using (2.5), (2.6) in (2.4), we have

$$(2.8) \quad \tilde{N}(X, Y) = N(X, Y)$$

Thus, we may conclude:

Theorem 2.2. *On an almost Hermite manifold, Nijenhuis tensor with respect to the semi-symmetric recurrent connection is equal to the Nijenhuis tensor with respect to the Riemannian connection.*

Now, we propose:

Theorem 2.3. *An almost Hermite manifold equipped with a semi-symmetric recurrent connection ∇ admits the following:*

$$(2.9) \quad \begin{cases} (a) \quad S(X, Y, \rho) = 0, \\ (b) \quad S(X, Y, Z) + S(Y, Z, X) + S(Z, X, Y) = 0, \\ (c) \quad S(\bar{X}, Y, \bar{Z}) + S(Y, \bar{Z}, \bar{X}) + S(\bar{Z}, \bar{X}, Y) = 0, \end{cases}$$

where $S(X, Y, Z)$ is defined as

$$(2.10) \quad S(X, Y, Z) = g(S(X, Y), Z)$$

Proof. Using equation (1.5) in equation (2.10), we get

$$(2.11) \quad S(X, Y, Z) = (a - b)\{\omega(X)g(Y, Z) - \omega(Y)g(X, Z)\}.$$

Putting $Z = \rho$ in (2.11) and using $\omega(X) = g(X, \rho)$, we have (2.9a).

Equation (2.11) can be written as

$$(2.12) \quad S(Y, Z, X) = (a - b)\{\omega(Y)g(Z, X) - \omega(Z)g(Y, X)\},$$

and

$$(2.13) \quad S(Z, X, Y) = (a - b)\{\omega(Z)g(X, Y) - \omega(X)g(Z, Y)\}.$$

Adding (2.11), (2.12) and (2.13), we get (2.9b). By adding equations (2.11), (2.12) and (2.13) after barring X and Z , we obtain (2.9c).

3. Covariant Almost Analytic Vector Field

In an almost Hermite manifold, if 1-form ω satisfies

$$(3.1) \quad \omega((\nabla_X F)(Y) - (\nabla_Y F)(X)) = (\nabla_{\bar{X}} \omega)(Y) - (\nabla_X \omega)(\bar{Y}),$$

then the 1-form ω is said to be covariant almost analytic vector field.

Subtracting (2.7) from (2.6), we have

$$(3.2) \quad (\nabla_X F)Y - (\nabla_Y F)X = (D_X F)Y - (D_Y F)X + b[\omega(\bar{Y})X - \omega(Y)\bar{X} - \omega(\bar{X})Y + \omega(X)\bar{Y}].$$

From equation (3.2), we may write

$$(3.3) \quad \omega((\nabla_X F)(Y) - (\nabla_Y F)(X)) = \omega((D_X F)(Y) - (D_Y F)(X)) \\ + 2b[\omega(\bar{Y})\omega(X) - \omega(Y)\omega(\bar{X})].$$

Now we define

$$(3.4) \quad (\nabla_X \omega)Y = g(\nabla_X \rho, Y) \quad \text{and} \quad (D_X \omega)Y = g(D_X \rho, Y).$$

Then ω satisfies

$$(3.5) \quad (\nabla_X \omega)Y = (D_X \omega)Y + a\omega(X)\omega(Y) + b\omega(\rho)g(X, Y).$$

Barring X in equation (3.5), we have

$$(3.6) \quad (\nabla_{\bar{X}} \omega)Y = (D_{\bar{X}} \omega)Y + b\omega(\rho)g(\bar{X}, Y) + a\omega(\bar{X})\omega(Y).$$

Replacing Y by $\bar{Y}$ in equation (3.5), we get

$$(3.7) \quad (\nabla_X \omega)\bar{Y} = (D_X \omega)\bar{Y} + b\omega(\rho)g(X, \bar{Y}) + a\omega(X)\omega(\bar{Y}).$$

Subtracting (3.7) from (3.6), we obtain

$$(3.8) \quad (\nabla_{\bar{X}} \omega)Y - (\nabla_X \omega)\bar{Y} = (D_{\bar{X}} \omega)Y - (D_X \omega)\bar{Y} + 2b\omega(\rho)g(\bar{X}, Y) \\ + a[\omega(\bar{X})\omega(Y) - \omega(X)\omega(\bar{Y})].$$

Subtracting (3.8) from (3.3), we have

$$(3.9) \quad \omega((\nabla_X F)(Y) - (\nabla_Y F)(X)) - \{(\nabla_{\bar{X}} \omega)Y - (\nabla_X \omega)\bar{Y}\} = \omega((D_X F)(Y) - (D_Y F)(X)) \\ - \{(\nabla_{\bar{X}} \omega)(Y) - (\nabla_X \omega)(\bar{Y})\} - 2b\omega(\rho)g(\bar{X}, Y) - (a + 2b)[\omega(\bar{X})\omega(Y) - \omega(X)\omega(\bar{Y})].$$

Thus, we may conclude:

Theorem 3.1. *On an almost Hermite manifold equipped with a semi-symmetric recurrent connection ∇ , a covariant almost analytic vector ω with respect to Riemannian connection D is also covariant almost analytic with respect to semi-symmetric recurrent connection ∇ if and only if*

$$2b\omega(\rho)g(\bar{X}, Y) + (a + 2b)[\omega(\bar{X})\omega(Y) - \omega(X)\omega(\bar{Y})] = 0.$$

Now, we propose:

Theorem 3.2 *On an almost Hermite manifold equipped with a semi-symmetric recurrent connection ∇ , we have*

$$(3.10) \quad \tilde{d}\omega(X, Y) = d\omega(X, Y),$$

where $\tilde{d}\omega(X, Y) = (\nabla_X \omega)Y - (\nabla_Y \omega)X$, $d\omega(X, Y) = (D_X \omega)Y - (D_Y \omega)X$.

Proof. Interchanging X and Y in equation (3.5), we have

$$(3.11) \quad (\nabla_Y \omega)X = (D_Y \omega)X + a\omega(\rho)g(X, Y) + b\omega(X)\omega(Y).$$

Subtracting (3.11) from (3.5), we get (3.10).

In view of (3.10), we conclude:

Corollary. *In an almost Hermite manifold equipped with a semi-symmetric recurrent connection ∇ , if ω is closed with respect to Riemannian connection D , it is also closed with respect to the semi-symmetric recurrent connection ∇ .*

4. Geodesics on an Almost Hermite Manifold

Definition. Let σ be a curve in M with the tangent field T . A vector field Y is σ parallel if and only if $\nabla_T Y = 0$ on σ and the curve is geodesic if and only if $\nabla_T T = 0$ on σ .

Now, we propose:

Theorem 4.1. *In an almost Hermite manifold equipped with a semi-symmetric recurrent connection ∇ , the Riemannian connection D and the semi-symmetric recurrent connection ∇ have same geodesics if and only if $a = -b$.*

Proof. Putting $X = Y = T$ in equation, (1.3), we have

$$(4.1) \quad \nabla_T T = D_T T + (a + b)\omega(T)T.$$

From equation (4.1), $D_T T = 0$ implies $\nabla_T T = 0$ if and only if $a = -b$.

In view of equation (2.3) and theorem (4.1), we conclude:

Theorem 4.2. *In an almost Hermite manifold equipped with a semi-symmetric recurrent connection ∇ , if the Riemannian connection D and semi-symmetric recurrent connection ∇ have same geodesics then symmetric part of the tensor $B(X, Y)$ vanishes and $B(X, Y) = A(X, Y)$.*

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