

On HGF- Structure Metric Manifold

S.B. Pandey and Meenakshi Pant

Department of Mathematics, Kumaun University
S.S.J. Campus, Almora (Uttaranchal)

(Received May 29, 2006)

Abstract. In this paper, we have defined various associated curvature tensors viz associated Pseudo Projective curvature tensor, associated Pseudo Conformal curvature tensor, associated Pseudo Conharmonic curvature tensor, associated Pseudo Concircular curvature tensor, associated Pseudo H-Projective curvature tensor and associated Pseudo Bochner curvature tensor in HGF-metric structure manifold. Recurrence and recurrence symmetry of (1), (2), (3), (4), (12), (13), (14), (23), (24), (34), (123), (124), (234), and (1234) kinds have been defined related to various associated curvature tensors. Theorems of different kinds in recurrent and recurrent symmetric HGF- metric manifolds involving equivalent conditions related to various types of associated curvature tensors have also been discussed.

1. Introduction

Let us consider a differentiable manifold M_n of differentiability class C^∞ . Let there exists a vector valued linear function F of class C^∞ satisfying

$$(1.1) \quad \bar{X} = -a^2 X$$

for arbitrary vector field X , where $\bar{X} = FX$ and a is a complex number. Then $\{F\}$ is said to be a hyperbolic differentiable structure, briefly known as HGF - structure, defined by the equation (1.1) and then M_n is called HGF- manifold¹.

The equation (1.1) gives different structures for different Values of a .

- (i) If $a \neq 0$, it is hyperbolic π -structure.
- (ii) If $a = \pm 1$, it is an almost complex or a hyperbolic almost product manifold.
- (iii) If $a = \pm i$, it is an almost product or a hyperbolic almost complex structure.
- (iv) If $a = 0$, it is an almost tangent or a hyperbolic almost tangent structure.

Let the HGF-structure be endowed with the Hermitian metric g , such that

$$(1.2) \quad g(\bar{X}, \bar{Y}) - a^2 g(X, Y) = 0.$$

Then $\{F, G\}$ is said to be a Hyperbolic differentiable metric structure, briefly known as HGF-metric structure and M_n is called a Hyperbolic differentiable metric structure manifold or HGF-metric structure manifold. Let us put

$$(1.3) \quad {}^1F(X, Y) = g(\bar{X}, Y).$$

Then the following equations hold:

$$(1.4 a) \quad {}^1F(X, Y) = -{}^1F(Y, X)$$

i.e., ${}^1F$ is skew symmetric in X and Y ,

$$(1.4 b) \quad {}^1F(\bar{X}, \bar{Y}) = -a^2 {}^1F(Y, X),$$

$$(1.4 c) \quad {}^1F(\bar{X}, Y) = -{}^1F(X, \bar{Y}).$$

A bilinear function B in HGF-metric manifold is said to be pure in the two slots, if

$$(1.5) \quad B(\bar{X}, \bar{Y}) + a^2 B(X, Y) = 0.$$

It is said to be hybrid in the slots, if

$$(1.6) \quad B(\bar{X}, \bar{Y}) - a^2 B(X, Y) = 0.$$

The equations (1.4 b) and (1.6) show that ${}^1F$ is hybrid in X and Y .

2. Associated Curvature Tensors of Different Kinds

Agreement 1.1: The discussions which follow will hold for manifolds other than almost tangent or Hyperbolic almost tangent manifolds.

In the HGF-metric structure manifold, Pseudo Projective curvature tensor W^* , Pseudo conformal curvature tensor C^* , Pseudo conharmonic curvature tensor L^* , Pseudo concircular curvature tensor V^* , Pseudo H-projective curvature tensor P^* , Pseudo H-conharmonic curvature tensor S^* and Pseudo Bochner curvature tensor B^* are define by

$$(2.1) \quad W^*(X, Y, Z) = K(X, Y, Z) - \frac{1}{a^2(n-1)}[a^2 Ric(Y, Z)X - a^2 Ric(X, Z)Y],$$

$$(2.2) \quad C^*(X, Y, Z) = K(X, Y, Z) + \frac{a^2}{(n+2)}[a^2 Ric(X, Z)Y - a^2 Ric(Y, Z)X + g(X, Z)r(Y) - g(Y, Z)r(X)] - \frac{a^2 R}{(n-1)(n-2)}[g(X, Z)Y - g(Y, Z)X],$$

$$(2.3) \quad L^*(X, Y, Z) = K(X, Y, Z) + \frac{a^2}{(n-2)} [a^2 Ric(X, Z)Y - a^2 Ric(Y, Z)X \\ + g(X, Z)r(Y) - g(Y, Z)r(X)],$$

$$(2.4) \quad V^*(X, Y, Z) = K(X, Y, Z) + \frac{a^2 R}{n(n-1)} [g(X, Z)Y - g(Y, Z)X],$$

$$(2.5) \quad P^*(X, Y, Z) = K(X, Y, Z) + \frac{1}{a^2(n+2)} [a^2 Ric(Y, Z)X - a^2 Ric(X, Z)Y - Ric(\bar{Y}, Z)\bar{X} \\ + Ric(\bar{X}, Z)\bar{Y} - 2Ric(X, \bar{Y})\bar{Z}],$$

$$(2.6) \quad S^*(X, Y, Z) = K(X, Y, Z) - \frac{a^2}{(n+4)} [a^2 Ric(Y, Z)X - a^2 Ric(X, Z)Y - g(X, Z)r(Y) \\ + g(Y, Z)r(X) + Ric(X, \bar{Z})\bar{Y} - Ric(Y, \bar{Z})\bar{X} - 2Ric(\bar{X}, Y)\bar{Z} \\ + g(X, \bar{Z})r(\bar{Y}) - g(Y, \bar{Z})r(\bar{X}) - 2g(\bar{X}, Y)r(\bar{Z})],$$

$$(2.7) \quad B^*(X, Y, Z) = K(X, Y, Z) - \frac{a^2}{(n+4)} [a^2 Ric(Y, Z)g(X, T) - a^2 Ric(X, Z)g(Y, T) \\ - g(X, Z)r(Y) + g(Y, Z)r(X) + Ric(X, \bar{Z})\bar{Y} - Ric(Y, \bar{Z})\bar{X} \\ - 2Ric(\bar{X}, Y)\bar{Z} + g(X, \bar{Z})r(\bar{Y}) - g(Y, \bar{Z})r(\bar{X}) - 2g(\bar{X}, Y)r(\bar{Z}) \\ + \frac{a^2 R}{(n+2)(n+4)} [g(Y, Z)X - g(X, Z)Y + g(X, \bar{Z})\bar{Y} - g(Y, \bar{Z})\bar{X} - 2g(\bar{X}, Y)\bar{Z}].$$

The associated tensors of $W^*, C^*, L^*, V^*, P^*, S^*$ and B^* are defined by

$$(2.8 \text{ a}) \quad 'W^*(X, Y, Z, T) = g(W^*(X, Y, Z), T),$$

$$(2.8 \text{ b}) \quad 'C^*(X, Y, Z, T) = g(C^*(X, Y, Z), T),$$

$$(2.8 \text{ c}) \quad 'L^*(X, Y, Z, T) = g(L^*(X, Y, Z), T),$$

$$(2.8 \text{ d}) \quad 'V^*(X, Y, Z, T) = g(V^*(X, Y, Z), T),$$

$$(2.8 \text{ e}) \quad 'P^*(X, Y, Z, T) = g(P^*(X, Y, Z), T),$$

$$(2.8 \text{ f}) \quad 'S^*(X, Y, Z, T) = g(S^*(X, Y, Z), T),$$

$$(2.8 \text{ g}) \quad 'B^*(X, Y, Z, T) = g(B^*(X, Y, Z), T).$$

Consequently

$$(2.9) \quad 'W^*(X, Y, Z, T) = 'K(X, Y, Z, T) - \frac{1}{a^2(n-1)} [a^2 Ric(Y, Z)g(X, T) \\ - a^2 Ric(X, Z)g(Y, T)],$$

$$(2.10) \quad 'C^*(X, Y, Z, T) = 'K(X, Y, Z, T) + \frac{a^2}{(n+2)} [a^2 Ric(X, Z)g(Y, T) - a^2 Ric(Y, Z)g(X, T) \\ + g(X, Z)Ric(Y, T) - g(Y, Z)Ric(X, T)] \\ - \frac{a^2 R}{(n-1)(n-2)} [g(X, Z)g(Y, T) - g(Y, Z)g(X, T)],$$

$$(2.11) \quad 'L^*(X, Y, Z, T) = 'K(X, Y, Z, T) + \frac{a^2}{(n-2)} [a^2 Ric(X, Z)g(Y, T) \\ - a^2 Ric(Y, Z)g(X, T) + g(X, Z)Ric(Y, T) - g(Y, Z)Ric(X, T)],$$

$$(2.12) \quad 'V^*(X, Y, Z, T) = 'K(X, Y, Z, T) + \frac{a^2 R}{n(n-1)} [g(X, Z)g(Y, T) - g(Y, Z)g(X, T)],$$

$$(2.13) \quad 'P^*(X, Y, Z) = 'K(X, Y, Z, T) + \frac{1}{a^2(n+2)} [a^2 Ric(Y, Z)g(X, T) - a^2 Ric(X, Z)g(Y, T) \\ - Ric(\bar{Y}, Z)g(\bar{X}, T) + Ric(\bar{X}, Z)g(\bar{Y}, T) - 2Ric(X, \bar{Y})g(\bar{Z}, T)],$$

$$(2.14) \quad 'S^*(X, Y, Z, T) = 'K(X, Y, Z, T) - \frac{a^2}{(n+4)} [a^2 Ric(Y, Z)g(X, T) - a^2 Ric(X, Z)g(Y, T) \\ + Ric(X, \bar{Z})g(\bar{Y}, T) - Ric(Y, \bar{Z})g(\bar{X}, T) - 2Ric(\bar{X}, Y)\bar{Z} \\ - g(X, Z)Ric(Y, T) + g(Y, Z)Ric(X, T) + g(X, \bar{Z})Ric(\bar{Y}, T) \\ - g(Y, \bar{Z})Ric(\bar{X}, T) - 2g(\bar{X}, Y)Ric(\bar{Z}, T)],$$

$$(2.15) \quad 'B^*(X, Y, Z, T) = 'K(X, Y, Z, T) - \frac{a^2}{(n+4)} [a^2 Ric(Y, Z)g(X, T) - a^2 Ric(X, Z)g(Y, T) \\ - g(X, Z)Ric(Y, T) + g(Y, Z)Ric(X, T) + Ric(X, \bar{Z})g(\bar{Y}, Z) \\ - Ric(Y, \bar{Z})g(\bar{X}, T) - 2Ric(\bar{X}, Y)g(\bar{Z}, T) + g(X, \bar{Z})Ric(\bar{Y}, T) \\ - g(Y, \bar{Z})Ric(\bar{X}, T) - 2g(\bar{X}, Y)Ric(\bar{Z}, T)].$$

Using $'K(X, Y, Z, T) = -'K(Y, X, Z, T)$ in (2.9), we have

$$\begin{aligned} 'W^*(X, Y, Z, T) &= -'K(Y, X, Z, T) - \frac{1}{a^2(n-1)}[a^2 Ric(Y, Z)g(X, T) - a^2 Ric(X, Z)g(Y, T)] \\ &= -'W^*(Y, X, Z, T). \end{aligned}$$

Thus, we get

$$(2.16) \quad 'W^*(X, Y, Z, T) = -'W^*(Y, X, Z, T),$$

Which shows that ' W^* ' is skew-symmetric in first two slots.

Now,

$$\begin{aligned} (2.17) \quad &'W^*(X, Y, Z, T) + 'W^*(Y, Z, X, T) + 'W^*(Z, X, Y, T) \\ &= 'K(X, Y, Z, T) + 'K(Y, Z, X, T) + 'K(Z, X, Y, T) - \frac{1}{a^2(n-1)}[a^2 Ric(Y, Z)g(X, T) \\ &\quad - a^2 Ric(X, Z)g(Y, T) + a^2 Ric(Z, X)g(Y, T) - a^2 Ric(Y, Z)g(Z, T) \\ &\quad + a^2 Ric(X, Y)g(Z, T) - a^2 Ric(Z, Y)g(X, T)]. \end{aligned}$$

Using Bianchi's first identity for ' K ' in (2.17), we have

$$(2.18) \quad 'W^*(X, Y, Z, T) + 'W^*(Y, Z, X, T) + 'W^*(Z, X, Y, T) = 0.$$

Using ' $K(X, Y, Z, T) = -'K(Y, X, Z, T)$ ' in (2.10), we get

$$\begin{aligned} (2.19) \quad 'C^*(X, Y, Z, T) &= -'K(Y, X, Z, T) - \frac{a^2}{(n-2)}[a^2 Ric(X, Z)g(Y, T) - a^2 Ric(Y, Z)g(X, T) \\ &\quad + g(Y, Z)Ric(X, T) - g(X, Z)Ric(Y, T)] \\ &\quad + \frac{a^2 R}{(n-1)(n-2)}[g(Y, Z)g(X, T) - g(X, Z)g(Y, T)] \\ &= -'C^*(Y, X, Z, T), \end{aligned}$$

i.e., ' C^* ' is skew-symmetric in first two slots.

Now,

$$\begin{aligned} (2.20) \quad &'C^*(X, Y, Z, T) + 'C^*(Y, Z, X, T) + 'C^*(Z, X, Y, T) \\ &= 'K(X, Y, Z, T) + 'K(Y, Z, X, T) + 'K(Z, X, Y, T) \\ &\quad + \frac{a^2}{(n-2)}[a^2 Ric(Y, Z)g(X, T) - a^2 Ric(X, Z)g(Y, T) + g(X, Z)Ric(Y, T) \\ &\quad - g(Y, Z)Ric(X, T) + a^2 Ric(Z, X)g(Y, T) - a^2 Ric(Y, X)g(Z, T) \end{aligned}$$

$$\begin{aligned}
& + g(Y, X)Ric(Z, T) - g(Z, X)Ric(Y, T) + a^2 Ric(X, Y)g(Z, T) \\
& - a^2 Ric(Z, Y)g(X, T) + g(Z, Y)Ric(X, T) - g(X, Y)Ric(Z, T) \\
& - \frac{a^2 R}{(n-1)(n-2)} [g(X, Z)g(Y, T) - g(Y, Z)g(X, T) + g(Y, X)g(Z, T) \\
& - g(Z, X)g(Y, T) + g(Z, Y)g(X, T) - g(X, Y)g(Z, T)].
\end{aligned}$$

Using Bianchi's identities for ' K ' and $g(X, Y) = g(Y, X)$ in (2.20), we have

$$(2.21) \quad 'C^*(X, Y, Z, T) + 'C^*(Y, Z, X, T) + 'C^*(Z, X, Y, T) = 0.$$

Using $g(X, Y) = g(Y, X)$ in (2.11), we have

$$\begin{aligned}
(2.22) \quad 'L^*(X, Y, Z, T) &= -'K(Y, X, Z, T) - \frac{a^2}{(n-2)} [a^2 Ric(Y, Z)g(X, T) - a^2 Ric(X, Z)g(Y, Z) \\
&+ g(Y, Z)Ric(X, T) - g(X, Z)Ric(Y, T)] \\
&= -'L^*(Y, X, Z, T),
\end{aligned}$$

i.e., ' L^* ' is skew-symmetric in first two slots. Now

$$\begin{aligned}
(2.23) \quad 'L^*(X, Y, Z, T) + 'L^*(Y, Z, X, T) + 'L(Z, X, Y, T) \\
&= 'K(X, Y, Z, T) + 'K(Y, Z, X, T) + 'K(Z, X, Y, T) \\
&+ \frac{a^2}{(n-2)} [a^2 Ric(X, Z)g(Y, Z) \\
&- a^2 Ric(Y, Z)g(X, T) + g(X, Z)Ric(Y, T) - g(Y, Z)Ric(X, T)].
\end{aligned}$$

Using Bianchi's first identities for ' K ' in the equation (2.23), we get

$$(2.24) \quad 'L^*(X, Y, Z, T) + 'L^*(Y, Z, X, T) + 'L(Z, X, Y, T) = 0.$$

Using ' $K(X, Y, Z, T) = -'K(X, Y, T, Z)$ ' in (2.12), we obtain

$$\begin{aligned}
(2.25) \quad 'V^*(X, Y, Z, T) &= -'K(X, Y, T, Z) - \frac{a^2 R}{n(n-1)} [g(X, T)g(Y, Z) - g(Y, T)g(X, Z)] \\
&= -'V^*(X, Y, T, Z).
\end{aligned}$$

i.e. ' V^* ' is skew-symmetric in last two slots. Now

$$\begin{aligned}
(2.26) \quad 'V^*(X, Y, Z, T) + 'V^*(Y, Z, X, T) + 'V^*(Z, X, Y, T) \\
&= 'K(X, Y, Z, T) + 'K(Y, Z, X, T) + 'K(Z, X, Y, T)
\end{aligned}$$

$$+ \frac{a^2 R}{n(n-1)} [g(X, Z)g(Y, T) - g(Y, Z)g(X, T) + g(Y, X)g(Z, T) - g(Z, X)g(Y, T) + g(Z, Y)g(X, T) - g(X, Y)g(Z, T)].$$

Using Bianchi's first identity for $'K$ and $g(X, Y) = g(Y, X)$ in (2.26), we get

$$(2.27) \quad 'V^*(X, Y, Z, T) + 'V^*(Y, Z, X, T) + 'V^*(Z, X, Y, T) = 0.$$

3. Recurrence and Recurrence Symmetry of Different Kinds

Let $'A$ be any of the curvature tensors $'W^*, 'C^*, 'L^*, 'V^*, 'P^*, 'S^*$ and $'B^*$. Then we define recurrence of different kinds in $'A$ as follows :

Definition 3.1: A HGF- metric structure manifold is said to be (1)- recurrent in A , if

$$(3.1) \quad a^2 (\nabla' A)(X, Y, Z, T, U) - 'A((\nabla F)(\bar{X}, U), Y, Z, T) = a^2 B_1(U) 'A(X, Y, Z, T),$$

where $B_1(U)$ is a C^∞ function, called recurrence parameter.

Definition 3.2: The HGF- metric manifold is said to be (2)- recurrent in $'A$ if

$$(3.2) \quad a^2 (\nabla' A)(X, Y, Z, T, U) - 'A(X, (\nabla F)(\bar{Y}, U), Z, T) = a^2 B_1(U) 'A(X, Y, Z, T).$$

Definition 3.3: The manifold is said to be (3)- recurrent in $'A$ if

$$(3.3) \quad a^2 (\nabla' A)(X, Y, Z, T, U) - 'A(X, Y, (\nabla F)(\bar{Z}, U), T) = a^2 B_1(U) 'A(X, Y, Z, T).$$

Definition 3.4: The HGF- metric manifold is said to be (4)- recurrent in $'A$ if

$$(3.4) \quad a^2 (\nabla' A)(X, Y, Z, T, U) - 'A(X, Y, Z, (\nabla F)(\bar{T}, U), T) = a^2 B_1(U) 'A(X, Y, Z, T).$$

Definition 3.5: The HGF-metric manifold is said to be (12)- recurrent in $'A$, if

$$(3.5 a) \quad a^2 (\nabla' A)(\bar{X}, Y, Z, T, U) - 'A(\bar{X}, (\nabla F)(\bar{Y}, U), Z, T) + a^2 'A((\nabla F)(X, U), Y, Z, T) \\ = a^2 B_1(U) 'A(\bar{X}, Y, Z, T),$$

or equivalently

$$(3.5 b) \quad a^2 (\nabla' A)(X, \bar{Y}, Z, T, U) - 'A((\nabla F)(\bar{X}, U), \bar{Y}, Z, T) + a^2 'A(X, (\nabla F)(Y, U), Z, T) \\ = a^2 B_1(U) 'A(X, \bar{Y}, Z, T).$$

Definition 3.6: The HGF-metric manifold is said to be (13)- recurrent in $'A$, if

$$(3.6 a) \quad a^2 (\nabla' A)(X, Y, \bar{Z}, T, U) - 'A((\nabla F)(\bar{X}, U), Y, \bar{Z}, T) + a^2 'A(X, Y, (\nabla F)(Z, U), T) \\ = a^2 B_1(U) 'A(X, Y, \bar{Z}, T),$$

or equivalently

$$(3.6 \text{ b}) \quad a^2 (\nabla' A)(\bar{X}, Y, Z, T, U) - A(\bar{X}, Y, (\nabla F), (\bar{Z}, U), T) + a^2 A((\nabla F)(X, U), Y, Z, T) \\ = a^2 B_1(U)' A(\bar{X}, Y, Z, T).$$

Definition 3.7: The manifold is said to be (14)- recurrent in 'A, if

$$(3.7 \text{ a}) \quad a^2 (\nabla' A)(X, Y, Z, \bar{T}, U) - A((\nabla F)(\bar{X}, U), Y, Z, \bar{T}) + a^2 A(X, Y, Z, (\nabla F)(T, U)) \\ = a^2 B_1(U)' A(X, Y, Z, \bar{T}),$$

or equivalently

$$(3.7 \text{ b}) \quad a^2 (\nabla' A)(\bar{X}, Y, Z, T, U) - A(\bar{X}, Y, Z, (\nabla F), (\bar{T}, U)) + a^2 A((\nabla F)(X, U), Y, Z, T) \\ = a^2 B_1(U)' A(\bar{X}, Y, Z, T).$$

Definition 3.8: The HGF-metric manifold is said to be (23)- recurrent in 'A if

$$(3.8 \text{ a}) \quad a^2 (\nabla' A)(X, Y, \bar{Z}, T, U) - A(X, (\nabla F)(\bar{Y}, U), \bar{Z}, T) + a^2 A(X, Y, (\nabla F)(Z, U), T) \\ = a^2 B_1(U)' A(X, Y, \bar{Z}, T),$$

or equivalently

$$(3.8 \text{ b}) \quad a^2 (\nabla' A)(X, \bar{Y}, Z, T, U) - A(X, Y, (\nabla F)(\bar{Z}, U), T) + a^2 A(X, (\nabla F)(Y, U, Z, T) \\ = a^2 B_1(U)' A(X, \bar{Y}, Z, T).$$

Definition 3.9: The HGF-metric manifold is said to be (24)- recurrent in 'A, if

$$(3.9 \text{ a}) \quad a^2 (\nabla' A)(X, Y, Z, \bar{T}, U) - A(X, (\nabla F)(\bar{Y}, U), Z, \bar{T}) + a^2 A(X, Y, Z, (\nabla F)(T, U)) \\ = a^2 B_1(U)' A(X, Y, Z, \bar{T}),$$

or equivalently

$$(3.9 \text{ b}) \quad a^2 (\nabla' A)(X, \bar{Y}, Z, T, U) - A(X, \bar{Y}, Z, (\nabla F)(\bar{T}, U)) + a^2 A(X, (\nabla F)(Y, U), Z, T) \\ = a^2 B_1(U)' A(X, \bar{Y}, Z, T).$$

Definition 3.10: The HGF-metric manifold is said to be (34)- recurrent in 'A, if

$$(3.10 \text{ a}) \quad a^2 (\nabla' A)(X, Y, Z, \bar{T}, U) - A(X, Y, (\nabla F)(\bar{Z}, U), \bar{T}) + a^2 A(X, Y, Z, (\nabla F)(T, U)) \\ = a^2 B_1(U)' A(X, Y, Z, \bar{T}),$$

or equivalently

$$(3.10 \text{ b}) \quad a^2 (\nabla' A)(X, Y, \bar{Z}, T, U) - A(X, Y, \bar{Z}, (\nabla F)(\bar{T}, U)) + a^2 A(X, Y, (\nabla F)(Z, U), T) \\ = a^2 B_1(U)' A(X, Y, \bar{Z}, T).$$

Definition 3.11: The HGF-metric manifold is said to be (123)- recurrent in 'A' if

$$(3.11 \text{ a}) \quad a^2(\nabla' A)(X, \bar{Y}, \bar{Z}, T, U) - A((\nabla F)(\bar{X}, U), \bar{Y}, \bar{Z}, T) + a^2 A(X, (\nabla F)(Y, U), \bar{Z}, T) \\ + a^2 A(X, \bar{Y}, (\nabla F)(Z, U), T) = a^2 B_1(U)' A(X, \bar{Y}, \bar{Z}, T),$$

or equivalently

$$(3.11 \text{ b}) \quad a^2(\nabla' A)(\bar{X}, Y, \bar{Z}, T, U) - A(\bar{X}, (\nabla F)(\bar{Y}, U), \bar{Z}, T) + a^2 A((\nabla F)(X, U), Y, \bar{Z}, T) \\ + a^2 A(\bar{X}, Y, (\nabla F)(Z, U), T) = a^2 B_1(U)' A(\bar{X}, Y, \bar{Z}, T),$$

or equivalently

$$(3.11 \text{ c}) \quad a^2(\nabla' A)(\bar{X}, \bar{Y}, Z, T, U) - A(\bar{X}, \bar{Y}, (\nabla F)(\bar{Z}, U), T) + a^2 A((\nabla F)(X, U), \bar{Y}, Z, T) \\ + a^2 A(\bar{X}, (\nabla F)(Y, U), Z, T) = a^2 B_1(U)' A(\bar{X}, \bar{Y}, Z, T).$$

Definition 3.12: The HGF-metric manifold is said to be (124)- recurrent in 'A' if

$$(3.12 \text{ a}) \quad a^2(\nabla' A)(X, \bar{Y}, Z, \bar{T}, U) - A((\nabla F)(\bar{X}, U), \bar{Y}, Z, \bar{T}) + a^2 A(X, (\nabla F)(Y, U), Z, \bar{T}) \\ + a^2 A(X, \bar{Y}, Z, (\nabla F)(T, U)) = a^2 B_1(U)' A(X, \bar{Y}, Z, \bar{T}),$$

or equivalently

$$(3.12 \text{ b}) \quad a^2(\nabla' A)(\bar{X}, Y, Z, \bar{T}, U) - A(\bar{X}, (\nabla F)(\bar{Y}, U), Z, \bar{T}) + a^2 A((\nabla F)(X, U), Y, Z, \bar{T}) \\ + a^2 A(\bar{X}, Y, Z, (\nabla F)(T, U)) = a^2 B_1(U)' A(\bar{X}, Y, Z, \bar{T}),$$

or equivalently

$$(3.12 \text{ c}) \quad a^2(\nabla' A)(\bar{X}, \bar{Y}, Z, T, U) - A(\bar{X}, \bar{Y}, Z, (\nabla F)(\bar{T}, U)) + a^2 A((\nabla F)(X, U), \bar{Y}, Z, T) \\ + a^2 A(\bar{X}, (\nabla F)(Y, U), Z, T) = a^2 B_1(U)' A(\bar{X}, \bar{Y}, Z, T).$$

Definition 3.13: The HGF-metric manifold is said to be (134)- recurrent in 'A' if

$$(3.13 \text{ a}) \quad a^2(\nabla' A)(X, Y, \bar{Z}, \bar{T}, U) - A((\nabla F)(\bar{X}, U), Y, \bar{Z}, \bar{T}) + a^2 A(X, Y, (\nabla F)(Z, U), \bar{T}) \\ + a^2 A(X, Y, \bar{Z}, (\nabla F)(T, U)) = a^2 B_1(U)' A(X, Y, \bar{Z}, \bar{T}),$$

or equivalently

$$(3.13 \text{ b}) \quad a^2(\nabla' A)(\bar{X}, Y, Z, \bar{T}, U) - A(\bar{X}, Y, (\nabla F)(\bar{Z}, U), \bar{T}) + a^2 A((\nabla F)(X, U), Y, Z, \bar{T}) \\ + a^2 A(\bar{X}, Y, Z, (\nabla F)(T, U)) = a^2 B_1(U)' A(\bar{X}, Y, Z, \bar{T}),$$

or equivalently

$$(3.13 \text{ c}) \quad a^2(\nabla' A)(\bar{X}, Y, \bar{Z}, T, U) - A(\bar{X}, Y, \bar{Z}, (\nabla F)(\bar{T}, U)) + a^2 A((\nabla F)(X, U), Y, \bar{Z}, T) \\ + a^2 A(\bar{X}, Y, (\nabla F)(Z, U), T) = a^2 B_1(U)' A(\bar{X}, Y, \bar{Z}, T).$$

Definition 3.14: The HGF-metric manifold is said to be (234)- recurrent in 'A if

$$(3.14 \text{ a}) \quad a^2 (\nabla' A)(X, \bar{Y}, Z, \bar{T}, U) - 'A(X, \bar{Y}, (\nabla F)(\bar{Z}, U), \bar{T}) + a^2 'A(X, (\nabla F)(Y, U), Z, \bar{T}) \\ + a^2 'A(X, \bar{Y}, Z, (\nabla F)(T, U)) = a^2 B_1(U) 'A(X, \bar{Y}, Z, \bar{T}),$$

or equivalently

$$(3.14 \text{ b}) \quad a^2 (\nabla' A)(X, Y, \bar{Z}, \bar{T}, U) - 'A(X, (\nabla F)(\bar{Y}, U), \bar{Z}, \bar{T}) + a^2 'A(X, Y, (\nabla F)(Z, U), \bar{T}) \\ + a^2 'A(X, Y, \bar{Z}, (\nabla F)(T, U)) = a^2 B_1(U) 'A(X, Y, \bar{Z}, \bar{T}),$$

or equivalently

$$(3.14 \text{ c}) \quad a^2 (\nabla' A)(X, \bar{Y}, \bar{Z}, T, U) - 'A(X, \bar{Y}, \bar{Z}, (\nabla F)(\bar{T}, U) + a^2 'A(X, (\nabla F)(Y, U), \bar{Z}, T) \\ + a^2 'A(X, \bar{Y}, (\nabla F)(Z, U), T) = a^2 B_1(U) 'A(X, \bar{Y}, Z, T).$$

Definition 3.15: The HGF-metric manifold is said to be (1234)- recurrent in 'A if

$$(3.15 \text{ a}) \quad a^2 (\nabla' A)(X, \bar{Y}, \bar{Z}, \bar{T}, U) - 'A((\nabla F)(\bar{X}, U), \bar{Y}, \bar{Z}, \bar{T}) + a^2 'A(X, (\nabla F)(Y, U), \bar{Z}, \bar{T}) \\ + a^2 'A(X, \bar{Y}, (\nabla F)(Z, U), \bar{T}) + a^2 'A(X, \bar{Y}, \bar{Z}, (\nabla F)(T, U)) = a^2 B_1(U) 'A(X, \bar{Y}, \bar{Z}, \bar{T}),$$

or equivalently

$$(3.15 \text{ b}) \quad a^2 (\nabla' A)(\bar{X}, Y, \bar{Z}, \bar{T}, U) - 'A(\bar{X}, (\nabla F)(\bar{Y}, U), \bar{Z}, \bar{T}) + a^2 'A(\bar{X}, Y, \bar{Z}, (\nabla F)(T, U)) \\ + a^2 'A((\nabla F)(X, U), Y, \bar{Z}, \bar{T}) + a^2 'A(\bar{X}, Y, (\nabla F)(Z, U), \bar{T}) = a^2 B_1(U) 'A(\bar{X}, Y, \bar{Z}, \bar{T}),$$

or equivalently

$$(3.15 \text{ c}) \quad a^2 (\nabla' A)(\bar{X}, \bar{Y}, \bar{Z}, T, U) - 'A(\bar{X}, \bar{Y}, \bar{Z}, (\nabla F)(\bar{T}, U)) + a^2 'A((\nabla F)(X, U), \bar{Y}, \bar{Z}, T) \\ + a^2 'A(\bar{X}, (\nabla F)(Y, U), \bar{Z}, T) + a^2 'A(\bar{X}, \bar{Y}, (\nabla F)(Z, U), T) = a^2 B_1(U) 'A(\bar{X}, \bar{Y}, \bar{Z}, T),$$

or equivalently

$$(3.15 \text{ d}) \quad a^2 (\nabla' A)(\bar{X}, \bar{Y}, Z, \bar{T}, U) - 'A(\bar{X}, \bar{Y}, (\nabla F)(\bar{Z}, U), \bar{T}) + a^2 'A((\nabla F)(X, U), \bar{Y}, Z, \bar{T}) \\ + a^2 'A(\bar{X}, (\nabla F)(Y, U), Z, \bar{T}) + a^2 'A(\bar{X}, \bar{Y}, Z, (\nabla F)(T, U)) = a^2 B_1(U) 'A(\bar{X}, \bar{Y}, Z, \bar{T}).$$

Definition 3.16: The (1), (2), (3), (4), (12), (13), (14), (23), (24), (34), (123), (124), (234) and (1234)-recurrent HGF-metric manifolds are said to be 'A-symmetric if

$$(3.16) \quad B_1(U) = 0$$

in the above equations.

Theorem 3.1: The 'A-(1)-recurrent HGF-metric structure manifold is 'A-(4)-recurrent for the same recurrence parameter, if

$$(3.17) \quad 'A(X, Y, Z, (\nabla F)(\bar{T}, U)) = 'A((\nabla F)(\bar{X}, U), Y, Z, T).$$

Proof: Suppose that the $'A$ -(1)- recurrent HGF-metric structure manifold is $'A$ -(4)- recurrent then comparing the equations (3.1) and (3.4), we get the equation (3.17). Conversely, suppose that the equation (3.17) holds and the manifold is $'A$ -(1)- recurrent then using the equation (3.17) in (3.1), we get

$$a^2(\nabla' A)(X, Y, Z, T, U) - 'A(X, Y, Z, (\nabla F)(\bar{T}, U)) = a^2 B_1(U) 'A(X, Y, Z, T),$$

which shows that the manifold is $'A$ -(4)- recurrent.

Theorem 3.2: The $'A$ -(2)- recurrent HGF-metric structure manifold is $'A$ -(4)- recurrent for the same recurrence parameter, if

$$(3.18) \quad 'A(X, (\nabla F)(\bar{Y}, U), Z, T) = 'A(X, Y, Z, (\nabla F)(\bar{T}, U)).$$

Proof: Suppose that the $'A$ -(2)- recurrent manifold is $'A$ -(4)- recurrent for the same recurrence parameter, then comparing the equations (3.2) and (3.4), we get the equation (3.18). Conversely, let the equation (3.18) holds and the manifold is $'A$ -(2)- recurrent then using the equation (3.18) in (3.2), we get

$$a^2(\nabla' A)(X, Y, Z, T, U) - 'A(X, Y, Z, (\nabla F)(\bar{T}, U)) = a^2 B_1(U) 'A(X, Y, Z, T),$$

which shows that the manifold is $'A$ -(4)- recurrent

Theorem 3.3: The $'A$ -(3)- recurrent HGF-metric structure manifold is $'A$ -(4)- recurrent for the same recurrence parameter, if

$$(3.19) \quad 'A(X, Y, (\nabla F)(\bar{Z}, U), T) = 'A(X, Y, Z, (\nabla F)(\bar{T}, U)).$$

Proof: Suppose that the $'A$ -(3)- recurrent HGF-metric structure manifold is $'A$ -(4)- recurrent for the same recurrence parameter, then comparing the equations (3.3) and (3.4), we get the equation (3.19). Conversely, let the equation (3.19) holds and the manifold is $'A$ -(3)- recurrent then using the equation (3.19) in (3.3), we get

$$a^2(\nabla' A)(X, Y, Z, T, U) - 'A(X, Y, Z, (\nabla F)(\bar{T}, U)) = a^2 B_1(U) 'A(X, Y, Z, T),$$

which shows that the manifold is $'A$ -(4)- recurrent.

Note 3.1: Similarly, it can be proved that the $'A$ -(1)-recurrent manifold is $'A$ -(2)- recurrent or $'A$ -(1)-recurrent manifold is $'A$ -(3)-recurrent or $'A$ -(2)-recurrent manifold is $'A$ -(3)-recurrent for the same recurrence parameter, if

$$'A(\nabla F)(\bar{X}, U), Y, Z, T) = 'A(X, (\nabla F)(\bar{Y}, U), Z, T), \text{ or}$$

$$'A(\nabla F)(\bar{X}, U), Y, Z, T) = 'A(X, Y, (\nabla F)(\bar{Z}, U), T), \text{ or}$$

$$'A(X, (\nabla F)(\bar{Y}, U), Z, T) = 'A(X, Y, (\nabla F)(\bar{Z}, U), T).$$

Theorem 3.4. The $'A$ -(12)- recurrent HGF-metric structure manifold is $'A$ -(14)- recurrent for the same recurrence parameter, if

$$(3.20) \quad 'A(X, (\nabla F)(\bar{Y}, U), Z, T) = 'A(X, Y, Z, (\nabla F)(\bar{T}, U)).$$

Proof: Suppose that the $'A$ -(12)- recurrent HGF-metric structure manifold is $'A$ -(14)-recurrent for the same recurrence parameter, then comparing the equations (3.5)a and (3.7)b, we get the equation (3.20). Conversely, let the equation (3.20) holds and the manifold is $'A$ -(12)- recurrent then using the equation (3.20) in (3.5)a, we get

$$a^2(\nabla' A)(\bar{X}, Y, Z, T, U) - 'A(\bar{X}, Y, Z, (\nabla F)(\bar{T}, U)) + a^2 'A((\nabla F)(X, U), Y, Z, T) = a^2 B_1(U) 'A(\bar{X}, Y, Z, T),$$

which shows that the manifold is $'A$ -(14)- recurrent.

Theorem 3.5: The $'A$ -(12)- recurrent HGF-metric structure manifold is $'A$ -(24)-recurrent for the same recurrence parameter, if

$$(3.21) \quad 'A((\nabla F)(\bar{X}, U), Y, Z, T) = 'A(X, Y, Z, (\nabla F)(\bar{T}, U)).$$

Proof: Suppose that the $'A$ -(12)- recurrent HGF- manifold is $'A$ -(24)-recurrent then comparing the equations (3.5)b and (3.9)b, we get the equations (3.21). Conversely, suppose that the equation (3.21) holds and the manifold is $'A$ -(12)- recurrent then using the equation (3.21) in (3.5)b, we get

$$a^2(\nabla' A)(X, \bar{Y}, Z, T, U) - 'A(X, Y, Z, (\nabla F)(\bar{T}, U)) + a^2 'A(X, (\nabla F)(Y, U), Z, T) = a^2 B_1(U) 'A(X, \bar{Y}, Z, T),$$

which shows that the manifold is $'A$ -(24)- recurrent.

Note 3.2: Similarly, it can be proved that the $'A$ -(13)-recurrent HGF-metric manifold is $'A$ -(14)-recurrent for the same recurrence parameter, if

$$'A(X, Y, (\nabla F)(\bar{Z}, U), T) = 'A(X, Y, Z, (\nabla F)(\bar{T}, U)),$$

$'A$ -(12)-recurrent manifold is $'A$ -(13)-recurrent, if

$$'A(X, (\nabla F)(\bar{Y}, U), Z, T) = 'A(X, Y, (\nabla F)(\bar{Z}, U), T),$$

$'A$ -(12)-recurrent manifold is $'A$ -(23)-recurrent, if

$$'A((\nabla F)(\bar{X}, U), Y, Z, T) = 'A(X, Y, (\nabla F)(\bar{Z}, U), T),$$

$'A$ -(13)-recurrent manifold is $'A$ -(34)-recurrent, if

$$'A((\nabla F)(\bar{X}, U), Y, Z, T) = 'A(X, Y, Z, (\nabla F)(\bar{T}, U)),$$

$'A$ -(13)-recurrent manifold is $'A$ -(23)-recurrent, if

$$'A((\nabla F)(\bar{X}, U), Y, Z, T) = 'A(X, (\nabla F)(\bar{Y}, U), Z, T),$$

$'A$ -(14)-recurrent manifold is $'A$ -(24)-recurrent, if

$$'A((\nabla F)(\bar{X}, U), Y, Z, T) = 'A(X, (\nabla F)(\bar{Y}, U), Z, T),$$

$'A$ -(14)-recurrent manifold is $'A$ -(34)-recurrent, if

$$'A((\nabla F)(\bar{X}, U), Y, Z, T) = 'A(X, Y, (\nabla F)(\bar{Z}, U), T).$$

Theorem 3.6: The $'A$ -(14)- recurrent HGF-metric structure manifold is $'A$ -(1)-recurrent for the same recurrence parameter, if

$$(3.22) \quad 'A(X, Y, Z, (\nabla F)(T, U)) = 0.$$

Proof: Let the manifold is (1)-recurrent in $'A$ then barring T in the equation (3.1), we get

$$(3.23) \quad a^2(\nabla' A)(X, Y, Z, \bar{T}, U) - 'A((\nabla F)(\bar{X}, U), Y, Z, \bar{T}) = a^2 B_1(U) 'A(X, Y, Z, \bar{T}).$$

Now, suppose that the $'A$ -(14)- recurrent HGF-metric manifold is $'A$ -(1)-recurrent then comparing the equations (3.23) and (3.7)a, we get the equation (3.22).

Theorem 3.7: The $'A$ -(14)- recurrent HGF-metric structure manifold is $'A$ -(4)-recurrent for the same recurrence parameter, if

$$(3.24) \quad 'A((\nabla F)(X, U), Y, Z, T) = 0.$$

Proof: Let the manifold is (4)-recurrent in $'A$ then barring X in the equation (3.4), we get

$$(3.25) \quad a^2(\nabla' A)(\bar{X}, Y, Z, T, U) - 'A(\bar{X}, Y, Z, (\nabla F)(\bar{T}, U)) = a^2 B_1(U) 'A(\bar{X}, Y, Z, T).$$

Now, suppose that the $'A$ -(14)- recurrent HGF-metric manifold is $'A$ -(4)-recurrent then comparing the equations (3.25) and (3.7)b, we get the equation (3.24).

Theorem 3.8: The $'A$ -(24)- recurrent HGF-metric structure manifold is $'A$ -(2)-recurrent for the same recurrence parameter, if

$$(3.26) \quad 'A(X, Y, Z, (\nabla F)(T, U)) = 0.$$

Proof: Let the HGF-metric manifold is (2)-recurrent in $'A$ then barring T in the equation (3.2), we get

$$(3.27) \quad a^2(\nabla' A)(X, Y, Z, \bar{T}, U) - 'A(X, (\nabla F)(\bar{Y}, U), Z, \bar{T}) = a^2 B_1(U) 'A(X, Y, Z, \bar{T}).$$

Now, suppose that the $'A$ -(24)- recurrent HGF-metric manifold is $'A$ -(2)-recurrent then comparing the equations (3.27) and (3.9)a, we get the equation (3.26).

Theorem 3.9: The $'A$ -(24)- recurrent HGF-metric manifold is $'A$ -(4)-recurrent for the same recurrence parameter, if

$$(3.28) \quad 'A(X, (\nabla F)(Y, U), Z, U) = 0.$$

Proof: Let the HGF-metric manifold is (4)-recurrent in $'A$ then barring Y in the equation (3.4), we get

$$(3.29) \quad a^2(\nabla' A)(X, \bar{Y}, Z, T, U) - 'A(X, \bar{Y}, Z, (\nabla F)(\bar{T}, U)) = a^2 B_1(U) 'A(X, \bar{Y}, Z, T).$$

Now, suppose that the $'A$ -(24)- recurrent HGF-metric manifold is $'A$ -(4)-recurrent then comparing the equations (3.29) and (3.9)b, we get the equation (3.28).

Note 3.3: Similarly, it can be proved that the $'A$ -(34)-recurrent HGF-metric manifold is $'A$ -(3) and $'A$ -(4) -recurrent for the same recurrence parameter, if

' $A(X, Y, Z, (\nabla F)(T, U)) = 0$ and ' $A(X, Y, (\nabla F)(Z, U), T) = 0$ respectively.

' A -(12)-recurrent manifold is ' A -(1) and ' A -(2)-recurrent, if

' $A(X, (\nabla F)(Y, U), Z, T) = 0$ and ' $A((\nabla F)(X, U), Y, Z, T) = 0$, respectively.

' A -(13)-recurrent HGF-metric manifold is ' A -(1) and ' A -(3)-recurrent, if

' $A(X, Y, (\nabla F)(Z, U), T) = 0$, and ' $A((\nabla F)(X, U), Y, Z, T) = 0$, respectively and

' A -(23)-recurrent HGF-metric manifold is ' A -(2) and ' A -(3)-recurrent, if

' $A(X, Y, (\nabla F)(Z, U), T) = 0$, and ' $A(X, (\nabla F)(Y, U), Z, T) = 0$, respectively.

Theorem 3.10: The ' A -(124)- recurrent HGF-metric manifold is ' A -(4)-recurrent for the same recurrence parameter, if

$$(3.30) \quad 'A((\nabla F)(X, U), \bar{Y}, Z, T) + 'A(\bar{X}, (\nabla F)(Y, U), Z, T) = 0.$$

Proof: Let the HGF-metric manifold is (4)-recurrent in ' A then barring X and Y in the equation (3.4), we have

$$(3.31) \quad a^2(\nabla' A)(\bar{X}, \bar{Y}, Z, T, U) - 'A(\bar{X}, \bar{Y}, Z, (\nabla F)(\bar{T}, U)) = a^2 B_1(U) 'A(\bar{X}, \bar{Y}, Z, T).$$

Now, suppose that the ' A -(124)- recurrent HGF-metric manifold is ' A -(4)-recurrent for the same recurrence parameter, then comparing the equations (3.31) and (3.12)c, we get the equation (3.30).

Theorem 3.11: The ' A -(124)- recurrent HGF-metric manifold is ' A -(24)-recurrent for the same recurrence parameter, if

$$(3.32) \quad 'A((\nabla F)(X, U), \bar{Y}, Z, T) = 0.$$

Proof: Let the HGF-metric manifold is (24)-recurrent in ' A then barring X in the equation (3.9)b, we obtain

$$(3.33) \quad a^2(\nabla' A)(\bar{X}, \bar{Y}, Z, T, U) - 'A(\bar{X}, \bar{Y}, Z, (\nabla F)(\bar{T}, U)) + a^2 'A(X, (\nabla F)(Y, U), Z, T) \\ = a^2 B_1(U) 'A(\bar{X}, \bar{Y}, Z, T).$$

Now, suppose that the ' A -(124)- recurrent HGF-metric manifold is ' A -(24)-recurrent for the same recurrence parameter, then comparing the equations (3.33) and (3.12)c, we get the equation (3.32).

Theorem 3.12: The ' A -(124)- recurrent HGF-metric manifold is ' A -(12)-recurrent for the same recurrence parameter, if

$$(3.34) \quad 'A(\bar{X}, Y, Z, (\nabla F)(T, U)) = 0.$$

Proof: Let the HGF-metric manifold is (12)-recurrent in ' A then barring T in the equation (3.5)a, we get

$$(3.35) \quad a^2(\nabla' A)(\bar{X}, Y, Z, \bar{T}, U) - 'A(X, (\nabla F)(\bar{Y}, U), Z, \bar{T}) + a^2 'A((\nabla F)(X, U), Y, Z, \bar{T})$$

$$= a^2 B_1(U) A(\bar{X}, Y, Z, \bar{T}).$$

Now, suppose that the $'A$ -(124)- recurrent HGF-metric manifold is $'A$ -(12)-recurrent then comparing the equations (3.35) and (3.12)b, we obtain the equation (3.35).

Note 3.4: Similarly, it can be proved that the $'A$ -(124)-recurrent HGF-metric manifold is $'A$ -(1) or $'A$ -(2) or $'A$ -(14)-recurrent, if

$$'A((\nabla F)(X, U), Y, Z, \bar{T}) + 'A(X, \bar{Y}, Z, (\nabla F)(T, U)) = 0,$$

$$\text{or } 'A((\nabla F)(X, U), Y, Z, \bar{T}) + 'A(\bar{X}, Y, Z, (\nabla F)(T, U)) = 0,$$

$$'A(X, (\nabla F)(Y, U), Z, \bar{T}) = 0.$$

Note 3.5: Theorems for $'A$ -(123), $'A$ -(234) and $'A$ -(134)-recurrent HGF-metric manifold can also be stated and proved similarly.

Theorem 3.13: The $'A$ -(1234)- recurrent HGF-metric manifold is $'A$ -(4)-recurrent for the same recurrence parameter, if

$$(3.36) \quad 'A((\nabla F)(X, U), \bar{Y}, \bar{Z}, T) + 'A(\bar{X}, (\nabla F)(Y, U), \bar{Z}, T) + 'A(\bar{X}, \bar{Y}, (\nabla F)(Z, U), T) = 0.$$

Proof: Let the HGF-metric manifold is (4)-recurrent in $'A$ then barring X, Y and Z in the equation (3.4), we have

$$(3.37) \quad a^2 (\nabla' A)(\bar{X}, \bar{Y}, \bar{Z}, T, U) - 'A(\bar{X}, \bar{Y}, \bar{Z}, (\nabla F)(\bar{T}, U)) = a^2 B_1(U) A(\bar{X}, \bar{Y}, \bar{Z}, T).$$

Now, suppose that the $'A$ -(1234)- recurrent HGF-metric manifold is $'A$ -(4)-recurrent for the same recurrence parameter, then comparing the equations (3.37) and (3.15)c, we get (3.36).

Theorem 3.14: The $'A$ -(1234)- recurrent HGF-metric manifold is $'A$ -(34)-recurrent for the same recurrence parameter, if

$$(3.38) \quad 'A((\nabla F)(X, U), \bar{Y}, Z, \bar{T}) + 'A(\bar{X}, (\nabla F)(Y, U), Z, \bar{T}) = 0.$$

Proof: Let the HGF-metric manifold is (34)-recurrent in $'A$ then barring X and Y in the equation (3.10)a, we have

$$(3.39) \quad a^2 (\nabla' A)(\bar{X}, \bar{Y}, Z, \bar{T}, U) - 'A(\bar{X}, \bar{Y}, (\nabla F)(\bar{Z}, U), T) + a^2 'A(\bar{X}, \bar{Y}, Z, (\nabla F)(T, U)) \\ = a^2 B_1(U) A(\bar{X}, \bar{Y}, \bar{Z}, T).$$

Now, suppose that the $'A$ -(1234)- recurrent HGF-metric manifold is $'A$ -(34)-recurrent for the same recurrence parameter, then comparing the equations (3.39) and (2.15)d, we get the (3.38).

Theorem 3.15: The $'A$ -(1234)- recurrent HGF-metric manifold is $'A$ -(234)-recurrent for the same recurrence parameter, if

$$(3.40) \quad 'A((\nabla F)(X, U), Y, Z, T) = 0.$$

Proof: Let the HGF-metric manifold is (234)-recurrent in $'A$ then barring X in the equation (3.14)a, we obtain

$$(3.41) \quad a^2(\nabla' A)(\bar{X}, \bar{Y}, Z, \bar{T}, U) - 'A(\bar{X}, \bar{Y}, (\nabla F)(Z, U), \bar{T}) + a^2 'A(\bar{X}, (\nabla F)(Y, U), Z, \bar{T}) \\ + a^2 'A(\bar{X}, \bar{Y}, Z, (\nabla F)(T, U)) = a^2 B_1(U) 'A(\bar{X}, \bar{Y}, \bar{Z}, T).$$

Now, suppose that the $'A$ -(1234)- recurrent HGF-metric manifold is $'A$ -(234)-recurrent then comparing the equations (3.41) and (3.15)d, we get the (3.40).

Theorem 3.16: The $'A$ -(1234)- recurrent HGF-metric manifold is $'A$ -(123)-recurrent for the same recurrence parameter, if

$$(3.42) \quad 'A(X, \bar{Y}, \bar{Z}, (\nabla F)(T, U)) = 0.$$

Proof: Let the HGF-metric manifold is (134)-recurrent in $'A$ then barring T in the equation (3.11)a, we get

$$(3.43) \quad a^2(\nabla' A)(X, \bar{Y}, \bar{Z}, \bar{T}, U) - 'A((\nabla F)(\bar{X}, U), \bar{Y}, \bar{Z}, \bar{T}) + a^2 'A(X, (\nabla F)(Y, U), \bar{Z}, \bar{T}) \\ + a^2 'A(X, \bar{Y}, (\nabla F)(Z, U), \bar{T}) = a^2 B_1(U) 'A(X, \bar{Y}, \bar{Z}, \bar{T}).$$

Now, suppose that the $'A$ -(1234)- recurrent HGF-metric manifold is $'A$ -(123)-recurrent for the same recurrence parameter, then comparing the equations (3.43) and (3.15)a, we get (3.42).

Note 3.5: Similarly, it can be proved that the $'A$ -(1234)-recurrent HGF-metric manifold is $'A$ -(1) or $'A$ -(2) or $'A$ -(3) or $'A$ -(12) or $'A$ -(23) or $'A$ -(13) or $'A$ -(24) or $'A$ -(14) and $'A$ -(124) or $'A$ -(134) recurrent, if

$$'A(X, (\nabla F)(Y, U), \bar{Z}, \bar{T}) + 'A(X, \bar{Y}, (\nabla F)(Z, U), \bar{T}) + 'A(X, \bar{Y}, \bar{Z}, (\nabla F)(T, U)) = 0,$$

$$\text{or } 'A((\nabla F)(X, U), Y, \bar{Z}, \bar{T}) + 'A(\bar{X}, Y, (\nabla F)(Z, U), \bar{T}) + 'A(\bar{X}, Y, \bar{Z}, (\nabla F)(T, U)) = 0,$$

$$\text{or } 'A((\nabla F)(X, U), \bar{Y}, Z, \bar{T}) + 'A(\bar{X}, (\nabla F)(Y, U), Z, \bar{T}) + 'A(\bar{X}, \bar{Y}, Z, (\nabla F)(T, U)) = 0,$$

$$\text{or } 'A(X, \bar{Y}, (\nabla F)(Z, U), \bar{T}) + 'A(X, \bar{Y}, \bar{Z}, (\nabla F)(T, U)) = 0,$$

$$\text{or } 'A((\nabla F)(X, U), Y, \bar{Z}, \bar{T}) + 'A(\bar{X}, Y, \bar{Z}, (\nabla F)(T, U)) = 0,$$

$$\text{or } 'A(X, (\nabla F)(Y, U), \bar{Z}, \bar{T}) + 'A(X, \bar{Y}, \bar{Z}, (\nabla F)(T, U)) = 0,$$

$$\text{or } 'A((\nabla F)(X, U), Y, \bar{Z}, \bar{T}) + 'A(\bar{X}, Y, (\nabla F)(Z, U), \bar{T}) = 0,$$

$$\text{or } 'A(X, (\nabla F)(Y, U), \bar{Z}, \bar{T}) + 'A(X, \bar{Y}, (\nabla F)(Z, U), \bar{T}) = 0,$$

$$\text{and } 'A(X, \bar{Y}, (\nabla F)(Z, U), \bar{T}) = 0,$$

$$\text{or } 'A(X, (\nabla F)(Y, U), \bar{Z}, \bar{T}) = 0 \text{ respectively.}$$

Theorem 3.17: *In the (4)-recurrent HGF-metric manifold, if any two of the following conditions hold for the same recurrent parameter then the third also holds:*

- (i) *It is associated Pseudo conformal -(4)- recurrent,*
- (ii) *It is associated Pseudo conharmonic -(4)- recurrent,*
- (iii) *It is associated Pseudo concircular -(4)- recurrent.*

Proof: From the equations (2.10), (2.11) and (2.12), we have

$$(3.44) \quad 'L^*(X, Y, Z, T) = 'C^*(X, Y, Z, T) + \frac{n}{n-2} \{ 'K(X, Y, Z, T) - 'V^*(X, Y, Z, T) \}.$$

Barring T in the equation (3.44), we obtain

$$(3.45) \quad 'L^*(X, Y, Z, \bar{T}) = 'C^*(X, Y, Z, \bar{T}) + \frac{n}{n-2} \{ 'K(X, Y, Z, \bar{T}) - 'V^*(X, Y, Z, \bar{T}) \}.$$

From the equation (3.44), we have

$$(3.46) \quad a^2 B_1(U) 'L^*(X, Y, Z, T) = a^2 B_1(U) 'C^*(X, Y, Z, T) + \frac{n}{n-2} a^2 B_1(U) \{ 'K(X, Y, Z, T) - 'V^*(X, Y, Z, T) \}.$$

Differentiating (3.44) with respect to U, using (3.44) and then barring T in the resulting equation, we have

$$(3.47) \quad a^2 (\nabla 'L^*)(X, Y, Z, T, U) - 'L^*(X, Y, Z, (\nabla F)(\bar{T}, U)) = a^2 (\nabla 'C^*)(X, Y, Z, T, U) - 'C^*(X, Y, Z, (\nabla F)(\bar{T}, U)) + \frac{n}{n-2} \{ a^2 (\nabla 'K)(X, Y, Z, T, U) - 'K(X, Y, Z, (\nabla F)(\bar{T}, U)) + a^2 (\nabla 'V^*)(X, Y, Z, T, U) - 'V^*(X, Y, Z, (\nabla F)(\bar{T}, U)) \}$$

Subtracting (3.46) from (3.47), we have

$$(3.48) \quad a^2 (\nabla 'L^*)(X, Y, Z, T, U) - 'L^*(X, Y, Z, (\nabla F)(\bar{T}, U)) - a^2 B_1(U) 'L^*(X, Y, Z, T) = a^2 (\nabla 'C^*)(X, Y, Z, T, U) - 'C^*(X, Y, Z, (\nabla F)(\bar{T}, U)) - a^2 B_1(U) 'C^*(X, Y, Z, T) + \frac{n}{n-2} \{ a^2 (\nabla 'K)(X, Y, Z, T, U) - 'K(X, Y, Z, (\nabla F)(\bar{T}, U)) - a^2 B_1(U) 'K(X, Y, Z, T) - a^2 (\nabla 'V^*)(X, Y, Z, T, U) + 'V^*(X, Y, Z, (\nabla F)(\bar{T}, U)) - a^2 B_1(U) a^2 B_1(U) 'V^*(X, Y, Z, T) \}.$$

Now, if the (4)-recurrent HGF-metric manifold is associated Pseudo conformal -(4)-recurrent and associated Pseudo concircular -(4)- recurrent, for the same recurrent parameter then from the equation (3.48), we have.

$$a^2 (\nabla 'L^*)(X, Y, Z, T, U) - 'L^*(X, Y, Z, (\nabla F)(\bar{T}, U)) = a^2 B_1(U) 'A(X, Y, Z, T),$$

which shows that the manifold is associated Pseudo conharmonic -(4)- recurrent.

Similarly, it can be stated and proved that if the (4)-recurrent HGF-metric manifold is either associated Pseudo conformal $-(4)$ - recurrent and associated Pseudo conharmonic $-(4)$ - recurrent or associated Pseudo concircular $-(4)$ - recurrent and associated Pseudo conharmonic $-(4)$ - recurrent for the same recurrent parameter then it is either associated Pseudo concircular $-(4)$ - recurrent or associated Pseudo conformal $-(4)$ - recurrent.

Theorem 3.18: *In the (4)-symmetric HGF-metric manifold if any two of the following conditions hold for the same recurrent parameter then the third also holds:*

- (i) *It is associated Pseudo concircular $-(4)$ - symmetric,*
- (ii) *It is associated Pseudo conharmonic $-(4)$ - symmetric,*
- (iii) *It is associated Pseudo conformal $-(4)$ - symmetric.*

Proof: If the (4)-symmetric HGF-metric manifold is associated Pseudo conformal $-(4)$ - symmetric and associated Pseudo concircular $-(4)$ - symmetric for the same recurrent parameter then from the equation (3.48), we have

$$(3.49) \quad a^2 (\nabla' L^*)(X, Y, Z, T, U) - 'L^*(X, Y, Z, (\nabla F)(\bar{T}, U)) = 0,$$

which shows that the manifold is associated Pseudo conharmonic $-(4)$ - symmetric.

Similarly, it can be shown that if the (4)- symmetric HGF-metric manifold is either associated Pseudo conformal $-(4)$ - symmetric and associated Pseudo conharmonic $-(4)$ - symmetric or associated Pseudo concircular $-(4)$ - symmetric and associated Pseudo conharmonic $-(4)$ - symmetric then it is either associated Pseudo concircular $-(4)$ - symmetric or associated Pseudo conformal $-(4)$ - symmetric manifold.

Remark 3.1: Theorems of the type (3.17) and (3.18) can also be stated and proved for $'A$ -(1), $'A$ -(2), $'A$ -(3)-recurrent and $'A$ -(1), $'A$ -(2), $'A$ -(3)-symmetric manifolds.

Theorem 3.19: *In the (14)-recurrent HGF-metric manifold if any two of the following conditions hold for the same recurrent parameter then the third also holds:*

- (i) *It is associated Pseudo concircular $-(14)$ - recurrent,*
- (ii) *It is associated Pseudo conharmonic $-(14)$ - recurrent,*
- (iii) *It is associated Pseudo conformal $-(14)$ - recurrent.*

Proof: Barring X in the equation (3.44) we have

$$(3.50) \quad 'L^*(\bar{X}, Y, Z, T) = 'C^*(\bar{X}, Y, Z, T) + \frac{n}{n-2} \{ 'K(\bar{X}, Y, Z, T) - 'V^*(\bar{X}, Y, Z, T) \}.$$

From the equation (3.50), we obtain

$$(3.51) \quad a^2 B_1(U) 'L^*(X, Y, Z, T) \\ = a^2 B_1(U) 'C^*(X, Y, Z, T) + \frac{n}{n-2} a^2 B_1(U) \{ 'K(\bar{X}, Y, Z, T) - 'V^*(\bar{X}, Y, Z, T) \}.$$

Differentiating (3.50) with respect to U, using (3.50) and then barring T in the resulting equation, we have

Proof: Let the HGF-metric manifold is (134)-recurrent in $'A$ then barring T in the equation (3.11)a, we get

$$(3.43) \quad a^2 (\nabla' A)(X, \bar{Y}, \bar{Z}, \bar{T}, U) - 'A((\nabla F)(\bar{X}, U), \bar{Y}, \bar{Z}, \bar{T}) + a^2 'A(X, (\nabla F)(Y, U), \bar{Z}, \bar{T}) \\ + a^2 'A(X, \bar{Y}, (\nabla F)(Z, U), \bar{T}) = a^2 B_1(U) 'A(X, \bar{Y}, \bar{Z}, \bar{T}).$$

Now, suppose that the $'A$ -(1234)- recurrent HGF-metric manifold is $'A$ -(123)-recurrent for the same recurrence parameter, then comparing the equations (3.43) and (3.15)a, we get (3.42).

Note 3.5: Similarly, it can be proved that the $'A$ -(1234)-recurrent HGF-metric manifold is $'A$ -(1) or $'A$ -(2) or $'A$ -(3) or $'A$ -(12) or $'A$ -(23) or $'A$ -(13) or $'A$ -(24) or $'A$ -(14) and $'A$ -(124) or $'A$ -(134) recurrent, if

$$'A(X, (\nabla F)(Y, U), \bar{Z}, \bar{T}) + 'A(X, \bar{Y}, (\nabla F)(Z, U), \bar{T}) + 'A(X, \bar{Y}, \bar{Z}, (\nabla F)(T, U)) = 0.$$

$$\begin{aligned}
(3.52) \quad & a^2(\nabla' L^*)(\bar{X}, Y, Z, T, U) + a^2 L^*((\nabla F)(X, U), Y, Z, T) - L^*(\bar{X}, Y, Z, (\nabla F)(\bar{T}, U)) \\
& = a^2(\nabla' C^*)(\bar{X}, Y, Z, T, U) + a^2 C^*((\nabla F)(X, U), Y, Z, T) - C^*(\bar{X}, Y, Z, (\nabla F)(\bar{T}, U)) \\
& + \frac{n}{(n-2)} \{ -a^2(\nabla' K^*)(\bar{X}, Y, Z, T, U) + a^2 K^*((\nabla F)(X, U), Y, Z, T) \\
& - K^*(\bar{X}, Y, Z, (\nabla F)(\bar{T}, U)) - a^2(\nabla' V^*)(\bar{X}, Y, Z, T, U) \\
& - a^2 V^*((\nabla F)(X, U), Y, Z, T) + V^*(\bar{X}, Y, Z, (\nabla F)(\bar{T}, U)) \}.
\end{aligned}$$

Subtracting (3.51) from (3.52), we have

$$\begin{aligned}
(3.53) \quad & a^2(\nabla' L^*)(\bar{X}, Y, Z, T, U) + a^2 L^*((\nabla F)(X, U), Y, Z, T) - L^*(\bar{X}, Y, Z, (\nabla F)(\bar{T}, U)) \\
& - a^2 B_1(U)' L^*(\bar{X}, Y, Z, T) = a^2(\nabla' C^*)(\bar{X}, Y, Z, T, U) + a^2 C^*((\nabla F)(X, U), Y, Z, T) \\
& - C^*(\bar{X}, Y, Z, (\nabla F)(\bar{T}, U)) - a^2 B_1(U)' C^*(\bar{X}, Y, Z, T) + \frac{n}{(n-2)} \{ a^2(\nabla' K^*)(\bar{X}, Y, Z, T, U) \\
& + a^2 K^*((\nabla F)(X, U), Y, Z, T) - K^*(\bar{X}, Y, Z, (\nabla F)(\bar{T}, U)) - a^2 B_1(U)' K^*(\bar{X}, Y, Z, T) \\
& - a^2(\nabla' V^*)(\bar{X}, Y, Z, T, U) - a^2 V^*((\nabla F)(X, U), Y, Z, T) + V^*(\bar{X}, Y, Z, (\nabla F)(\bar{T}, U)) \\
& - a^2 B_1(U)' V^*(\bar{X}, Y, Z, T) \}.
\end{aligned}$$

Now, if the (14)-recurrent HGF-metric manifold is associated Pseudo conformal -(14)-recurrent and associated Pseudo concircular -(14)- recurrent, for the same recurrence parameter then from the equation (3.52), we have

$$\begin{aligned}
& a^2(\nabla' L^*)(\bar{X}, Y, Z, T, U) + a^2 L^*((\nabla F)(X, U), Y, Z, T) - L^*(\bar{X}, Y, Z, (\nabla F)(\bar{T}, U)) \\
& = a^2 B_1(U)' L^*(\bar{X}, Y, Z, T),
\end{aligned}$$

which shows that manifold is associated Pseudo conharmonic -(14)- recurrent manifold.

Similarly, it can be shown that if the (14)-recurrent HGF-metric manifold is either associated Pseudo conharmonic -(14)- recurrent and associated Pseudo conformal -(14)-recurrent or associated Pseudo conharmonic -(14)- recurrent and associated Pseudo concircular (14)- recurrent for the same recurrence parameter then it is either associated Pseudo concircular -(14)- recurrent or associated Pseudo conformal -(14)- recurrent.

Theorem 3.20: *In the (14)-symmetric HGF-metric manifold if any two of the following conditions hold then the third also holds:*

- (i) *It is associated Pseudo concircular -(14)- symmetric,*
- (ii) *It is associated Pseudo conharmonic -(14)- symmetric,*
- (iii) *It is associated Pseudo conformal -(14)- symmetric.*

Proof: If the (14)-symmetric HGF-metric manifold is associated Pseudo concircular $-(14)$ - symmetric and associated Pseudo conformal $-(14)$ - symmetric then from the equation (3.52), we have

$$a^2(\nabla' L^*)(\bar{X}, Y, Z, T, U) + a^2 L^*(\nabla F)(X, U), Y, Z, T - L^*(\bar{X}, Y, Z, (\nabla F)(\bar{T}, U)) = 0,$$

which shows that the manifold is associated Pseudo conharmonic $-(14)$ - symmetric.

Similarly it can be shown that if the (14)- symmetric HGF-metric manifold is either associated Pseudo conharmonic $-(14)$ - symmetric and associated Pseudo conformal $-(14)$ - symmetric or associated Pseudo conharmonic $-(14)$ - symmetric and associated Pseudo concircular $-(14)$ - symmetric then it is either associated Pseudo concircular $-(14)$ - symmetric or associated Pseudo conformal $-(14)$ - symmetric .

Remark 3.2: Theorems of the type (3.19) and (3.20) can also be stated and proved for $'A$ -(12), $'A$ -(13), $'A$ -(23), $'A$ -(24) and $'A$ -(34)-recurrent or $'A$ -(12), $'A$ -(13), $'A$ -(23) and $'A$ -(34)-symmetric manifolds.

Theorem 3.21: In the (124)-recurrent HGF-metric manifold if any two of the following conditions hold for the same recurrence parameter then the third also holds:

- (i) It is associated Pseudo concircular $-(124)$ - recurrent,
- (ii) It is associated Pseudo conharmonic $-(124)$ - recurrent,
- (iii) It is associated Pseudo conformal $-(124)$ - recurrent.

Proof: Barring X, Y and T in the equation (3.44), we have

$$(3.54) \quad 'L^*(\bar{X}, \bar{Y}, Z, \bar{T}) = 'C^*(\bar{X}, \bar{Y}, Z, \bar{T}) + \frac{n}{n-2} \{ 'K(\bar{X}, \bar{Y}, Z, \bar{T}) - 'V^*(\bar{X}, \bar{Y}, Z, \bar{T}) \}.$$

From the equation (3.54), we obtain

$$(3.55) \quad a^2 B_1(U) 'L^*(\bar{X}, \bar{Y}, Z, T) = a^2 B_1(U) 'C^*(\bar{X}, \bar{Y}, Z, T) + a^2 B_1(U) \frac{n}{n-2} \{ 'K(\bar{X}, \bar{Y}, Z, T) - 'V^*(\bar{X}, \bar{Y}, Z, T) \}.$$

Differentiating (3.54) with respect to U, using (3.54) and then barring T in the resulting equation, we have

$$(3.56) \quad a^2(\nabla' L^*)(\bar{X}, \bar{Y}, Z, T, U) + a^2 L^*((\nabla F)(X, U), \bar{Y}, Z, T) - a^2 L^*(\bar{X}, (\nabla F)(Y, U), Z, T) - 'L^*(\bar{X}, \bar{Y}, Z, (\nabla F)(\bar{T}, U)) = a^2(\nabla' C^*)(\bar{X}, \bar{Y}, Z, T, U) + a^2 C^*((\nabla F)(X, U), \bar{Y}, Z, T) + a^2 C^*(\bar{X}, (\nabla F)(Y, U), Z, T) - 'C^*(\bar{X}, \bar{Y}, Z, (\nabla F)(\bar{T}, U)) + \frac{n}{(n-2)} \{ a^2(\nabla' K)(\bar{X}, \bar{Y}, Z, T) + a^2 K((\nabla F)(X, U), \bar{Y}, Z, T) + a^2 K(\bar{X}, (\nabla F)(Y, U), Z, T) - 'K(\bar{X}, \bar{Y}, Z, (\nabla F)(\bar{T}, U)) - a^2(\nabla' V^*)(\bar{X}, \bar{Y}, Z, T, U) - a^2 V^*((\nabla F)(X, U), \bar{Y}, Z, T) - a^2 V^*(\bar{X}, (\nabla F)(Y, U), Z, T) + 'V^*(\bar{X}, \bar{Y}, Z, (\nabla F)(\bar{T}, U)) \}.$$

Subtracting (3.55) from (3.56), we have

$$\begin{aligned}
 (3.57) \quad & a^2(\nabla' L^*)(\bar{X}, \bar{Y}, Z, T, U) + a^2 L^*((\nabla F)(X, U), \bar{Y}, Z, T) + a^2 L^*(\bar{X}, (\nabla F)(Y, U), Z, T) \\
 & - L^*(\bar{X}, \bar{Y}, Z, (\nabla F)(\bar{T}, U)) - a^2 B_1(U)' L^*(\bar{X}, \bar{Y}, Z, T) \\
 & = a^2(\nabla' C^*)(\bar{X}, \bar{Y}, Z, T, U) + a^2 C^*((\nabla F)(X, U), \bar{Y}, Z, T) + a^2 C^*(\bar{X}, (\nabla F)(Y, U), Z, T) \\
 & - C^*(\bar{X}, \bar{Y}, Z, (\nabla F)(\bar{T}, U)) - a^2 B_1(U)' C^*(\bar{X}, \bar{Y}, Z, T) + \frac{n}{(n-2)} \{a^2(\nabla' K)((\bar{X}, \bar{Y}, Z, T, U) \\
 & + a^2 K((\nabla F)(X, U), \bar{Y}, Z, T) + a^2 K(\bar{X}, (\nabla F)(Y, U), Z, T) - K(\bar{X}, \bar{Y}, Z, (\nabla F)(\bar{T}, U)) \\
 & - a^2 B_1(U)' K(\bar{X}, \bar{Y}, Z, T) - a^2(\nabla' V^*)(\bar{X}, \bar{Y}, Z, T, U) - a^2 V^*((\nabla F)(X, U), \bar{Y}, Z, T) \\
 & - a^2 V^*(\bar{X}, (\nabla F)(Y, U), Z, T) + V^*(\bar{X}, \bar{Y}, Z, (\nabla F)(\bar{T}, U)) + a^2 B_1(U)' V^*(\bar{X}, \bar{Y}, Z, T)\}.
 \end{aligned}$$

Now, if the (124)-recurrent HGF-metric manifold is associated Pseudo conformal – (124)- recurrent and associated Pseudo concircular –(124)- recurrent for the same recurrence parameter then from the equation (3.57), we have

$$\begin{aligned}
 & a^2(\nabla' L^*)(\bar{X}, \bar{Y}, Z, T, U) + a^2 L^*((\nabla F)(X, U), \bar{Y}, Z, T) \\
 & + a^2 L^*(\bar{X}, (\nabla F)(Y, U), Z, T) - L^*(\bar{X}, \bar{Y}, Z, (\nabla F)(\bar{T}, U)) \\
 & = a^2 B_1(U)' L^*(\bar{X}, \bar{Y}, Z, T).
 \end{aligned}$$

Similarly, it can be shown that if the (124)-recurrent HGF-metric manifold is either associated Pseudo conharmonic –(124)- recurrent and associated Pseudo conformal –(124)- recurrent or associated Pseudo concircular –(124)- recurrent and associated Pseudo conharmonic –(124)- recurrent then it is either associated Pseudo concircular –(124)- recurrent or associated Pseudo conformal –(124)- recurrent.

Theorem 3.22: *In the (124)-recurrent HGF-metric manifold if any two of the following conditions hold for the same recurrence parameter then the third also holds:*

- (i) *It is associated Pseudo concircular –(124)- symmetric,*
- (ii) *It is associated Pseudo conharmonic –(124)- symmetric,*
- (iii) *It is associated Pseudo conformal –(124)- symmetric.*

Proof: If the (124)-symmetric HGF-metric manifold is associated Pseudo conformal –(124)- symmetric and associated Pseudo concircular –(124)- symmetric then from the equation (3.56), we have

$$\begin{aligned}
 & a^2(\nabla' L^*)(\bar{X}, \bar{Y}, Z, T, U) - L^*(\bar{X}, \bar{Y}, Z, (\nabla F)(\bar{T}, U)) \\
 & + a^2 L^*((\nabla F)(X, U), \bar{Y}, Z, T) + a^2 L^*(\bar{X}, (\nabla F)(Y, U), Z, T) \\
 & = a^2 B_1(U)' L^*(\bar{X}, \bar{Y}, Z, T),
 \end{aligned}$$

which follows that the manifold is associated Pseudo conharmonic –(124)- symmetric.

Similarly, it can be shown that if the (124)- symmetric HGF-metric manifold is either associated Pseudo conharmonic –(124)- symmetric and associated Pseudo conformal –(124)- symmetric or associated Pseudo conharmonic –(124)- symmetric and associated Pseudo concircular –(124)- symmetric then it is either associated Pseudo conformal –(124)- symmetric or associated Pseudo concircular –(124)- symmetric.

Remark 3.3: Theorems of the type (3.21) and (3.22) can also be stated and proved for 'A'-(134), 'A'-(123), 'A'-(234)-recurrent and 'A'-(134), 'A'-(123), 'A'-(234)-symmetric manifolds.

Theorem 3.23: In the (1234)-recurrent HGF-metric manifold if any two of the following conditions hold for the same recurrence parameter then the third also holds:

- (i) It is associated Pseudo concircular –(1234)- recurrent,
- (ii) It is associated Pseudo conharmonic –(1234)- recurrent,
- (iii) It is associated Pseudo conformal –(1234)- recurrent.

Proof: Barring X, Y, Z and T in the equation (3.45) we have

$$(3.58) \quad 'L^*(\bar{X}, \bar{Y}, \bar{Z}, \bar{T}) = 'C^*(\bar{X}, \bar{Y}, \bar{Z}, \bar{T}) + \frac{n}{(n-2)} \{ 'K(\bar{X}, \bar{Y}, \bar{Z}, \bar{T}) - 'V^*(\bar{X}, \bar{Y}, \bar{Z}, \bar{T}) \}.$$

From the equation (3.58), we obtain

$$(3.59) \quad \begin{aligned} a^2 B_1(U) 'L^*(\bar{X}, \bar{Y}, \bar{Z}, T) &= a^2 B_1(U) 'C^*(\bar{X}, \bar{Y}, \bar{Z}, T) \\ &+ \frac{n}{n-2} a^2 B_1(U) \{ 'K(\bar{X}, \bar{Y}, \bar{Z}, T) - 'V^*(\bar{X}, \bar{Y}, \bar{Z}, T) \}. \end{aligned}$$

Differentiating (3.58) with respect to U, using (3.58) and then barring T in the resulting equation, we get

$$(3.60) \quad \begin{aligned} &a^2 (\nabla 'L^*)(\bar{X}, \bar{Y}, \bar{Z}, T, U) + a^2 'L^*((\nabla F)(X, U), \bar{Y}, \bar{Z}, T) + a^2 'L^*(\bar{X}, (\nabla F)(Y, U), \bar{Z}, T) \\ &+ a^2 'L^*(\bar{X}, \bar{Y}, (\nabla F)(Z, U), T) - 'L^*(\bar{X}, \bar{Y}, \bar{Z}, (\nabla F)(\bar{T}, U),) \\ &= a^2 (\nabla 'C^*)(\bar{X}, \bar{Y}, \bar{Z}, T, U) + a^2 'C^*((\nabla F)(X, U), \bar{Y}, \bar{Z}, T) + a^2 'C^*(\bar{X}, (\nabla F)(Y, U), \bar{Z}, T) \\ &+ a^2 'C^*(\bar{X}, \bar{Y}, (\nabla F)(Z, U), T) - 'C^*(\bar{X}, \bar{Y}, \bar{Z}, (\nabla F)(\bar{T}, U)) + \frac{n}{(n-2)} \{ a^2 (\nabla 'K)(\bar{X}, \bar{Y}, \bar{Z}, T) \\ &+ a^2 'K(\nabla F)(X, U), \bar{Y}, \bar{Z}, T) + a^2 'K(\bar{X}, \bar{Y}, (\nabla F)(Z, U), T) + a^2 'K(\bar{X}, (\nabla F)(Y, U), \bar{Z}, T) \\ &- 'K(\bar{X}, \bar{Y}, \bar{Z}, (\nabla F)(\bar{T}, U)) - a^2 (\nabla 'V^*)(\bar{X}, \bar{Y}, \bar{Z}, T, U) - a^2 'V^*((\nabla F)(X, U), \bar{Y}, \bar{Z}, T) \\ &- a^2 'V^*(\bar{X}, (\nabla F)(Y, U), \bar{Z}, T) - a^2 'V^*(\bar{X}, \bar{Y}, (\nabla F)(Z, U), T) + a^2 'V^*(\bar{X}, \bar{Y}, \bar{Z}, (\nabla F)(\bar{T}, U)) \}. \end{aligned}$$

Now subtracting (3.59) from (3.60), we obtain

$$(3.61) \quad a^2 (\nabla 'L^*)(\bar{X}, \bar{Y}, \bar{Z}, T, U) + a^2 'L^*((\nabla F)(X, U), \bar{Y}, \bar{Z}, T) + a^2 'L^*(\bar{X}, (\nabla F)(Y, U), \bar{Z}, T)$$

$$\begin{aligned}
& + a^2 L^*(\bar{X}, \bar{Y}, (\nabla F)(Z, U), T) - L^*(\bar{X}, \bar{Y}, \bar{Z}, (\nabla F)(\bar{T}, U),) - a^2 B_1(U) L^*(\bar{X}, \bar{Y}, \bar{Z}, T) \\
& = a^2 (\nabla C^*)(\bar{X}, \bar{Y}, \bar{Z}, T, U) + a^2 C^*((\nabla F)(X, U), \bar{Y}, \bar{Z}, T) + a^2 C^*(\bar{X}, (\nabla F)(Y, U), \bar{Z}, T) \\
& + a^2 C^*(\bar{X}, \bar{Y}, (\nabla F)(Z, U), T) - C^*(\bar{X}, \bar{Y}, \bar{Z}, (\nabla F)(\bar{T}, U)) - a^2 B_1(U) C^*(\bar{X}, \bar{Y}, \bar{Z}, T) \\
& + \frac{n}{(n-2)} \{a^2 (\nabla K)(\bar{X}, \bar{Y}, \bar{Z}, T, U) + a^2 K((\nabla F)(X, U), \bar{Y}, \bar{Z}, T) \\
& + a^2 K(\bar{X}, (\nabla F)(Y, U), \bar{Z}, T) + a^2 K(\bar{X}, \bar{Y}, (\nabla F)(Z, U), T) - a^2 K(\bar{X}, \bar{Y}, \bar{Z}, (\nabla F)(\bar{T}, U)) \\
& - a^2 B_1(U) K(\bar{X}, \bar{Y}, \bar{Z}, T) - a^2 (\nabla V^*)(\bar{X}, \bar{Y}, \bar{Z}, T, U) - a^2 V^*((\nabla F)(X, U), \bar{Y}, \bar{Z}, T) \\
& - a^2 V^*(\bar{X}, \bar{Y}, (\nabla F)(Z, U), T) + a^2 V^*(\bar{X}, \bar{Y}, \bar{Z}, (\nabla F)(\bar{T}, U)) + a^2 B_1(U) V^*(\bar{X}, \bar{Y}, \bar{Z}, T) \\
& - a^2 V^*(\bar{X}, (\nabla F)(Y, U), \bar{Z}, T)
\end{aligned}$$

Now, If the (1234)-recurrent HGF-metric manifold is associated Pseudo conformal -(1234)- recurrent and associated Pseudo concircular -(1234)- recurrent for the same recurrence parameter then from the equation (3.61), we have

$$\begin{aligned}
(3.62) \quad & a^2 (\nabla L^*)(\bar{X}, \bar{Y}, \bar{Z}, T, U) + a^2 L^*((\nabla F)(X, U), \bar{Y}, \bar{Z}, T) + a^2 L^*(\bar{X}, (\nabla F)(Y, U), \bar{Z}, T) \\
& + a^2 L^*(\bar{X}, \bar{Y}, (\nabla F)(Z, U), T) - a^2 L^*(\bar{X}, \bar{Y}, \bar{Z}, (\nabla F)(\bar{T}, U)) = a^2 B_1(U) L^*(\bar{X}, \bar{Y}, \bar{Z}, T),
\end{aligned}$$

which shows that the manifold is associated Pseudo conharmonic -(1234)- recurrent.

Similarly, it can be proved that if the (1234)- recurrent HGF-metric manifold is either associated Pseudo conharmonic -(1234)- recurrent and associated Pseudo conformal -(1234)- recurrent or associated Pseudo conharmonic and associated Pseudo concircular -(1234)- recurrent then it is either associated Pseudo concircular -(1234)- recurrent or associated Pseudo conformal -(1234)- recurrent.

Theorem 3.24: In the (1234)-symmetric HGF-metric manifold if any two of the following conditions hold then the third also holds:

- (i) It is associated Pseudo concircular -(1234)- symmetric,
- (ii) It is associated Pseudo conformal -(1234)- symmetric
- (iii) It is associated Pseudo conharmonic -(1234)- symmetric.

Proof: If the (1234)-symmetric HGF-metric manifold is associated Pseudo conformal -(1234)- symmetric and associated Pseudo concircular -(1234)- symmetric then from the equation (3.60), we have

$$\begin{aligned}
(3.63) \quad & a^2 (\nabla L^*)(\bar{X}, \bar{Y}, \bar{Z}, T, U) + a^2 L^*((\nabla F)(X, U), \bar{Y}, \bar{Z}, T) + a^2 L^*(\bar{X}, (\nabla F)(Y, U), \bar{Z}, T) \\
& + a^2 L^*(\bar{X}, \bar{Y}, (\nabla F)(Z, U), T) - L^*(\bar{X}, \bar{Y}, \bar{Z}, (\nabla F)(\bar{T}, U)) = 0,
\end{aligned}$$

which shows that the manifold is associated Pseudo conharmonic -(1234)- symmetric.

Similarly, it can be proved that if the (1234)- symmetric HGF-metric manifold is either associated Pseudo conharmonic -(1234)- symmetric and associated Pseudo conformal -(1234)- symmetric or associated Pseudo conharmonic -(1234)- symmetric and

associated Pseudo concircular $-(1234)$ - symmetric then it is either associated Pseudo concircular $-(1234)$ -symmetric or associated Pseudo conformal $-(1234)$ - symmetric.

References

1. S. B. Pandey and B. C. Joshi, Hyperbolic general Differentiable Manifold-I, *Jour. Sci. Res.*, **9(1)** (1987) 43-44.
2. R. S. Mishra and S. B. Pandey, On H-Conharmonic Curvature Tensor in Kähler Manifolds, *Indian J. pure applied Mathematics*, **7(11)** (1974) 1277-1287.
3. R. S. Mishra, S. B. Pandey and U. Sharma, Pseudo H-Projective n-Recurrent Manifold with KH-Structure, *Indian J. pure applied Mathematics*, **7(10)** (1986) 1187-1195.
4. S. B. Pandey and L. Dasila, On general Differentiable Manifold, *U. Scientist Phyl. Sciences*, **7(2)** (1995) 147-152.
5. R. S. Mishra, *Structures on a Differentiable Manifold and their applications*, Chandrama Prakashan, 50- A, Balrampur House, Allahabad, 1984.
6. S. B. Pandey and Meenakshi Pant, On HGF- Structure Metric Manifold (Recurrence Parameters), *Proc. Nat. Conf. Mod. Math. Appl. I.T.R.B.S.C., Agra* (2003).
7. S. B. Pandey and Puran Singh, On general contact Hypersurfaces of HGF- Manifold, *U. Scientist, Phyl. Sciences*, **5(2)** (1993) 142-147.