

Absolute Cesàro Summability Factors of Fourier Series

Sarojini Pati

Bishop Johnson College of Science and Commerce, Allahabad -211001
(A constituent college of AAIDU)
Email : patisarojini@yahoo.co.in

(Received January 5, 2006)

Abstract : In the present paper the case $\beta = -\alpha$, where $\alpha \geq 0$, has been considered in a Theorem of Dixit¹ and the result of his theorem has been extended by replacing $\log\left(\frac{2\pi}{t}\right)$ by $\rho\left(\frac{1}{t}\right)$ under certain conditions on $\rho(x)$ assumed to be monotonically increasing for $x > 0$, tending to ∞ with x .

1. Introduction

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series of real or complex numbers and let $\{s_n\}$ be the sequence of partial sums, i.e.,

$$s_n = u_0 + u_1 + \cdots + u_n \quad (n = 0, 1, 2, \dots).$$

The (C, α) mean s_n^α of the sequence $\{s_n\}$, or the series $\sum u_n$, where $\alpha > -1$, is defined as²

$$s_n^\alpha = \frac{S_n^\alpha}{A_n^\alpha} = \frac{1}{A_n^\alpha} \sum_{\nu=0}^n A_{n-\nu}^{\alpha-1} s_\nu \quad (n = 0, 1, 2, \dots)$$

A sequence $\{\xi_n\}$ is said to be of 'bounded variation' or $\{\xi_n\} \in BV$, when $\sum_n |\xi_n - \xi_{n-1}| < \infty$. If $\{s_n^\alpha\} \in BV$, i.e. if $\sum_{n=1}^{\infty} |s_n^\alpha - s_{n-1}^\alpha| < \infty$, where $\alpha > -1$, then we say that the series $\sum u_n$, or the sequence $\{s_n\}$, is absolutely summable by Cesàro means of order α , or summable $|C, \alpha|^3$.

Let $f(t)$ be a periodic function with period 2π , integrable in the sense of Lebesgue over $(-\pi, \pi)$. Let

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos nt \, dt \quad (n = 0, 1, 2, \dots),$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin nt \, dt \quad (n=1, 2, 3, \dots).$$

Then the series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$ is called the Fourier series of $f(t)$.

Without loss of generality we can take $a_0 = 0$, i.e., $\int_{-\pi}^{\pi} f(t) \, dt = 0$. Thus the above Fourier series becomes

$$\sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum A_n(t) \text{ (say).}$$

We write

$$s_n(x) = \sum_{v=1}^n A_v(x) \quad (n=1, 2, 3, \dots);$$

$$\phi(t) = \phi_x(t) = \frac{1}{2} \{f(x+t) + f(x-t)\} - f(x);$$

$$\Phi(t) = \int_0^t |\phi(u)| \, du;$$

$$D_n(t) = \frac{1}{2} + \sum_{v=1}^n \cos vt \quad (n=1, 2, \dots);$$

$$D_0(t) = \frac{1}{2}.$$

We know that $D_n(t) = \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}$. A sequence $\{\lambda_n\}$ is said to be 'convex' when

$$\Delta^2 \lambda_n \geq 0 \quad (n=1, 2, \dots), \text{ where } \Delta \lambda_n = \lambda_n - \lambda_{n+1} \text{ and } \Delta^2 \lambda_n = \Delta(\Delta \lambda_n).$$

In 1963 Pati⁴ established the following important theorem:

Theorem A: If $\{\lambda_n\}$ is a convex sequence such that

$$\sum_{n=1}^{\infty} n^{-1} \lambda_n (\log n)^{\frac{1}{2}} < \infty,$$

then $\sum \lambda_n A_n(\cdot)$ is summable $|C, 1|$ at every point $t=x$, at which, as $t \rightarrow 0$,

$$(1.1) \quad \Phi(t) = \int_0^t |\phi(u)| \, du = O(t)$$

In an attempt at extending Theorem A, Liu⁵ proved the following:

Theorem B: If $\{\lambda_n\}$ is a convex sequence such that

$$\sum n^{-1} \lambda_n (\log n)^{\frac{1}{2}(1-\alpha)} < \infty \quad (0 \leq \alpha < 1),$$

then $\sum_{n=1}^{\infty} \lambda_n A_n(t)$ is summable $|C, 1|$ at every point $t = x$ at which, as $t \rightarrow 0$,

$$(1.2) \quad \Phi(t) = \int_0^t |\phi(u)| du = O \left(\frac{t}{\left(\log \frac{1}{t} \right)^\alpha} \right).$$

Theorem C. If $\{\lambda_n\}$ is a convex sequence such that

$$\sum n^{-1} \lambda_n (\log \log n)^{\frac{1}{2}} < \infty,$$

then $\sum \lambda_n A_n(t)$ is summable $|C, 1|$ at every point $t = x$ at which, as $t \rightarrow 0$,

$$(1.3) \quad \Phi(t) = \int_0^t |\phi(u)| du = O \left(\frac{t}{\log \frac{1}{t}} \right).$$

It can be seen from Liu's analysis that 'o' used by him can be replaced by 'O' and thus when $\alpha = 0$, Theorem B will reduce to Theorem A. It is worth mentioning that Hsiang⁶ proved the following:

Theorem D: If $\Phi(t) = O \left(\frac{t}{\log \frac{1}{t}} \right)$ and $\lambda_n = \frac{1}{(\log n)^{1+\varepsilon}}$, $\varepsilon > 0$, then $\sum \lambda_n A_n(x)$ is

summable $|C, 1|$.

Improving this theorem, Sharma and Gupta⁷ proved the following:

Theorem E: If $\Phi(t) = O \left(\frac{t}{\left(\log \frac{1}{t} \right)^\alpha} \right)$, $\alpha > \frac{1}{2}$, and $\{\lambda_n\}$ is a convex sequence

such that $\sum n^{-1} \lambda_n < \infty$, then $\sum \lambda_n A_n(x)$ is summable $|C, 1|$.

The results of Liu⁵ and Hsiang⁶ were generalized by Dixit¹ in 1974, in the following form:

Theorem F: Let β be any real number and

$$\Phi(t) = O \left(t \left(\log \frac{2\pi}{t} \right)^\beta \right), \text{ as } t \rightarrow 0.$$

Let $\{\lambda_n\}$ satisfy $\lambda_n h_n = O(1)$, $\sum \frac{|\lambda_n|}{n} h_n < \infty$, $\sum |\Delta \lambda_n| h_n < \infty$, where

$$h_n = \begin{cases} (\log n)^{\beta + \frac{1}{2}}, & \text{where } \beta > -\frac{1}{2}, \\ (\log \log n)^{\frac{1}{2}}, & \text{where } \beta = -\frac{1}{2}, \\ 1 & \text{where } \beta < -\frac{1}{2}. \end{cases}$$

Then $\sum \lambda_n A_n(t)$ at $t = x$ is summable $[C, 1]$.

The object of the present paper is to consider the case $\beta = -\alpha$, where $\alpha \geq 0$ in Theorem F and to extend the result of Theorem F by replacing $\log \frac{2\pi}{t}$ by $\rho\left(\frac{1}{t}\right)$ under certain conditions on $\rho(x)$, assumed to be monotonically increasing for $x > 0$, tending to ∞ with x . It can be seen that our result extends Theorem F, for $\beta \leq 0$, and also theorems B, C, D and E. We have given more precise 'estimates' and thereby improved upon Liu's work especially by considering separately the cases: $0 \leq \alpha < \frac{1}{2}$, $\alpha = \frac{1}{2}$, $\alpha > \frac{1}{2}$. In Liu's analysis there occurs a singularity of $\frac{1}{(\log \frac{1}{t})^\alpha}$ ($0 < \alpha < 1$) at the point $t = 1$, which lies in the interval of integration in an integral.

2. The Theorem

For the statement of our theorem the following notations are required:

$$h_n = \begin{cases} (\rho(n))^{\frac{1}{2}-\alpha}, & \text{if } 0 \leq \alpha < \frac{1}{2}; \\ (\log \rho(n))^{\frac{1}{2}}, & \text{if } \alpha = \frac{1}{2} \\ 1 & \text{if } \alpha > \frac{1}{2} \end{cases}$$

and

$$k_{n,\gamma} = \begin{cases} (\rho(n))^{1-\gamma}, & \text{if } 0 \leq \gamma < 1 \\ \log \rho(n), & \text{if } \gamma = 1 \\ 1 & \text{if } \gamma > 1. \end{cases}$$

Theorem: Let $\alpha \geq 0$ and $\rho(x)$ be a positive monotonically increasing function of $x(>0)$ such that $\rho(x) = O(x)$, $\frac{x}{(\rho(x))^{\alpha+1}}$ is monotonically increasing as x increases,

$$(2.1) \quad \int_{1/n}^{\pi} \frac{dt}{t \left(\rho\left(\frac{1}{t}\right) \right)^{\gamma}} = O(k_{n,\gamma}) (\gamma \geq 0),$$

where $\sum_{n=\nu}^{\infty} \frac{k_{n,\gamma}}{n^2} = O\left(\frac{k_{\nu,\gamma}}{\nu}\right)$ ($\gamma \geq 0$). Let $\{\lambda_n\}$ be a sequence of real numbers such that

$$\sum_{n=1}^{\infty} |\Delta \lambda_n| k_{n,\gamma} < \infty (\gamma \geq 0), \quad \sum_{n=1}^{\infty} \frac{|\lambda_n|}{n} h_n < \infty, \quad \sum_{n=1}^{\infty} |\Delta \lambda_n| h_n < \infty, \quad \lambda_n h_n = O(1). \text{ If}$$

$$(2.2) \quad \Phi(t) = O\left(\frac{t}{\left(\rho\left(\frac{1}{t}\right)\right)^{\alpha}}\right), \text{ as } t \rightarrow 0, \text{ then } \sum \lambda_n A_n(t), \text{ at } t = x, \text{ is summable } |C, 1|.$$

3. Lemmas

We require the following lemmas for the proof of our theorem.

Lemma 1: Let $\psi(t)$ be a positive function and let $\Phi(t) = O\left(t\psi\left(\frac{1}{t}\right)\right)$ as $t \rightarrow 0$. If

$\{\lambda_n\}$ is such that

$$(a) \quad \sum_{n=1}^{\infty} \frac{|\lambda_n|}{n} \psi(n) < \infty, \quad \sum_{\nu=1}^{\infty} |\Delta \lambda_{\nu}| \nu \sum_{n=\nu+1}^{\infty} \psi(n) \Delta\left(\frac{1}{n}\right) < \infty$$

$$(b) \quad \sum_{\nu=1}^{\infty} |\Delta \lambda_{\nu}| \nu \sum_{n=\nu+1}^{\infty} p_n \Delta\left(\frac{1}{n}\right) < \infty \quad (p_n = \int_{1/n}^{\pi} \frac{|\phi(t)|}{t} dt)$$

$$(c) \quad \sum_{n=1}^{\infty} \Delta\left(\frac{1}{n}\right) \left| \lambda_n \sum_{\nu=1}^n \int_{1/\nu}^{\pi} \phi(t) D_{\nu-1}(t) dt \right| < \infty,$$

then $\sum \lambda_n A_n(x)$ is summable $|C, 1|$ iff

$$\sum_{n=1}^{\infty} \frac{|\lambda_n|}{n} |s_n(x) - f(x)| < \infty.$$

Proof. This is the particular case of Theorem F of Dixit, in the case: $\lambda_n = n$, where λ_n is to be interpreted as per the notations of Dixit, since summability $|R, n, 1|$ is

equivalent to summability $|C, 1|$. Note that we have used $\{\lambda_n\}$ as the sequence of factors for which Dixit uses the notation $\{\varepsilon_n\}$.

Lemma 2. *If (2.2) holds, then as $n \rightarrow \infty$*

$$\sum_{\nu=1}^n |s_{\nu}(x) - f(x)| = \begin{cases} O(n(\rho(n))^{\frac{1}{2}-\alpha}), & (0 \leq \alpha < 1/2) \\ O(n(\log \rho(n))^{1/2}), & (\alpha = 1/2), \\ O(n), & (\alpha > 1/2). \end{cases}$$

Proof. For $\nu \geq 1$, we have

$$s_{\nu}(x) - f(x) = \frac{2}{\pi} \int_0^{\pi} \phi(t) \frac{\sin \nu t}{t} dt + O(1).$$

Hence, for $1 \leq \nu \leq n$,

$$\begin{aligned} |s_{\nu}(x) - f(x)| &\leq \frac{2}{\pi} \left| \int_0^{\pi} \phi(t) \frac{\sin \nu t}{t} dt \right| + O(1) \\ &\leq \frac{2}{\pi} \int_0^{1/n} |\phi(t)| \frac{|\sin \nu t|}{t} dt + \frac{2}{\pi} \left| \int_{1/n}^{\pi} \phi(t) \frac{\sin \nu t}{t} dt \right| + O(1) \\ &\leq \frac{2\nu}{\pi} \int_0^{1/n} |\phi(t)| dt + \frac{2}{\pi} \left| \int_{1/n}^{\pi} \phi(t) \frac{\sin \nu t}{t} dt \right| + O(1) \\ &= O\left(\frac{\nu}{n(\rho(n))^{\alpha}}\right) + \frac{2}{\pi} \left| \int_{1/n}^{\pi} \phi(t) \frac{\sin \nu t}{t} dt \right| + O(1) \\ &= O\left(\frac{1}{(\rho(n))^{\alpha}}\right) + \frac{2}{\pi} \left| \int_{1/n}^{\pi} \phi(t) \frac{\sin \nu t}{t} dt \right| + O(1). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{\nu=1}^n |s_{\nu}(x) - f(x)|^2 &\leq K \sum_{\nu=1}^n \left\{ \int_{1/n}^{\pi} \phi(t) \frac{\sin \nu t}{t} dt \right\}^2 + O\left(\frac{n}{(\rho(n))^{2\alpha}}\right) + O(n) \\ &= K \int_{1/n}^{\pi} \int_{1/n}^{\pi} \frac{\phi(t)}{t} \frac{\phi(u)}{u} \left(\sum_{\nu=1}^n \sin \nu t \cdot \sin \nu u \right) dt du + O\left(\frac{n}{(\rho(n))^{2\alpha}}\right) + O(n) \\ &= \frac{K}{2} \int_{1/n}^{\pi} \int_{1/n}^{\pi} \frac{\phi(t)}{t} \frac{\phi(u)}{u} \left(\sum_{\nu=1}^n 2 \sin \nu t \cdot \sin \nu u \right) dt du + O\left(\frac{n}{(\rho(n))^{2\alpha}}\right) + O(n). \end{aligned}$$

Now $2 \sin \nu t \cdot \sin \nu u = \cos \nu(u-t) - \cos \nu(u+t)$ and

$$\begin{aligned} \sum_{\nu=1}^n 2 \sin \nu t \sin \nu u &= \sum_{\nu=1}^n \cos \nu (u-t) - \sum_{\nu=1}^n \cos \nu (u+t) = D_n(u-t) - D_n(u+t) \\ &= \frac{\sin(n+\frac{1}{2})(u-t)}{2 \sin \frac{u-t}{2}} - \frac{\sin(n+\frac{1}{2})(u+t)}{2 \sin \frac{u+t}{2}}. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{\nu=1}^n |s_{\nu}(x) - f(x)|^2 &\leq \frac{k}{2} \int_{1/n}^{\pi} \int_{1/n}^{\pi} \frac{\phi(t)}{t} \frac{\phi(u)}{u} \left\{ \frac{\sin(n+\frac{1}{2})(u-t)}{2 \sin \frac{u-t}{2}} - \frac{\sin(n+\frac{1}{2})(u+t)}{2 \sin \frac{u+t}{2}} \right\} dt du \\ &\quad + O\left(\frac{n}{(\rho(n))^{2\alpha}}\right) + O(n). \\ &= \frac{K}{2} T_n + O\left(\frac{n}{(\rho(n))^{2\alpha}}\right) + O(n), \text{ say.} \end{aligned}$$

Now

$$\begin{aligned} T_n &= \int_{1/n}^{\pi} \int_{1/n}^{\pi} \frac{\phi(t)}{t} \frac{\phi(u)}{u} \left\{ \frac{\sin(n+\frac{1}{2})(u-t)}{2 \sin \frac{u-t}{2}} - \frac{\sin(n+\frac{1}{2})(u+t)}{2 \sin \frac{u+t}{2}} \right\} dt du \\ &= \int_{1/n}^{\pi} \int_{1/n}^{\pi} \frac{\phi(t)}{t} \frac{\phi(u)}{u} \frac{\sin n(u-t)}{u-t} dt du - \int_{1/n}^{\pi} \frac{\phi(t)}{t} \frac{\phi(u)}{u} \frac{\sin n(u+t)}{u+t} dt du \\ &\quad + O\left(\int_{1/n}^{\pi} \int_{1/n}^{\pi} \frac{|\phi(t)|}{t} \frac{|\phi(u)|}{u} dt du\right). \\ &= T_{n,1} - T_{n,2} + O(T_{n,3}), \text{ say.} \end{aligned}$$

Now

$$\begin{aligned} |T_{n,1}| &\leq \left| \int_{1/n}^{\pi} \frac{\phi(t)}{t} \int_{1/n}^t \frac{\phi(u)}{u} \frac{\sin n(t-u)}{t-u} dt du \right| \\ &\quad + \left| \int_{1/n}^{\pi} \frac{\phi(t)}{t} \int_t^{\pi} \frac{\phi(u)}{u} \frac{\sin n(u-t)}{u-t} dt du \right| \\ &= \left| \int_{1/n}^{\pi} \frac{\phi(t)}{t} \int_{1/n}^t \frac{\phi(u)}{u} \frac{\sin n(t-u)}{t-u} dt du \right| \end{aligned}$$

$$\begin{aligned}
& + \left| \int_{1/n}^{\pi} \frac{\phi(u)}{u} \int_{1/n}^u \frac{\phi(t)}{t} \frac{\sin n(u-t)}{u-t} du dt \right| \\
& = 2 \left| \int_{1/n}^{\pi} \frac{\phi(t)}{t} \int_{1/n}^t \frac{\phi(u)}{u} \frac{\sin n(t-u)}{t-u} dt du \right| \\
& = 2 \left| \int_{1/n}^{\pi} \frac{\phi(t)}{t^2} \int_{1/n}^t \frac{t\phi(u)}{u} \frac{\sin n(t-u)}{t-u} dt du \right| \\
& = 2 \left| \int_{1/n}^{\pi} \frac{\phi(t)}{t^2} \int_{1/n}^t \phi(u) \frac{\sin n(t-u)}{t-u} dt du \right| [t = u + (t-u)] \\
& \quad + 2 \left| \int_{1/n}^{\pi} \frac{\phi(t)}{t^2} \int_{1/n}^t \frac{\phi(u)}{u} \sin n(t-u) dt du \right| \\
& \leq 2 \int_{1/n}^{\pi} \frac{|\phi(t)|}{t^2} \int_{1/n}^t |\phi(u)| \frac{|\sin n(t-u)|}{t-u} dt du \\
& \quad + 2 \int_{1/n}^{\pi} \frac{|\phi(t)|}{t^2} \int_{1/n}^t \frac{|\phi(u)|}{u} |\sin n(t-u)| dt du \\
& \leq 2n \int_{1/n}^{\pi} \frac{|\phi(t)|}{t^2} \int_{1/n}^t |\phi(u)| dt du + 2 \int_{1/n}^{\pi} \frac{|\phi(t)|}{t^2} \int_{1/n}^t \frac{|\phi(u)|}{u} dt du \\
& \quad \left[\frac{|\sin n(t-u)|}{t-u} \leq n; |\sin n(t-u)| \leq 1 \right] \\
& \leq 2n \int_{1/n}^{\pi} \frac{|\phi(t)|}{t^2} \int_{1/n}^t |\phi(u)| dt du + 2n \int_{1/n}^{\pi} \frac{|\phi(t)|}{t^2} \int_{1/n}^t |\phi(u)| dt du \\
& = 4n \int_{1/n}^{\pi} \frac{|\phi(t)|}{t^2} \int_{1/n}^t |\phi(u)| dt du \\
& \leq 4n \int_{1/n}^{\pi} \frac{|\phi(t)|}{t^2} \int_0^t |\phi(u)| dt du \\
& = 4n \int_{1/n}^{\pi} \frac{|\phi(t)|}{t^2} \Phi(t) dt \\
& = O \left(n \int_{1/n}^{\pi} \frac{|\phi(t)|}{t} \frac{1}{\left(\rho\left(\frac{1}{t}\right) \right)^{\alpha}} dt \right)
\end{aligned}$$

$$= O \left(n \int_{1/n}^{\pi} \frac{|\phi(t)|}{t} \frac{1}{\left(\rho\left(\frac{1}{t}\right)\right)^{\alpha}} dt \right).$$

Now, by integrating by parts

$$\begin{aligned} \int_{1/n}^{\pi} \frac{|\phi(t)|}{t} \frac{1}{\left(\rho\left(\frac{1}{t}\right)\right)^{\alpha}} dt &= \left[\frac{\Phi(t)}{t \left(\rho\left(\frac{1}{t}\right)\right)^{\alpha}} \right]_{1/n}^{\pi} - \int_{1/n}^{\pi} \Phi(t) \frac{d}{dt} \left(\frac{1}{t \left(\rho\left(\frac{1}{t}\right)\right)^{\alpha}} \right) dt \\ &= O(1) + O \left(\frac{1}{(\rho(n))^{2\alpha}} \right) - \int_{1/n}^{\pi} \frac{\Phi(t)}{t} \cdot d \left(\frac{1}{\left(\rho\left(\frac{1}{t}\right)\right)^{\alpha}} \right) + \int_{1/n}^{\pi} \frac{\Phi(t)}{t^2} \frac{dt}{\left(\rho\left(\frac{1}{t}\right)\right)^{\alpha}} \\ &= O(1) + O \left(\int_{1/n}^{\pi} \frac{1}{\left(\rho\left(\frac{1}{t}\right)\right)^{\alpha}} d \left(\frac{1}{\left(\rho\left(\frac{1}{t}\right)\right)^{\alpha}} \right) \right) + O \left(\int_{1/n}^{\pi} \frac{1}{t} \frac{dt}{\left(\rho\left(\frac{1}{t}\right)\right)^{2\alpha}} \right) \\ &\quad \text{(since } \alpha > 0 \text{ and } \rho \text{ is monotonically increasing)} \\ &= O(1) + O \left\{ \left[\frac{1}{\left(\rho\left(\frac{1}{t}\right)\right)^{\alpha}} \right]^2 \right\}_{1/n}^{\pi} + O \left(\int_{1/n}^{\pi} \frac{dt}{t \left(\rho\left(\frac{1}{t}\right)\right)^{2\alpha}} \right) \\ &= O(1) + O(1) + O \left(\frac{1}{(\rho(n))^{2\alpha}} \right) + O \left(\int_{1/n}^{\pi} \frac{dt}{t \left(\rho\left(\frac{1}{t}\right)\right)^{2\alpha}} \right) \end{aligned}$$

$$= O(1) + O \left(\int_{1/n}^{\pi} \frac{dt}{t \left(\rho \left(\frac{1}{t} \right) \right)^{2\alpha}} \right).$$

By assumption (2.1) of our theorem,

$$\begin{aligned} \int_{1/n}^{\pi} \frac{dt}{t \left(\rho \left(\frac{1}{t} \right) \right)^{2\alpha}} &= O \left\{ (\rho(n))^{1-2\alpha} \right\} \text{ if } 0 \leq \alpha < \frac{1}{2} \\ &= O(\log \rho(n)), \text{ if } \alpha = \frac{1}{2}; = O(1), \text{ if } \alpha > \frac{1}{2}. \end{aligned}$$

Thus

$$|T_{n,1}| = \begin{cases} O(n(\rho(n))^{1-2\alpha}), & \text{if } 0 \leq \alpha < \frac{1}{2} \\ O(n \log \rho(n)), & \text{if } \alpha = \frac{1}{2} \\ O(n), & \text{if } \alpha > \frac{1}{2}. \end{cases}$$

Also by similar reasoning we see that

$$\begin{aligned} |T_{n,2}| &\leq 2 \left| \int_{1/n}^{\pi} \frac{\phi(t)}{t} \int_{1/n}^t \frac{\phi(u) \sin n(t+u)}{u} \frac{1}{t+u} du dt \right| \\ &\leq 4n \int_{1/n}^{\pi} \frac{|\phi(t)|}{t^2} \Phi(t) dt. \end{aligned}$$

Hence, arguing as for $|T_{n,1}|$, we have

$$|T_{n,2}| = \begin{cases} O(n(\rho(n))^{1-2\alpha}), & \text{if } 0 \leq \alpha < \frac{1}{2} \\ O(n \log \rho(n)), & \text{if } \alpha = \frac{1}{2} \\ O(n), & \text{if } \alpha > \frac{1}{2}. \end{cases}$$

Also, integrating by parts

$$\int_{1/n}^{\pi} \frac{|\phi(t)|}{t} dt = \frac{\Phi(t)}{t} \Big|_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi(t)}{t^2} dt$$

$$\begin{aligned}
&= O(1) + O\left(\frac{1}{(\rho(n))^\alpha}\right) + O\left(\int_{1/n}^{\pi} \frac{1}{t \left(\rho\left(\frac{1}{t}\right)\right)^\alpha} dt\right) \\
&= O(1) + O\left(\int_{1/n}^{\pi} \frac{dt}{t \left(\rho\left(\frac{1}{t}\right)\right)^\alpha}\right).
\end{aligned}$$

Hence

$$\begin{aligned}
T_{n,3} &= \left(\int_{1/n}^{\pi} \frac{|\phi(t)|}{t} dt\right)^2 = O((\rho(n))^{2-2\alpha}), \text{ if } 0 \leq \alpha < 1; \\
&= O((\log(\rho(n)))^2), \text{ if } \alpha = 1; = O(1), \text{ if } \alpha > 1.
\end{aligned}$$

Thus, if $0 \leq \alpha < \frac{1}{2}$,

$$\begin{aligned}
T_n &= O(n(\rho(n))^{1-2\alpha}) + O((\rho(n))^{2-2\alpha}) \\
&= O(n(\rho(n))^{1-2\alpha}), \text{ since } \rho(n) = O(n) \text{ (by hypothesis)}
\end{aligned}$$

and if $\alpha = \frac{1}{2}$

$$\begin{aligned}
T_n &= O(n \log \rho(n)) + O((\rho(n))^{2-2\alpha}) = O(n \log \rho(n)) + O(\rho(n)) \\
&= O(n \log \rho(n)) + O(n) \text{ [since } \rho(n) = O(n) \text{ by hypothesis]} \\
&= O(n \log \rho(n)).
\end{aligned}$$

If $\alpha > 1/2$, but $\alpha < 1$,

$$\begin{aligned}
T_n &= O(n) + O((\rho(n))^{2-2\alpha}) \\
&= O(n) + O(\rho(n) \cdot (\rho(n))^{1-2\alpha}) \\
&= O(n), \text{ since } \rho(n) = O(n), \text{ by hypothesis.}
\end{aligned}$$

Also if $\alpha = 1$,

$$\begin{aligned}
T_n &= O(n) + O((\log \rho(n))^2) \\
&= O(n), \text{ since } (\log \rho(n))^2 = O(\rho(n))
\end{aligned}$$

$$= O(n).$$

If $\alpha > 1$,

$$T_n = O(n) + O(1) = O(n).$$

Thus

$$\begin{aligned} T_n &= O(n(\rho(n))^{1-2\alpha}), \quad \text{if } 0 \leq \alpha < \frac{1}{2}; \\ &= O(n \log \rho(n)), \quad \text{if } \alpha = \frac{1}{2}; \\ &= O(n), \quad \text{if } \alpha > 1/2. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{v=1}^n |s_v(x) - f(x)|^2 &= \frac{K}{2} T_n + O\left(\frac{n}{(\rho(n))^{2\alpha}}\right) + O(n) \\ &= O(T_n) + O(n). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{v=1}^n |s_v(x) - f(x)|^2 &= O(n(\rho(n))^{1-2\alpha}), \quad \text{if } 0 \leq \alpha < \frac{1}{2}; \\ &= O(n \log \rho(n)) \quad \text{if } \alpha = \frac{1}{2}; \\ &= O(n) \quad \text{if } \alpha > \frac{1}{2}. \end{aligned}$$

By Cauchy's inequality

$$\sum_{v=1}^n |s_v(x) - f(x)| \leq n^{1/2} \cdot \left(\sum_{v=1}^n |s_v(x) - f(x)|^2 \right)^{1/2}.$$

Hence using the above estimates,

$$\begin{aligned} \sum_{v=1}^n |s_v(x) - f(x)| &= O(n(\rho(n))^{\frac{1}{2}-\alpha}), \quad \text{if } 0 \leq \alpha < \frac{1}{2}; \\ &= O((\log \rho(n))^{\frac{1}{2}}), \quad \text{if } \alpha = \frac{1}{2}; \\ &= O(n), \quad \text{if } \alpha > \frac{1}{2}. \end{aligned}$$

This completes the proof of lemma 2.

4. Proof of the Theorem

For proving our Theorem, it is enough to show that under the hypotheses assumed by us, the conditions (a), (b), (c) of Lemma 1 are satisfied and that

$$\sum_{n=1}^{\infty} \frac{|\lambda_n|}{n} |s_{\nu}(x) - f(x)| < \infty,$$

because, by appealing to the sufficiency part of Lemma 1, it will then follow that $\sum \lambda_n A_n(x)$ is summable $|C, 1|$.

Now conditions (a) of Lemma 1 are:

$$\sum_{n=1}^{\infty} \frac{|\lambda_n|}{n} \psi(n) < \infty,$$

$$\sum_{\nu=1}^{\infty} |\Delta \lambda_{\nu}| \nu \sum_{n=\nu+1}^{\infty} \psi(n) \Delta\left(\frac{1}{n}\right) < \infty,$$

$$\text{our } \psi\left(\frac{1}{t}\right) = \left(\rho\left(\frac{1}{t}\right)\right)^{-\alpha}.$$

Hence

$$\psi(n) = (\rho(n))^{-\alpha}.$$

Hence conditions (a) and (b) of Lemma 1 become:

$$(4.1) \quad \sum_{n=1}^{\infty} \frac{|\lambda_n|}{n} \frac{1}{(\rho(n))^{\alpha}} < \infty$$

and

$$\sum_{\nu=1}^{\infty} |\Delta \lambda_{\nu}| \nu \sum_{n=\nu+1}^{\infty} \frac{1}{(\rho(n))^{\alpha}} \frac{1}{n(n+1)} < \infty,$$

which is satisfied if

$$(4.2) \quad \sum_{\nu=1}^{\infty} \frac{|\Delta \lambda_{\nu}|}{(\rho(\nu))^{\alpha}} < \infty,$$

and

$$\sum_{\nu=1}^{\infty} |\Delta \lambda_{\nu}| \nu \sum_{n=\nu+1}^{\infty} p_n \Delta\left(\frac{1}{n}\right).$$

Now integrating by parts

$$p_n = \int_{1/n}^{\pi} \frac{|\phi(t)|}{t} dt$$

$$\begin{aligned}
&= \frac{\Phi(t)}{t} \Big|_{1/n}^{\pi} + \int_{1/n}^{\pi} \frac{\Phi(t)}{t^2} dt \\
&= O(1) + O\left(\frac{1}{(\rho(n))^{\alpha}}\right) + O\left\{\int_{1/n}^{\pi} \frac{dt}{t \left(\rho\left(\frac{1}{t}\right)\right)^{\alpha}}\right\} \\
&= O(1) + O(k_{n,\alpha}) = O(k_{n,\alpha}),
\end{aligned}$$

where

$$k_{n,\alpha} = \begin{cases} (\rho(n))^{1-\alpha}, & \text{if } 0 \leq \alpha < 1; \\ \log \rho(n), & \text{if } \alpha = 1; \\ 1, & \text{if } \alpha > 1. \end{cases}$$

Also

$$\begin{aligned}
\sum_{n=\nu+1}^{\infty} p_n \Delta\left(\frac{1}{n}\right) &= O\left[\sum_{n=\nu+1}^{\infty} k_{n,\alpha} \frac{1}{n(n+1)}\right] = O\left(\sum_{n=\nu}^{\infty} \frac{k_{n,\alpha}}{n^2}\right) \\
&= O\left(\frac{k_{\nu,\alpha}}{\nu}\right),
\end{aligned}$$

by hypothesis in case of $0 \leq \alpha \leq 1$. For $\alpha > 1$, this estimate is obviously true.

Thus condition (b) is satisfied if

$$(4.3) \quad \sum_{n=\nu}^{\infty} \frac{k_{n,\alpha}}{n^2} = O\left(\frac{k_{\nu,\alpha}}{\nu}\right)$$

and

$$\sum_{\nu=1}^{\infty} |\Delta \lambda_{\nu}| \nu \frac{k_{\nu,\alpha}}{\nu} < \infty,$$

or

$$(4.4) \quad \sum_{\nu=1}^{\infty} |\Delta \lambda_{\nu}| k_{\nu,\alpha} < \infty,$$

which holds by hypothesis.

Let us consider the condition:

$$(c) \quad \sum_{n=1}^{\infty} \Delta\left(\frac{1}{n}\right) \left| \lambda_n \sum_{\nu=1}^n \int_{1/n}^{\pi} \phi(t) D_{\nu-1}(t) dt \right| < \infty.$$

$$\begin{aligned}
\Sigma &= \sum_{n=1}^{\infty} \Delta\left(\frac{1}{n}\right) |\lambda_n| \left| \int_{1/n}^{\pi} \phi(t) \sum_{\nu=1}^n D_{\nu-1}(t) dt \right| \\
&\leq \sum_{n=1}^{\infty} \Delta\left(\frac{1}{n}\right) |\lambda_n| \int_{1/n}^{\pi} |\phi(t)| \left| \sum_{\nu=1}^n D_{\nu-1}(t) \right| dt \\
&\leq K \sum_{n=1}^{\infty} \Delta\left(\frac{1}{n}\right) |\lambda_n| \int_{1/n}^{\pi} \frac{|\phi(t)|}{t^2} dt.
\end{aligned}$$

Now integrating by parts, for $\alpha \geq 0$,

$$\begin{aligned}
\int_{1/n}^{\pi} \frac{|\phi(t)|}{t^2} dt &= \frac{\Phi(t)}{t^2} + 2 \int_{1/n}^{\pi} \frac{\Phi(t)}{t^3} dt \\
&= O(1) + O\left(\frac{n}{(\rho(n))^\alpha}\right) + O\left(\int_{1/n}^{\pi} \frac{1}{t^2} \frac{1}{\left(\rho\left(\frac{1}{t}\right)\right)^\alpha} dt\right) \\
&= O\left(\frac{n}{(\rho(n))^\alpha}\right) + O\left(\int_{1/n}^{\pi} \frac{dt}{t^2 \left(\rho\left(\frac{1}{t}\right)\right)^\alpha}\right) \\
&= O\left(\frac{n}{(\rho(n))^\alpha}\right),
\end{aligned}$$

since

$$\int_{1/n}^{\pi} \frac{dt}{t^2 \left(\rho\left(\frac{1}{t}\right)\right)^\alpha} = O\left(\frac{n}{(\rho(n))^\alpha}\right),$$

as shown below, assuming that $\frac{x}{(\rho(x))^{\alpha+1}}$ is monotonically increasing.

By integration by parts

$$\int_{1/n}^{\pi} \frac{dt}{t^2 \left(\rho\left(\frac{1}{t}\right)\right)^\alpha} = \left[-\frac{1}{t} \frac{1}{\left(\rho\left(\frac{1}{t}\right)\right)^\alpha} \right]_{1/n}^{\pi} + \alpha \int_{1/n}^{\pi} \frac{1}{t} \frac{\rho'\left(\frac{1}{t}\right) \frac{1}{t^2}}{\left(\rho\left(\frac{1}{t}\right)\right)^{\alpha+1}} dt$$

$$= O(1) + O\left(\frac{n}{(\rho(n))^\alpha}\right) + \alpha I, \text{ say.}$$

Now

$$\begin{aligned} I &= \int_{1/n}^{\pi} \frac{1}{t} \frac{\rho'\left(\frac{1}{t}\right) \left(\frac{1}{t^2}\right)}{\left(\rho\left(\frac{1}{t}\right)\right)^{\alpha+1}} dt = O\left(\frac{n}{(\rho(n))^{\alpha+1}}\right) \left| \int_{1/n}^{\pi} \rho'\left(\frac{1}{t}\right) \cdot \left(-\frac{1}{t^2}\right) dt \right| \\ &= O\left(\frac{n}{(\rho(n))^{\alpha+1}}\right) \left| \rho\left(\frac{1}{t}\right) \right|_{1/n}^{\pi} = O\left(\frac{n}{(\rho(n))^{\alpha+1}} \cdot |O(1) - \rho(n)|\right) \\ &= O\left(\frac{n}{(\rho(n))^{\alpha+1}} \cdot \rho(n)\right) = O\left(\frac{n}{(\rho(n))^\alpha}\right). \end{aligned}$$

Thus condition (c) is satisfied if

$$\sum_{n=1}^{\infty} \Delta\left(\frac{1}{n}\right) |\lambda_n| \frac{n}{(\rho(n))^\alpha} < \infty$$

or
$$\sum_{n=1}^{\infty} \frac{|\lambda_n|}{n(\rho(n))^\alpha} < \infty \quad (\alpha \geq 0),$$

which is the same as condition (4.1). So far we have required : (4.1), (4.2), (4.3), (4.4) and

$$(4.5) \quad \rho(x) = O(x), \quad \frac{x}{(\rho(x))^{\alpha+1}}$$

is monotonically increasing as x increases, $\alpha \geq 0$, from which it immediately follows that, for $\alpha \geq 0$, $\frac{x}{(\rho(x))^\alpha}$ is monotonically increasing as x increases.

By Lemma 2, we have

$$\sum_{\nu=1}^n |s_\nu(x) - f(x)| = O(n h_n),$$

where

$$h_n = \begin{cases} (\rho(n))^{\frac{1}{2}-\alpha}, & \text{if } 0 \leq \alpha < \frac{1}{2}, \\ (\log \rho(n))^{\frac{1}{2}}, & \text{if } \alpha = \frac{1}{2}, \\ 1, & \text{if } \alpha > \frac{1}{2}. \end{cases}$$

We proceed to show that, under our hypotheses,

$$\sum_{\nu=1}^n \frac{|\lambda_\nu|}{\nu} |s_\nu(x) - f(x)| = O(1).$$

Now

$$\begin{aligned} \sum_n &= \sum_{\nu=1}^n \frac{|\lambda_\nu|}{\nu} |s_\nu(x) - f(x)| \\ &= \sum_{\nu=1}^{n-1} \Delta \left(\frac{|\lambda_\nu|}{\nu} \right) |s_\nu(x) - f(x)| + \frac{\lambda_n}{n} \sum_{\nu=1}^n |s_\nu(x) - f(x)| \\ &= \sum_{\nu=1}^{n-1} \left(\frac{|\lambda_\nu|}{\nu(\nu+1)} + \frac{\Delta(|\lambda_\nu|)}{\nu+1} \right) \sum_{\mu=1}^{\nu} |s_\mu(x) - f(x)| \\ &\quad + \frac{|\lambda_n|}{n} \sum_{\nu=1}^n |s_\nu(x) - f(x)| \\ &= \sum_{\nu=1}^{n-1} \frac{|\lambda_\nu|}{\nu(\nu+1)} \sum_{\mu=1}^{\nu} |s_\mu(x) - f(x)| + \sum_{\nu=1}^{n-1} \frac{\Delta(|\lambda_\nu|)}{\nu+1} \sum_{\mu=1}^{\nu} |s_\mu(x) - f(x)| \\ &\quad + \frac{|\lambda_n|}{n} \sum_{\nu=1}^n |s_\nu(x) - f(x)| \\ &\leq \sum_{\nu=1}^{n-1} \frac{|\lambda_\nu|}{\nu(\nu+1)} \sum_{\mu=1}^{\nu} |s_\mu(x) - f(x)| + O \left(\sum_{\nu=1}^{n-1} \frac{|\Delta \lambda_\nu|}{\nu+1} \times \right. \\ &\quad \left. \times \sum_{\mu=1}^{\nu} |s_\mu(x) - f(x)| \right) + \frac{|\lambda_n|}{n} \sum_{\nu=1}^n |s_\nu(x) - f(x)|. \end{aligned}$$

Now, for $0 \leq \alpha < \frac{1}{2}$,

$$\begin{aligned} \sum_n &= O \left(\sum_{\nu=1}^{n-1} \frac{|\lambda_\nu|}{\nu(\nu+1)} \nu (\rho(\nu))^{\frac{1}{2}-\alpha} \right) + O \left(\sum_{\nu=1}^{n-1} \frac{|\Delta \lambda_\nu|}{\nu+1} \nu (\rho(\nu))^{\frac{1}{2}-\alpha} \right) \\ &\quad + O \left(\frac{|\lambda_n|}{n} n (\rho(n))^{\frac{1}{2}-\alpha} \right) \\ &= O \left(\sum_{\nu=1}^{n-1} \frac{|\lambda_\nu|}{\nu} (\rho(\nu))^{\frac{1}{2}-\alpha} \right) + O \left(\sum_{\nu=1}^{n-1} \frac{|\Delta \lambda_\nu|}{\nu} (\rho(\nu))^{\frac{1}{2}-\alpha} \right) + O \left(|\lambda_n| (\rho(n))^{\frac{1}{2}-\alpha} \right) \end{aligned}$$

$$= O\left(\sum_{v=1}^{n-1} \frac{|\lambda_v|}{v} h_v\right) + O\left(\sum_{v=1}^{n-1} |\Delta\lambda_v| h_v\right) + O(|\lambda_n| h_n)$$

Thus for $\sum_n$ to be $O(1)$, we need

$$(4.6) \quad (i) \sum_{v=1}^{\infty} \frac{|\lambda_v|}{v} h_v < \infty, \quad (ii) \sum_{v=1}^{\infty} |\Delta\lambda_v| h_v < \infty, \quad \lambda_n h_n = O(1)$$

If $\alpha = \frac{1}{2}$, then

$$\begin{aligned} \sum_n &= O\left(\sum_{v=1}^{n-1} \frac{|\lambda_v|}{v(v+1)} v (\log \rho(v))^{\frac{1}{2}}\right) + O\left(\sum_{v=1}^{n-1} \frac{|\Delta\lambda_v|}{v} v (\log \rho(v))^{\frac{1}{2}}\right) \\ &\quad + O\left(\frac{|\lambda_n|}{n} n (\log \rho(n))^{\frac{1}{2}}\right) \\ &= O\left(\sum_{v=1}^{n-1} \frac{|\lambda_v|}{v} h_v\right) + O\left(\sum_{v=1}^{n-1} |\Delta\lambda_v| h_v\right) + O(|\lambda_n| h_n) \end{aligned}$$

Thus for $\sum_n = O(1)$, we require

$$(4.7) \quad (i) \sum_{v=1}^{\infty} \frac{|\lambda_v|}{v} h_v < \infty \quad (ii) \sum_{v=1}^{\infty} |\Delta\lambda_v| h_v < \infty. \quad (iii) \lambda_n h_n = O(1).$$

If $\alpha > \frac{1}{2}$, then

$$\begin{aligned} \sum_n &= O\left(\sum_{v=1}^{n-1} \frac{|\lambda_v|}{v(v+1)} v + O\left(\sum_{v=1}^{n-1} \frac{|\Delta\lambda_v|}{v+1} v\right)\right) + O\left(\frac{|\lambda_n|}{n} n\right) \\ &= O\left(\sum_{v=1}^{n-1} \frac{|\lambda_v|}{v}\right) + O\left(\sum_{v=1}^{n-1} |\Delta\lambda_v|\right) + O(|\lambda_n|) \\ &= O\left(\sum_{v=1}^{n-1} \frac{|\lambda_v|}{v}\right) + O\left(\sum_{v=1}^{n-1} |\Delta\lambda_v| h_v\right) + O(|\lambda_v| h_v). \end{aligned}$$

Thus, for $\sum_n = O(1)$, we require

$$(4.8) \quad (i) \sum_{v=1}^{\infty} \frac{|\lambda_v|}{v} h_v < \infty \quad (ii) \sum_{v=1}^{\infty} |\Delta\lambda_v| h_v < \infty \quad (iii) \lambda_n h_n = O(1).$$

Now the conditions (4.6), (4.7) and (4.8) are all of the same form, given by (4.6), for all cases of α . Thus we need the conditions (4.1), (4.2), (4.3), (4.4), (4.5) and (4.6).

It is clear that (4.1) follows from (4.6) (i) and (4.2) follows from (4.6) (ii).

References

1. G.D. Dixit, On absolute Riesz summability Factors of Fourier series, *Report Series, Univ. of Auckland*, **54** (1974).
2. G.H. Hardy and J.E. Littlewood, The strong summability of Fourier series. *Fundamenta Mathematicae*, **25** (1935) 162-189.
3. M. Fekete, Zur Theorie der divergenten Reihen, *Math. és Termész. Ertesito (Budapest)*, **29** (1911) 719-726.
4. T. Pati, On an unsolved problem in the theory of absolute summability factors of Fourier series, *Mathematische Zeitschrift*, **82** (1963) 106-114.
5. T.S. Liu, On the absolute Cesàro summability factors of Fourier series, *Proceedings of Japan Academy*, **41** (1965) 757-762.
6. F.C. Hsiang, On the absolute summability factors of Fourier series at a given point, *Composito Mathematico*, **17** (1966) 156-160.
7. P.L. Sharma and B.L. Gupta, Absolute Cesàro summability of the factored Fourier series, *Kôdai Math. Sem. Rep.*, **22** (1970) 61-64.

