

On C-recurrent Finsler Space

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Abstract. Matsumoto¹ introduced a Finsler space whose (h)hv-torsion tensor C_{jk}^i is recurrent with respect to Cartan's connection Γ_{jk}^{*i} and called it as a C^h -recurrent Finsler space. Pandey and Verma² generalized this concept by introducing a C^h -birecurrent Finsler space. The C^h -birecurrent is characterized by the condition $C_{jk|h|m}^i = a_{hm} C_{jk}^i$, where a_{hm} are components of a non-null covariant tensor of order 2. In this paper, we introduce a Finsler space whose (h)hv-torsion tensor C_{jk}^i is recurrent and birecurrent with respect to Berwald's connection G_{jk}^i . We shall call such spaces as C-recurrent and C-birecurrent Finsler space.

1. Preliminaries

Let F_n be an n dimensional Finsler space equipped with the fundamental metric function $F(x, y)$ and the metric tensor $g_{ij}(x, y)$. The covariant derivatives of an arbitrary vector field X^i with respect to Berwald connection G_{jk}^i and Cartan connection Γ_{jk}^{*i} are given by

$$(1.1) \quad \mathcal{B}_k X^i = \partial_k X^i - (\partial_r X^i) G_k^r + X^r G_{rk}^i$$

and

$$(1.2) \quad X_{|k}^i = \partial_k X^i - (\partial_r X^i) G_k^r + X^r \Gamma_{rk}^{*i}$$

respectively.

The directional differential operator $\dot{\partial}_k$ commutes with the Berwald covariant differential operator $\mathcal{B}_h$ according as

$$(1.3) \quad (\dot{\partial}_k \mathcal{B}_h - \mathcal{B}_h \dot{\partial}_k) T_j^i = T_j^r G_{khr}^i - T_r^i G_{khj}^r,$$

where T_j^i is an arbitrary tensor field and $G_{jkh}^i = \dot{\partial}_h G_{jk}^i$. The tensor G_{jkh}^i also satisfies the following

$$(1.4) \quad \begin{aligned} (a) \quad & G_{jkh}^i y^h = 0, \\ (b) \quad & G_{jkh}^i = G_{khj}^i = G_{hjk}^i. \end{aligned}$$

The Ricci commutation formula for Berwald covariant differentiation is given by

$$(1.5) \quad \mathcal{B}_k \mathcal{B}_h X^i - \mathcal{B}_h \mathcal{B}_k X^i = H_{rk}^i - (\dot{\partial}_k X^i) H_{hk}^r,$$

where

$$(1.6) \quad H_{rhk}^i = \partial_k G_{rh}^i + G_{rh}^s G_{sk}^i + G_{srk}^i G_h^s - k/h$$

and

$$(1.7) \quad H_{hk}^r = \partial_k G_{hk}^r + G_{hk}^s G_{sk}^r - k/h$$

The notation $-k/h$ stands for the interchange of the indices k and h and subtraction from the former terms. The tensor H_{jkh}^i is Berwald curvature tensor and is related with the tensor H_{kh}^i by

$$(1.8) \quad (a) \quad H_{jkh}^i y^j = H_{kh}^i, \quad (b) \quad \dot{\partial}_j H_{kh}^i = H_{jkh}^i.$$

The (h)hv-torsion tensor C_{jk}^i is defined as

$$(1.9) \quad C_{jk}^i = g^{ir} C_{jrk},$$

where

$$(1.10) \quad C_{ijk} = \frac{1}{2} \dot{\partial}_i g_{jk}$$

and

$$(1.11) \quad g^{ir} g_{rk} = \delta_k^i.$$

The (h)hv-torsion tensor C_{jk}^i satisfies the following

$$(1.12) \quad \begin{aligned} (a) \quad & C_{jk}^i = C_{kj}^i, \\ (b) \quad & C_{jk}^i y^j = C_{kj}^i y^j = 0, \\ (c) \quad & y_i C_{jk}^i = 0 \\ \text{and } (d) \quad & y^r \dot{\partial}_r C_{jk}^i = -C_{jk}^i. \end{aligned}$$

2. A C-Recurrent Finsler Space

Let us consider a C-recurrent Finsler Space characterized by the following condition

$$(2.1) \quad \mathcal{B}_m C_{jk}^i = \lambda_m C_{jk}^i,$$

where λ_m is a non-null covariant vector field. As per definition of Berwald's covariant derivative

$$(2.2) \quad \mathcal{B}_m C_{jk}^i = \partial_m C_{jk}^i - (\dot{\partial}_r C_{jk}^i) G_m^r + C_{jk}^r G_{rm}^i - C_{rk}^i G_{jm}^r - C_{jr}^i G_{km}^r.$$

The Berwald connection G_{jk}^i and Cartan connection Γ_{jk}^{*i} are connected by

$$(2.3) \quad G_{jk}^i = \Gamma_{jk}^{*i} + (\dot{\partial}_j \Gamma_{rk}^{*i}) y^r$$

and

$$(2.4) \quad G_{jk}^i y^j = \Gamma_{jk}^{*i} y^j = G_k^i.$$

Substituting the value G_{jk}^i from the equation (2.3) in to equation (2.2), we find

$$(2.5) \quad \mathcal{B}_m C_{jk}^i = C_{jk|m}^i + C_{jk}^r (\dot{\partial}_r \Gamma_{sm}^{*i}) y^s - C_{jk}^i (\dot{\partial}_j \Gamma_{sm}^{*r}) y^s - C_{jr}^i (\dot{\partial}_k \Gamma_{sm}^{*r}) y^s.$$

Using (2.1) in this equation, we find

$$(2.6) \quad \lambda_m C_{jk}^i = C_{jk|m}^i + C_{jk}^r (\dot{\partial}_r \Gamma_{sm}^{*i}) y^s - C_{rk}^i (\dot{\partial}_j \Gamma_{sm}^{*r}) y^s - C_{jr}^i (\dot{\partial}_k \Gamma_{sm}^{*r}) y^s.$$

This leads to:

Theorem 2.1: A C-recurrent Finsler space is C^h -recurrent if and only if

$$C_{jk}^r (\dot{\partial}_r \Gamma_{sm}^{*i}) y^s - C_{rk}^i (\dot{\partial}_j \Gamma_{sm}^{*r}) y^s - C_{jr}^i (\dot{\partial}_k \Gamma_{sm}^{*r}) y^s = 0.$$

If the C-recurrent Finsler space is also a Landsberg spaces, we have $(\dot{\partial}_r \Gamma_{sh}^{*r}) y^s = 0$. Hence the equation (2.6) reduces to $C_{jk|m}^i = \lambda_m C_{jk}^i$, which shows that the space considered is C^h -recurrent.

Thus we have:

Theorem 2.2: A Landsberg C-recurrent Finsler space is necessarily C^h -recurrent.

Transvecting (2.5) by y^m , we get

$$(2.7) \quad y^m \mathcal{B}_m C_{jk}^i = y^m C_{jk|m}^i + y^m C_{jk}^r (\dot{\partial}_r \Gamma_{sm}^{*i}) y^s - C_{rk}^i y^m (\dot{\partial}_j \Gamma_{sm}^{*r}) y^s - C_{jr}^i y^m (\dot{\partial}_k \Gamma_{sm}^{*r}) y^s.$$

We know that,³

$$(2.8) \quad \dot{\partial}_r \Gamma_{sh}^* y^s = C_{rm|s}^i y^s.$$

Transvecting this equation by y^m , we get

$$(2.9) \quad \dot{\partial}_r \Gamma_{sh}^* y^s y^m = 0.$$

Using (2.9) in (2.7), we get

$$(2.10) \quad \mathcal{B}_m y^m C_{jk}^i = y^m C_{jk|m}^i = P_{jk}^i.$$

Transvecting (2.1) by y^m and using (2.10), we get

$$(2.11) \quad P_{jk}^i = \lambda C_{jk}^i,$$

where $\lambda = \lambda_m y^m$. This shows that the Finsler space considered is a $*P$ -Finsler space.

Thus we conclude:

Theorem 2.3: *A C-recurrent Finsler space is necessarily a $*P$ -Finsler space.*

Differentiating (2.1) covariantly with respect to x^ℓ in the sense of Berwald, we get

$$(2.12) \quad \mathcal{B}_\ell \mathcal{B}_m C_{jk}^i = (\mathcal{B}_\ell \lambda_m + \lambda_m \lambda_\ell) C_{jk}^i.$$

Putting $\mathcal{B}_\ell \lambda_m + \lambda_m \lambda_\ell = a_{\ell m}$, we get

$$(2.13) \quad \mathcal{B}_\ell \mathcal{B}_m C_{jk}^i = a_{\ell m} C_{jk}^i.$$

This shows that (2.1) implies (2.13). But (2.13) need not imply (2.1).

This leads to:

Theorem 2.4: *Every C-recurrent Finsler space is a C-birecurrent Finsler space but the converse need not be true.*

Transvecting (2.1) by g_{ih} , we get

$$(2.14) \quad \mathcal{B}_m C_{jkh} - (\mathcal{B}_m g_{ih}) C_{jk}^i = \lambda_m C_{jkh}.$$

This shows that in a C-recurrent space the tensor C_{jkh} is not necessarily recurrent.

However it is recurrent if $(\mathcal{B}_m g_{ih}) C_{jk}^i = 0$. In a Landsberg space $\mathcal{B}_m g_{ih}$ vanishes identically. Therefore, we conclude that in a Landsberg C-recurrent space the tensor C_{jkh} is recurrent.

Using Ricci commutation formula given by (1.5), we have

$$(2.15) \quad \mathcal{B}_h \mathcal{B}_m C_{jk}^i - \mathcal{B}_m \mathcal{B}_h C_{jk}^i = C_{jk}^r H_{rmh}^i - C_{rk}^i H_{jmh}^r - C_{jr}^i H_{kmh}^r \\ - (\dot{\partial}_r C_{jk}^i) H_{mh}^r.$$

Using (2.12) in (2.15), we get

$$(2.16) \quad (\mathcal{B}_h \lambda_m - \mathcal{B}_m \lambda_h) C_{jk}^i = C_{jk}^r H_{rmh}^i - C_{rk}^i H_{jmh}^r - C_{jr}^i H_{kmh}^r - (\dot{\partial}_r C_{jk}^i) H_{mh}^r.$$

Putting $\mathcal{B}_h \lambda_m - \mathcal{B}_m \lambda_h = A_{hm}$, we may rewrite the (2.16) as

$$(2.17) \quad A_{hm} C_{jk}^i = C_{jk}^r H_{rmh}^i - C_{rk}^i H_{jmh}^r - C_{jr}^i H_{kmh}^r - (\dot{\partial}_r C_{jk}^i) H_{mh}^r.$$

Contracting the indices i and j in (2.17), we find

$$(2.18) \quad A_{hm} C_k = -C_r H_{kmh}^r - (\dot{\partial}_r C_k) H_{mh}^r.$$

If there exist a vector field λ_k independent of y^i satisfying

$$(2.19) \quad \lambda_m H_{kh}^i + \lambda_k H_{hm}^i + \lambda_h H_{mk}^i = 0,$$

then the partial derivative of (2.19) with respect to y^ℓ gives

$$(2.20) \quad \lambda_m H_{\ell kh}^i + \lambda_k H_{\ell hm}^i + \lambda_h H_{\ell mk}^i = 0.$$

Multiplying (2.18) by λ_p and taking skew-symmetric part with respect to the indices p, h, m , we get

$$(2.21) \quad (\lambda_p A_{hm} + \lambda_h A_{mp} + \lambda_m A_{ph}) C_k = -C_r (\lambda_p H_{kmh}^r + \lambda_m H_{khp}^r + \lambda_h H_{kpm}^r) - (\dot{\partial}_r C_k) (\lambda_p H_{mh}^r + \lambda_m H_{hp}^r + \lambda_h H_{pm}^r).$$

Using (2.19) and (2.20) in (2.21), we get

$$(2.22) \quad (\lambda_p A_{hm} + \lambda_h A_{mp} + \lambda_m A_{ph}) C_k = 0.$$

The equation (2.22) implies at least one of the following:

$$(2.23) \quad (a) \quad \lambda_p A_{hm} + \lambda_h A_{mp} + \lambda_m A_{ph} = 0$$

$$(b) \quad C_k = 0.$$

In view of Deicke's theorem, the condition (2.23b) implies that the space is Riemannian. Thus we have:

Theorem 2.5: *If there exists a vector field λ_m independent of y^i satisfying (2.19) in a C-recurrent Finsler space, then either the space is Riemannian or the identity (2.23a) holds.*

Differentiating (2.1) covariantly with respect to y^h in the sense of Berwald, we get

$$(2.24) \quad \dot{\partial}_h \mathcal{B}_m C'_{jk} = (\dot{\partial}_h \lambda_m) C'_{jk} + \lambda_m (\dot{\partial}_h C'_{jk}).$$

Utilizing the commutation formula given by (1.3) in the equation (2.24), we find

$$(2.25) \quad \mathcal{B}_m \dot{\partial}_h C'_{jk} + C'_{jk} G_{mhr}^r - C'_{rk} G_{jmh}^r - C'_{jr} G_{mhh}^r = (\dot{\partial}_h \lambda_m) C'_{jk} + \lambda_m (\dot{\partial}_h C'_{jk}).$$

Contracting the indices i and j in the above equation, we get

$$(2.26) \quad \mathcal{B}_m \dot{\partial}_h C_k - C_r G_{mhh}^r = (\dot{\partial}_h \lambda_m) C_k + \lambda_m (\dot{\partial}_h C_k).$$

If the tensor $\dot{\partial}_h C_k$ is recurrent i.e. $\mathcal{B}_m \dot{\partial}_h C_k = \lambda_m (\dot{\partial}_h C_k)$, then

$$(2.27) \quad -C_r G_{mhh}^r = (\dot{\partial}_h \lambda_m) C_k.$$

Transvecting the above equation by y^m , we have $y^m (\dot{\partial}_h \lambda_m) C_k = 0$, which gives at least one of the following conditions

$$(2.28) \text{ (a) } y^m (\dot{\partial}_h \lambda_m) = 0$$

$$\text{(b) } C_k = 0.$$

(2.28a) may be rewritten as $\dot{\partial}_h \lambda = \lambda_{hk}$ where $\lambda = \lambda_h y^h$. Thus the condition implies that the space is Riemannian. Thus we conclude:

Theorem 2.6: *If the tensor $\dot{\partial}_h C_k$ is recurrent in a C-recurrent Finsler space, then either $\lambda_m = \dot{\partial}_m \lambda$ or the space is Riemannian.*

3. C-Recurrent Finsler Space with T-Condition

Let us consider a C-recurrent space satisfying T-condition. For such space

$$(3.1) \quad T = F^2 C_{hj} |_k + C_{kij} y_h + C_{hjk} y_i + C_{ihk} y_j + C_{jih} y_k = 0.$$

Transvecting this equation by g^{ij} and using $C_{hj} g^{ij} = C_h$ and $g^{ij} |_k = 0$, we get

$$(3.2) \quad F^2 C_h |_k + C_k y_h + C_h y_k + C'_{hk} y_i + C'_{jkh} y_j = 0.$$

But $C'_{hk} y_i = 0$. Therefore

$$(3.3) \quad F^2 C_h |_k + C_k y_h + C_h y_k = 0.$$

which, in view of (1.6), becomes

$$(3.4) \quad F^2 \dot{\partial}_k C_h - F^2 C_r C'_{hkk} + C_k y_h + C_h y_k = 0.$$

or

$$(3.5) \quad \dot{\partial}_k C_h = C_r C_{hk}^r - \frac{1}{F^2} (C_k y_h + C_h y_k).$$

Putting the value of $\dot{\partial}_k C_h$ in the equation (3.18), we get

$$(3.6) \quad A_{hm} C_k = -C_r H_{kmh}^r - \left\{ C_s C_{kr}^s - \frac{1}{F^2} (C_r y_k + C_k y_r) \right\} H_{mh}^r.$$

Since $y_r H_{mh}^r = 0$, the equation (3.6) takes the form

$$(3.7) \quad A_{hm} C_k = -C_r H_{kmh}^r - C_s C_{kr}^s H_{mh}^r + \frac{1}{F^2} C_r y_k H_{mh}^r.$$

The relation between the Berwald curvature tensor and Cartan's 3rd curvature tensor R_{jkh}^i is given by⁴

$$(3.8) \quad H_{ijk}^r = R_{ijk}^r - C_{is}^r H_{jk}^s.$$

or equivalently

$$(3.9) \quad H_{irjk} = R_{irjk} - C_{irs} H_{jk}^s.$$

Transvecting this equation by $C^i C^r$ and using the fact that R_{irjk} is skew-symmetric in first two indices, we have

$$C^i C^r H_{irjk} = -C^i C^r C_{irs} H_{jk}^s,$$

or

$$(3.10) \quad C^i C^r H_{irjk} + C^i C^r C_{irs} H_{jk}^s = 0.$$

Transvecting (3.7) by C^k , we have

$$(3.11) \quad A_{hm} C_k C^k = -C_r C^k H_{kmh}^r - C_s C^k C_{kr}^s H_{mh}^r + \frac{1}{F^2} C_r y_k C^k H_{mh}^r.$$

Because of (3.10) and $y_k C^k = 0$, we get

$$(3.12) \quad A_{hm} C_k C^k = 0.$$

This equation implies at least one of the following:

$$(3.13) \text{ (a) } A_{hm} = 0.$$

$$\text{(b) } C_k C^k = 0.$$

Since the metric of the Finsler space is positive definite, $C_k C^k = 0$ implies $C_k = 0$. According to Decieke's theorem the condition $C_k = 0$ implies that the space is Riemannian. Thus we have:

Theorem 3.1: *A C-recurrent Finsler space satisfy T-condition is either Riemannian or its recurrence vector satisfies $\mathcal{B}_h \lambda_m = \mathcal{B}_m \lambda_h$.*

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