

Fibonacci and Fibonacci-type Sequences: In Search of Innovative Pattern Recognition

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Abstract: The objective of this study is to explore an innovative pattern along with derivations connected to the Fibonacci sequences. Relevant proofs with supporting explanations have been provided in this study to derive the conceptualized pattern for this study which in turn will ensure reduction of the complexions and simplification of the Fibonacci formula while solving different problems linked to Fibonacci series and Fibonacci-type sequences. An attempt has also been made in this study to validate the conceptualized pattern with the help of induction method.

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Mathematics Subject Classification Code: 11B39, 11B50, 19A22, 35B36

1. Introduction

In Mathematics, the Fibonacci numbers, commonly denoted F_n , form a sequence, called the Fibonacci sequence, such that each number is the sum of the two preceding ones. A Fibonacci series is a series of numbers where except the first and second term, every next term is the sum of previous two terms (i.e. $F_n = F_{n-1} + F_{n-2}$, where $n > 2$)¹.

1, 1, 2, 3, 5, 8, 13, 21, 34, 55.....

Fibonacci numbers are named after the eminent Italian mathematician Leonardo of Pisa, later on known and popularized as 'Fibonacci'. In his 1202 book entitled 'Liber Abaci' (also spelled as "Liber Abbaci"; The Book of Calculation), Fibonacci introduced the sequence to Western European mathematics, although the sequence had also been explained earlier in Indian mathematics^{2,3}.

During the period of 200 BC in the work carried out by Pingala on counting possible patterns of Sanskrit poetry which was formed from the syllables of two lengths²⁻⁵.

India has also made a significant contribution in the area of Fibonacci sequence. The Fibonacci sequence also appears in Indian mathematics in the context with 'Sanskrit Prosody', as it was expressed by Parmanand Singh in 1986^{4,5}.

In the Sanskrit poetic tradition, there was an interest in counting all patterns of "long (L) syllables of 2 units duration", juxtaposed with short (S) syllables of 1 unit duration^{5,6}. Thus counting the different patterns of successive 'L' and 'S' with a given total duration results in the Fibonacci numbers, the number of patterns of duration 'm units' is F_{m+1} ^{2, 5-8}.

Outside India, the Fibonacci sequence had first appeared in the book "Liber Abaci" ("The Book of Calculation", 1202) by Fibonacci where it is used to calculate the growth of rabbit populations^{9,10}.

2. Objectives of the study

The objective of this study is listed in the following.

To search for an innovative pattern along with derivations connected to Fibonacci sequence and Fibonacci-type sequences followed by the validation through induction method.

The novelty of this study is to reduce the complexions and simplifying the Fibonacci formula while solving different problems which are linked to Fibonacci sequence and Fibonacci-type sequences.

3. Formula development with respect to Fibonacci sequence

In this paper, an attempt has been made to develop an innovative approach connected to Fibonacci sequence. The pattern which has been developed in this study is given in the following.

$$(3.1) \quad F_n + F_{n+4} = 3F_{n+2} \quad \text{where } n \geq 1.$$

Now in the following a detailed discussion has been made with appropriate logic, explanation and validation one after another. Here, a serious attempt has also been made to provide numerous explanations and validations in order to establish the idea which has been conceptualized as expressed above. A multi-dimensional approach has also been discussed herewith to derive the innovative pattern as we can find above. The step-wise discussion is presented below.

Proof:

$$\begin{aligned}
 F(n)+F(n+4) &= F(n)+[F(n+2)+F(n+3)] = F(n)+F(n+2)+[F(n+2)+F(n+1)] \\
 &= F(n)+F(n+1)+2F(n+2) = 3F(n+2)
 \end{aligned}$$

Now, let's consider the following series:

1, 1, 2, 3, 5, 8, **13**, 21, **34**, 55, **89**, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811

In the following some examples are provided below with respect to the application of conceptualized formula in this study.

Application of the conceptualized formula:

Suppose the question is:

- I. The sum of 7th term and 11th term (as made bold and highlighted above) of a Fibonacci series is 102. Determine the 9th term of the series.

Now how can we do this without much complication with the help of conceptualized formula?

Simply, apply the conceptualized formula: $F_n + F_{n+4} = 3F_{n+2}$

$$F_7 + F_{7+4} = 3F_{7+2}$$

$$F_7 + F_{11} = 3F_9$$

$$3F_9 = 102$$

$$F_9 = 102/3 = 34$$

- II. Let's again consider the series as follows:

If the sum of 3rd and 7th term is 48 more than the sum of 6th and 10th term in a Fibonacci series and 5th term is 5, determine the 8th term. In this context, again without much complexity we can apply the formula as conceptualized in this study, even if we do not know the series.

Given, $t_5 = 5$

We know, $F_n + F_{n+4} = 3F_{n+2}$

Or $F_3 + F_7 = 3F_5$

or $F_3 + F_7 = 3 \times 5 = 15$

Now, Given $F_6 + F_{10} = (F_3 + F_7) + 48 = 15 + 48 = 63$

We know that: $F_n + F_{n+4} = 3F_{n+2}$

$$\text{Or, } F_6 + F_{10} = 3F_8$$

$$\text{Or, } 3F_8 = 63 \quad \text{or, } F_8 = 63/3 = 21$$

This is a simple demonstration of the application of the derived formula which may diminish the complexity in solving sums as well as throws light upon the simplicity of the formula and here it carries the novelty of the work followed by contribution of new knowledge on the basis of existing knowledge.

4. Application of Conceptualized Formula in Fibonacci-type sequences

Fibonacci-type sequences can be considered to be such sequences which follow the same basic rule as Fibonacci series but where Term 1(T_1) and Term 2(T_2) can be of any natural value (natural number). For example: T_1 and T_2 are not at all bound to the values of 1 and 1 respectively rather, T_1 and T_2 can represent any natural number, as like 4 and 7.

Explanation and validation with respect to Fibonacci-type sequences formula-I

$$(4.1) \quad T_n + T_{n+4} = 3T_{n+2}$$

$$\text{Let } T_1 = a_1, \quad T_2 - T_1 = d$$

So the series is as follows:

$$a_1, a_1+d, 2a_1+d, 3a_1+2d, 5a_1+3d, 8a_1+5d, 13a_1+8d, 21a_1+13d, 34a_1+21d, 55a_1+34d, \dots$$

It has been observed that:

$$\text{a) } T_1 + T_5 = 3T_3$$

Explanation:

$$T_1 = a_1, \quad T_5 = 5a_1+3d, \quad T_1+T_5 = a_1+5a_1+3d = 6a_1+3d = 3(2a_1+d) = 3T_3$$

$$\text{b) } T_2 + T_6 = 3T_4$$

Explanation:

$$T_2+T_6 = (a_1+d) + (8a_1+5d) = 9a_1+6d = 3(3a_1+2d) = 3T_4$$

$$\text{c) } T_3 + T_7 = 3T_5$$

Explanation:

$$T_3+T_7 = (2a_1+d) + (13a_1+8d) = 15a_1+9d = 3(5a_1+3d) = 3T_5$$

$$\text{d) } T_4+T_8=3T_6$$

Explanation:

$$T_4+T_8 = (3a_1+2d) + (21a_1+13d) = 24a_1+15d = 3(8a_1+5d) = 3T_6$$

e) $T_5 + T_9 = 3T_7$

Explanation:

$$T_5 + T_9 = (5a_1 + 3d) + (34a_1 + 21d) = 39a_1 + 24d = 3(13a_1 + 8d) = 3T_7$$

Now, We can assume it to be applicable for Term T_1 to T_n .

So, by analysing the patterns,

The formula can be guessed as: $T_n + T_{n+4} = 3T_{n+2}$

Now consider any Fibonacci-type Series, for example:

(i) 2, 7, 9, 16, 25, 41, 66, 105, 171, 276, 447, 723.....

(ii) 7, 15, 22, 37, 59, 96, 155, 251, 406, 657.....

(iii) 11, 13, 24, 37, 61, 98, 159, 257, 416, 673.....

As per formula, in each series, we can observe the following:

(i) $T_1 + T_5 = 2 + 25 = 27 = 3 \times 9 = 3T_3$

(ii) $T_1 + T_5 = 7 + 59 = 66 = 3 \times 22 = 3T_3$

(iii) $T_1 + T_5 = 11 + 61 = 72 = 3 \times 24 = 3T_3$

5. Validation of Conceptualized Formula through the Induction Method

In the following an attempt has been made to validate the conceptualized formula connected to Fibonacci-type sequences with the help of an induction method. First, we have to notice if this formula is true for $n=1$ or not.

We observe that, in the all the above mentioned series given above as examples, $T_1 + T_5 = 3T_3$.

Now, we can proceed to induction step.

$T_n + T_{n+4} = 3T_{n+2}$, Let this expression be true for $n \geq 1$.

For every n , we shall have to prove this expression true for $n+1$.

On adding $T_{n-1} + T_{n+3}$ to both sides of the expression, we get:

$$(T_n + T_{n-1}) + (T_{n+4} + T_{n+3}) = 3T_{n+2} + T_{n-1} + T_{n+3}$$

$$T_{n+1} + T_{n+5} = 3T_{n+2} + T_{n-1} + T_{n+3}$$

$$T_{n+1} + T_{(n+1)+4} = T_{n+2} + T_{n+2} + T_{n+2} + T_{n-1} + T_{n+3}$$

$$= T_n + T_{n+1} + T_{n+2} + T_{n+2} + T_{n-1} + T_{n+3} \quad (\text{as } T_{n+2} = T_n + T_{n+1})$$

$$= T_n + (T_{n+1} + T_{n+2}) + T_{n+2} + T_{n-1} + T_{n+3}$$

$$\begin{aligned}
&= (T_n) + T_{n+3} + (T_{n+2}) + (T_{n-1}) + T_{n+3} \text{ [as } T_{n+1} + T_{n+2} = T_{n+3}] \\
&= ((T_n + T_{n-1}) + T_{n+2}) + 2T_{n+3} \text{ (as } T_n + T_{n-1} + T_{n+2}) \\
&= T_{n+1} + T_{n+2} + 2T_{n+3} = T_{n+3} + 2T_{n+3} = 3T_{n+3} = 3T_{(n+1)+2}
\end{aligned}$$

Here is one exemplary sum describing the usage of this concept.

Let's consider the series as:

4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364.....

i) The sum 5th term and 9th term is 228, determine the 7th term. Write a few terms of the series.

$$T_n + T_n + 4 = 3T_{n+2}$$

$$T_5 + T_9 = 3T_7$$

$$\text{Therefore, } T_7 = 228/3 = 76$$

Now we shall express, $T_1 = a$, $T_2 = b$

So, the series becomes:

$a, b, a+b, a+2b, 2a+3b, 3a+5b, 5a+8b, 8a+13b, 13a+21b, \dots$

$$(5.1) \quad T_5 = 2a + 3b$$

$$(5.2) \quad T_9 = 13a + 21b$$

$$T_5 + T_9 = (2a + 3b) + (13a + 21b) = 15a + 24b = 3(5a + 8b) = 3T_7$$

Therefore, $5a + 8b = 76$ or, $5|76 - 8b$ or, $5|(75 - 5b) + 1 - 3b$ or, $5|1 - 3b$

Or, $3b \equiv 1 \pmod{5}$ or $2 \times 3b \equiv 2 \times 1 \pmod{5}$ or $6b \equiv 2 \pmod{5}$ or $b \equiv 2 \pmod{5}$

As we want a series of whole numbers (0 and positive integers).

$5a + 8b = 76$ and $b \equiv 2 \pmod{5}$ gives only two values: $b = 2$, $b = 7$

[Next value of b satisfying the equation is 12 but when $b = 12$, $a = -4$, which is negative. But we want a series of positive terms].

Therefore, when $b = 2$, $a = 12$, series is:

12, 2, 14, 16, 30, 46, 76, 122, 198 [knowing that $5a + 8b = 76$]

Or, when $b = 7$, $a = 4$ series is:

4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843.....

Thus, we solve the problem so simply with the help of conceptualized formula.

6. Conclusion & Discussion

From this study we come to know an innovative formula connected to Fibonacci Numbers. Serious attempt has also been made in this study to show how in a very simple manner derivation can be made with respect to the innovative formula which has been conceptualized in this study. Further this study can encourage the other researchers to develop thought process in a very simple manner while deriving any formula connected to Fibonacci as well as Fibonacci-type Series.

7. Practical Implication

The practical implications of the study are listed below.

- i. The innovative formula can be used in the class room teaching programmes.
- ii. It may be useful for solving complex puzzles while developing algorithm & software programmes.

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