

On A Four-Dimensional Finsler Space with Vanishing ν -Connection Vectors

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Abstract: In 1977, M. Matsumoto and R. Miron¹ constructed an orthonormal frame for an n -dimensional Finsler space, called 'Miron frame'. The present authors² discussed four-dimensional Finsler spaces on the basis of Miron frame and find some interesting results. The aim of the present paper is to find sufficient conditions for a four-dimensional Finsler space to be S-3 like and to obtain some fruitful results for a four-dimensional Finsler space with vanishing ν -connection vectors.

Keywords: Finsler space, Miron frame, S-3 like space, ν -connection vectors.

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1. Orthonormal frame and connection vectors

Let M^4 be a four-dimensional smooth manifold and $F^4 = (M^4, L)$ be a four-dimensional Finsler space equipped with a metric function $L(x, y)$ on M^4 . The normalized supporting element, the metric tensor, the angular metric tensor and Cartan tensor are defined by

$$l_i = \dot{\partial}_i L, \quad g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2, \quad h_{ij} = L \dot{\partial}_i \dot{\partial}_j L \quad \text{and} \quad C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij} \quad \text{respectively.}$$

The torsion vector C^i is defined by $C^i = C^i_{jk} g^{jk}$. Throughout this paper, we use the symbols $\dot{\partial}_i$ and ∂_i for $\partial/\partial y^i$ and $\partial/\partial x^i$ respectively. The Cartan connection in the Finsler space is given as $\Gamma = (F^i_{jk}, G^i_j, C^i_{jk})$. The h - and ν -covariant derivatives of a covariant vector $X_i(x, y)$ with respect to the Cartan connection are given by

$$(1.1) \quad X_{i|j} = \partial_j X_i - (\dot{\partial}_h X_i) G^h_j - F^r_{ij} X_r,$$

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and

$$(1.2) \quad X_i|_j = \dot{\partial}_j X_i - C_{ij}^r X_r.$$

The Miron frame for a four-dimensional Finsler space is constructed by the unit vectors $(e_1^i, e_2^i, e_3^i, e_4^i)$. The first vector e_1^i is the normalized supporting element l^i and the second e_2^i is the normalized torsion vector $m^i = C^i / \mathbb{C}$, where $\mathbb{C}$ is the length of the vector C^i . The third vector $e_3^i = n^i$ and the fourth $e_4^i = p^i$ are constructed by the method of Matsumoto and Miron¹. With respect to this frame, the scalar components of an arbitrary tensor T_j^i are defined by

$$(1.3) \quad T_{\alpha\beta} = T_j^i e_{\alpha)i} e_{\beta)j},$$

then we get

$$(1.4) \quad T_j^i = T_{\alpha\beta} e_{\alpha)}^i e_{\beta)j},$$

where summation convention is also applied to Greek indices. The scalar components of the metric tensor g_{ij} are $\delta_{\alpha\beta}$. Therefore we get

$$(1.5) \quad g_{ij} = l_i l_j + m_i m_j + n_i n_j + p_i p_j.$$

Let $H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma} / L$ be scalar components of the h - and v -covariant derivatives $e_{\alpha)|j}^i$ and $e_{\alpha)}|_j^i$ respectively of the vectors $e_{\alpha)}^i$ then

$$(1.6) \quad e_{\alpha)|j}^i = H_{\alpha)\beta\gamma} e_{\beta)}^i e_{\gamma)j},$$

and

$$(1.7) \quad L e_{\alpha)}^i|_j = V_{\alpha)\beta\gamma} e_{\beta)}^i e_{\gamma)j}.$$

$H_{\alpha)\beta\gamma}$ and $V_{\alpha)\beta\gamma}$ are called h - and v -connection scalars respectively and are positively homogeneous of degree zero in y . Orthogonality of the Miron frame yields³ $H_{\alpha)\beta\gamma} = -H_{\beta)\alpha\gamma}$ and $V_{\alpha)\beta\gamma} = -V_{\beta)\alpha\gamma}$. Also we have $H_{1)\beta\gamma} = 0$ and $V_{1)\beta\gamma} = \delta_{\beta\gamma} - \delta_{1\beta} \delta_{1\gamma}$.

Now we define Finsler vector fields:

$$h_i = H_{2)3\gamma} e_{\gamma)i}, \quad j_i = H_{4)2\gamma} e_{\gamma)i}, \quad k_i = H_{3)4\gamma} e_{\gamma)i},$$

and

$$u_i = V_{2)3\gamma} e_{\gamma)i}, \quad v_i = V_{4)2\gamma} e_{\gamma)i}, \quad w_i = V_{3)4\gamma} e_{\gamma)i}.$$

The vector fields h_i, j_i, k_i are called h -connection vectors and the vector fields u_i, v_i, w_i are called ν -connection vectors. The scalars $H_{2)3\gamma}, H_{4)2\gamma}, H_{3)4\gamma}$ and $V_{2)3\gamma}, V_{4)2\gamma}, V_{3)4\gamma}$ are considered as the scalar components $h_\gamma, j_\gamma, k_\gamma$ and $u_\gamma, v_\gamma, w_\gamma$ of the h - and ν -connection vectors respectively.

From (1.7), we get

$$(1.8) \quad \begin{aligned} \text{a)} \quad & L e_{1)j}^i |_{j} = L l^i |_{j} = m^i m_j + n^i n_j + p^i p_j = h_j^i, \\ \text{b)} \quad & L e_{2)j}^i |_{j} = L m^i |_{j} = -l^i m_j + n^i u_j - p^i v_j, \\ \text{c)} \quad & L e_{3)j}^i |_{j} = L n^i |_{j} = -l^i n_j - m^i u_j + p^i w_j, \\ \text{d)} \quad & L e_{4)j}^i |_{j} = L p^i |_{j} = -l^i p_j + m^i v_j - n^i w_j. \end{aligned}$$

Because of the homogeneity of e_{α}^i , (1.8) gives

$$\begin{aligned} L m^i |_{j} l^j &= 0 = n^i u_j l^j - p^i v_j l^j, \\ L n^i |_{j} l^j &= 0 = -m^i u_j l^j + p^i w_j l^j. \end{aligned}$$

These imply $u_1 = u_j l^j = 0, v_1 = v_j l^j = 0, w_1 = w_j l^j = 0$. Consequently, we have:

Proposition 1.1- *The first scalar components u_1, v_1, w_1 of ν -connection vectors u_i, v_i, w_i vanish identically.*

2. Scalar derivatives

Taking h -covariant differentiation of both sides of (1.4), we get

$$(2.1) \quad T_{j|k}^i = (\delta_k T_{\alpha\beta}) e_{\alpha}^i e_{\beta)j} + T_{\alpha\beta} e_{\alpha) |k}^i e_{\beta)j} + T_{\alpha\beta} e_{\alpha}^i e_{\beta)j|k}.$$

If $T_{\alpha\beta}$ are the scalar components of $T_{j|k}^i$, i.e.

$$(2.2) \quad T_{j|k}^i = T_{\alpha\beta, \gamma} e_{\alpha}^i e_{\beta)j} e_{\gamma)k},$$

then we obtain

$$(2.3) \quad T_{\alpha\beta, \gamma} = (\delta_k T_{\alpha\beta}) e_{\gamma}^k + T_{\mu\beta} H_{\mu)\alpha\gamma} + T_{\alpha\mu} H_{\mu)\beta\gamma}.$$

Similarly, if we put

$$(2.4) \quad L T_j^i |_{k} = T_{\alpha\beta; \gamma} e_{\alpha}^i e_{\beta)j} e_{\gamma)k},$$

then the scalar components $T_{\alpha\beta;\gamma}$ of $L T_j^i|_k$ are given by

$$(2.5) \quad T_{\alpha\beta;\gamma} = L(\dot{\partial}_k T_{\alpha\beta}) e_{\gamma}^k + T_{\mu\beta} V_{\mu\alpha\gamma} + T_{\alpha\mu} V_{\mu\beta\gamma}.$$

The scalar components $T_{\alpha\beta,\gamma}$ and $T_{\alpha\beta;\gamma}$ respectively are called h - and v -scalar derivatives of the scalar components $T_{\alpha\beta}$ of T_j^i .

From (1.7), it follows that

$$L^2 e_{\alpha}^i |j|_k + L e_{\alpha}^i |j| l_k = L(L e_{\alpha}^i |j|)_k = V_{\alpha\beta\gamma;\delta} e_{\beta}^i e_{\gamma}^j e_{\delta}^k,$$

which implies

$$(2.6) \quad L^2 e_{\alpha}^i |j|_k = (V_{\alpha\beta\gamma;\delta} - V_{\alpha\beta\gamma} \delta_{1\delta}) e_{\beta}^i e_{\gamma}^j e_{\delta}^k.$$

According to the formula (2.5), $V_{\alpha\beta\gamma;\delta}$ are given by

$$V_{\alpha\beta\gamma;\delta} = L(\dot{\partial}_k V_{\alpha\beta\gamma}) e_{\delta}^k + V_{\alpha\mu\gamma} V_{\mu\beta\delta} + V_{\alpha\beta\mu} V_{\mu\gamma\delta}.$$

For $\alpha = 2, \beta = 3$, we get

$$\begin{aligned} V_{2)3\gamma;\delta} &= L(\dot{\partial}_k V_{2)3\gamma}) e_{\delta}^k + V_{2)\mu\gamma} V_{\mu)3\delta} + V_{2)3\mu} V_{\mu)\gamma\delta} \\ &= L(\dot{\partial}_k u_{\gamma}) e_{\delta}^k + V_{2)1\gamma} V_{1)3\delta} + V_{2)4\gamma} V_{4)3\delta} + u_{\mu} V_{\mu)\gamma\delta} \\ &= u_{\gamma;\delta} - \delta_{2\gamma} \delta_{3\delta} + v_{\gamma} w_{\delta}; \end{aligned}$$

where $u_{\gamma;\delta} = L(\dot{\partial}_k u_{\gamma}) e_{\delta}^k + u_{\mu} V_{\mu)\gamma\delta}$. Similarly, we get

$$V_{4)2\gamma;\delta} = v_{\gamma;\delta} - \delta_{4\gamma} \delta_{2\delta} + w_{\gamma} u_{\delta},$$

$$V_{3)4\gamma;\delta} = w_{\gamma;\delta} - \delta_{3\gamma} \delta_{4\delta} + u_{\gamma} v_{\delta}.$$

3. v -Curvature tensor

The v -curvature tensor is defined by

$$S_{hijk} = C_{hk}^r C_{ijr} - C_{hj}^r C_{ikr}.$$

The scalar components $S_{\alpha\beta\gamma\delta}$ of $L^2 S_{hijk}$ are written as

$$(3.1) \quad L^2 S_{hijk} = S_{\alpha\beta\gamma\delta} e_{\alpha}^h e_{\beta}^i e_{\gamma}^j e_{\delta}^k.$$

Since S_{hijk} is skew-symmetric in h and i as well as j and k and $S_{0ijk} = S_{hi0k} = 0$, the surviving independent components of $S_{\alpha\beta\gamma\delta}$ are six and these are S_{2323} , S_{2324} , S_{2334} , S_{2424} , S_{2434} and S_{3434} .

A Finsler space F^n ($n \geq 4$) is called S-3 like, if there exists a scalar S such that the curvature tensor S_{hijk} of F^n is written in the form

$$(3.2) \quad L^2 S_{hijk} = S(h_{hj} h_{ik} - h_{hk} h_{ij}),$$

where $h_{ij} = g_{ij} - l_i l_j$ is the angular metric tensor.

In terms of scalar components, the Ricci identity

$$e_{\alpha}^i |_{j|k} - e_{\alpha}^i |_{k|j} = e_{\alpha}^r S_{rjk}^i,$$

may be written as

$$(V_{\alpha})_{\beta\gamma;\delta} - V_{\alpha})_{\beta\gamma} \delta_{1\delta} - (V_{\alpha})_{\beta\delta;\gamma} - V_{\alpha})_{\beta\delta} \delta_{1\gamma} = S_{\alpha\beta\gamma\delta}.$$

For different values of α and β , we get

$$(3.3) \quad \left\{ \begin{array}{l} (u_{\gamma;\delta} + v_{\gamma} w_{\delta} - \delta_{2\gamma} \delta_{3\delta} - u_{\gamma} \delta_{1\delta}) - (u_{\delta;\gamma} + v_{\delta} w_{\gamma} - \delta_{2\delta} \delta_{3\gamma} - u_{\delta} \delta_{1\gamma}) = S_{23\gamma\delta}, \\ (v_{\gamma;\delta} + w_{\gamma} u_{\delta} - \delta_{4\gamma} \delta_{2\delta} - v_{\gamma} \delta_{1\delta}) - (v_{\delta;\gamma} + w_{\delta} u_{\gamma} - \delta_{4\delta} \delta_{2\gamma} - v_{\delta} \delta_{1\gamma}) = S_{42\gamma\delta}, \\ (w_{\gamma;\delta} + u_{\gamma} v_{\delta} - \delta_{3\gamma} \delta_{4\delta} - w_{\gamma} \delta_{1\delta}) - (w_{\delta;\gamma} + u_{\delta} v_{\gamma} - \delta_{3\delta} \delta_{4\gamma} - w_{\delta} \delta_{1\gamma}) = S_{34\gamma\delta}. \end{array} \right.$$

which gives us

$$(3.4) \quad \left\{ \begin{array}{l} S_{2323} = (u_{2;3} - u_{3;2} + v_2 w_3 - v_3 w_2) - 1, \\ S_{4242} = (v_{4;2} - v_{2;4} + w_4 u_2 - w_2 u_4) - 1, \\ S_{3434} = (w_{3;4} - w_{4;3} + u_3 v_4 - u_4 v_3) - 1, \\ S_{2324} = (u_{2;4} - u_{4;2} + v_2 w_4 - v_4 w_2), \\ S_{2334} = (u_{3;4} - u_{4;3} + v_3 w_4 - v_4 w_3), \\ S_{4234} = (v_{3;4} - v_{4;3} + w_3 u_4 - w_4 u_3). \end{array} \right.$$

If the Finsler space F^4 is S-3 like then

$$L^2 S_{hijk} = S(h_{hj} h_{ik} - h_{hk} h_{ij}), \text{ i.e.}$$

$$L^2 S_{hijk} = S\{(m_h n_i - m_i n_h)(m_j n_k - m_k n_j) + (m_h p_i - m_i p_h)(m_j p_k - m_k p_j) \\ + (n_h p_i - n_i p_h)(n_j p_k - n_k p_j)\},$$

which implies

$$(3.5) \quad S_{2323} = S, \quad S_{2324} = 0, \quad S_{2334} = 0, \quad S_{2424} = S, \quad S_{2434} = 0, \quad S_{3434} = S.$$

From (3.4) and (3.5), we get

$$(3.6) \quad \begin{aligned} (u_{2;3} - u_{3;2} + v_2 w_3 - v_3 w_2) &= S + 1, \\ (v_{4;2} - v_{2;4} + w_4 u_2 - w_2 u_4) &= S + 1, \\ (w_{3;4} - w_{4;3} + u_3 v_4 - u_4 v_3) &= S + 1, \\ (u_{2;4} - u_{4;2} + v_2 w_4 - v_4 w_2) &= 0, \\ (u_{3;4} - u_{4;3} + v_3 w_4 - v_4 w_3) &= 0, \\ (v_{3;4} - v_{4;3} + w_3 u_4 - w_4 u_3) &= 0. \end{aligned}$$

Thus, we have:

Theorem 3.1. *A four-dimensional Finsler space is S-3 like if and only if the conditions (3.6) are satisfied.*

Corollary 3.1. *A four-dimensional Finsler space is S-3 like if the v-connection vectors satisfy $u_{\gamma;\delta} = v_{\gamma;\delta} = w_{\gamma;\delta} = 0$ and*

$$\frac{u_{\gamma}}{u_{\delta}} = \frac{v_{\gamma}}{v_{\delta}} = \frac{w_{\gamma}}{w_{\delta}}.$$

Corollary 3.2. *A four-dimensional Finsler space with vanishing v-connection vectors is S-3 like with v-curvature -1 .*

From Corollary 3.2, we have

$$L^2 S_{hijk} = -1(h_{hj} h_{ik} - h_{hk} h_{ij}).$$

Taking h -covariant differentiation on both sides and using $L_{|r} = 0$ and $h_{ij|r} = 0$, we get $S_{hijk|r} = 0$, and then the identity $P_{hijk} - P_{hikj} = S_{hijk|r} y^r$ reduces to $P_{hijk} = P_{hikj}$, where P_{hijk} is the hv -curvature tensor defined by

$$P_{hijk} = C_{ijk|h} - C_{hjk|i} + C_{hj}^r C_{rik|s} y^s - C_{ij}^r C_{rhk|s} y^s.$$

Thus, we have:

Theorem 3.2. *The $h\nu$ -curvature tensor P_{hijk} of a four-dimensional Finsler space with vanishing ν -connection vectors is symmetric in j and k .*

A Finsler space is said to be a Landsberg space if $h\nu$ -curvature tensor P_{hijk} vanishes.

A $*P$ -Finsler space is characterized by $C_{ijk|r} y^r = \lambda C_{ijk}$, while a Finsler space is said to be a P -2 like Finsler space if $P_{hijk} = P_h C_{ijk} - P_i C_{hjk}$, where P_i is a covariant vector field.

M. Matsumoto³ proved that if the $h\nu$ -curvature tensor P_{hijk} is symmetric in j and k in a $*P$ -Finsler space then either $P_{hijk} = 0$ or $S_{hijk} = 0$. Therefore in view of Theorem 3.2, $P_{hijk} = 0$ or $S_{hijk} = 0$, in a four-dimensional $*P$ -Finsler space with vanishing ν -connection vectors. But for such space $S_{hijk} \neq 0$ in view of Corollary 3.2. Therefore $P_{hijk} = 0$. Hence the space is a Landsberg space. Thus, we have:

Theorem 3.3. *A four-dimensional $*P$ -Finsler space with vanishing ν -connection vectors is a Landsberg space.*

Since a P -2 like Finsler space is $*P$ -Finsler space, we may conclude:

Theorem 3.4. *A four-dimensional P -2 like Finsler space with vanishing ν -connection vectors is a Landsberg space.*

References

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