

Fuzzy Reliability of a Degradable System

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Abstract: Researches in classical reliability theory are mainly based on Probist reliability, having binary state assumption for states and governed by the probability distributions. In the present paper binary state assumption has been replaced by fuzzy state assumption thus obtaining Profust reliability estimates. In reliability estimation of any system, parameters like failure rates, and/or repair rates play a vgerly Impoortant role. Normally the tools used to measure these parameters are based on the actual observance of the functioning of the system supplemented by statistical techniques. The values that are used for these parameters inherit the uncertainties of statistical methods. In the present paper we have introduced a new approach based on fuzzy set theory to estimate these failure rates.

1. Introduction

Most of the researches in classical reliability theory are covered under Probist reliability, having binary state assumption for states and governed by the probability distributions. In degradable systems, it is unrealistic to assume that the system possesses only two states 'working' or 'failed'. The system may work to a tolerable limit even if it has been degraded to some degree. We consider a degradable system having two units $A(n)$ and $B(m)$ having n and m components respectively. The units are connected in series and both the units are degradable. We assume that system transits from one state to other fuzzily. Zadeh suggested a paradigm shift from the theory of total denial and affirmation to a theory of grading to give new concept of set called fuzzy set. Fuzzy sets can express the gradual transition of the system from working state to a failed state. The crisp set theory only dichotomizes the system in working state and failed state but fuzzy set theory can cover up all possible states between a fully working state. This approach to the reliability theory is known as profust reliability wherein the binary state assumption is replaced by fuzzy state assumption^{2,4,5}.

In the reliability evaluation of a very complex system, the measure of the involved parameters are obtained by statistical data available to us from testing and operating modes under different operatig conditions. It is well known to us that these data are

comprised of very uncertainties. So it is not realistic to assign a crisp number to these parameters by any statistical mean during their estimation. To a very common parameter known as failure rate of unit/component we can assign a fuzzy set i.e. the failure behaviour of the unit/component is characterized by the fuzzy sets. In this way we are able to cover up most of the uncertainties present in the statistical data. Also it is very realistic approach. Obviously since under different operating conditions the system imparts different failure characteristics and also different experts may assign different fuzzy sets to a particular parameter. Now since it is very tedious (or not possible) to evaluate the Profust reliability of a system by taking the failure rate as a fuzzy set. Therefore using an appropriate defuzzification method we try to find out a crisp number from these fuzzy sets (i.e. a crisp value of fuzzy rate). In another way we can say that by taking the input as a fuzzy set for the failure rate we can find a crisp output.

We have considered a system having two redundant (k out of n) units namely $A(n)$ and $B(m)$ connected in series and contains n and m componets in parallel respectively (also we consider the failure of the systems if and only if the unit A or B is degraded up to a threshold).

2. Notation

$A(n)$	:	Unit A having n components in parallel redundancy.
$B(m)$	:	Unit B having m components in parallel redundancy.
$P(t)$	:	Probability that a component is still working at time t .
n, m	:	Number of compoents in unit A and B respectively.
$\tilde{\lambda}_1, \tilde{\lambda}_2$	:	Fuzzy failure rate of components in unit A and B respectively.

3. Fuzzy sets

A fuzzy set is a set that consists of the elements having the varying degree of membership in the set. Classically a set is a well defined collection of objects i.e. the elements of universe of discourse are distinguished by a very sharp layer, but in many realistic situations it is not possible to distinct these elements by such a sharp layer due to the uncertainty involved in that situations. So these situations may be better explained by the fuzzy sets, the set which contains all the elements of the universe but with different degree of membership.

Since a crisp set A defined over a universal set X may be characterized by a function termed as characteristic function which is denoted by χ_A and defined as

$$\chi_A : X \rightarrow \{0,1\}$$

$$\text{where } \chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Analogous to this characteristic function we can find a function called membership function to characterize a fuzzy set, which is denoted by $\mu_{\tilde{A}}$

$$\mu_{\tilde{A}}: X \rightarrow [0, 1]$$

From this we can say that a fuzzy set is the generalization of a crisp set. Notation for a fuzzy set, defined over the universe X is as follows.

$$\tilde{A} = \{(x_i, \mu_{\tilde{A}}(x_i)) : x_i \in X\}$$

4. Defuzzification

There are many situations where for a fuzzy input we are required a crisp output. Defuzzification is the tool which makes it possible. To illustrate it let the output in a process is the union of two or more fuzzy sets defined on the universe of discourse. Then this output which is also a fuzzy set (need not to be normal) may be converted into crisp number by the defuzzification process. Still we have near about seven methods of defuzzification in the literature. Here we have opted the centroid method, which was firstly introduced by Sugeno and Lee and may be given by the following expression.

$$x^* = \frac{\int \mu_{\tilde{A}}(x) \cdot x dx}{\int \mu_{\tilde{A}}(x) \cdot dx}$$

where $\tilde{A}$ is a fuzzy set, which is the union of two or more fuzzy sets i.e.

$$\tilde{A} = \tilde{A}_1 \cup \tilde{A}_2 \cup \dots \cup \tilde{A}_n \text{ and } \cup \text{ is a standard fuzzy union.}$$

5. Failure rate estimation

Suppose we have some statistical data about the failure characteristic of the system obtained by conducting various tests in different operating modes for a given time frame. Now from the data available to the experts, the different fuzzy sets may be assigned to the failure rate of a unit/component by different experts.

Let λ_1 and λ_2 be the fuzzy failure rates of the components of unit A and unit B the given system respectively.

The fuzzy failure rate λ_1 to be assigned the fuzzy sets about (0.0003), about (0.0009) and about (0.0018) by different experts for a given range [0, 0.0025] of failure rate. These failure rates may be defined as below and shown in fig-1

$$\tilde{\lambda}'_1 = \begin{cases} \frac{\lambda_1}{0.0001} & 0 \leq \lambda_1 \leq 0.0001 \\ 1 & 0.0001 \leq \lambda_1 \leq 0.0004 \\ \frac{0.0007 - \lambda_1}{0.0003} & 0.0004 \leq \lambda_1 \leq 0.0007 \end{cases}$$

$$\tilde{\lambda}''_1 = \begin{cases} \frac{\lambda_1 - 0.0005}{0.0003} & 0.0005 \leq \lambda_1 \leq 0.0008 \\ 1 & 0.0008 \leq \lambda_1 \leq 0.0012 \\ \frac{0.0014 - \lambda_1}{0.0002} & 0.0012 \leq \lambda_1 \leq 0.0014 \end{cases}$$

$$\tilde{\lambda}'''_1 = \begin{cases} \frac{\lambda_1 - 0.0012}{0.0005} & 0.0012 \leq \lambda_1 \leq 0.0017 \\ 1 & 0.0017 \leq \lambda_1 \leq 0.0019 \\ \frac{0.0025 - \lambda_1}{0.0006} & 0.0019 \leq \lambda_1 \leq 0.0025 \end{cases}$$

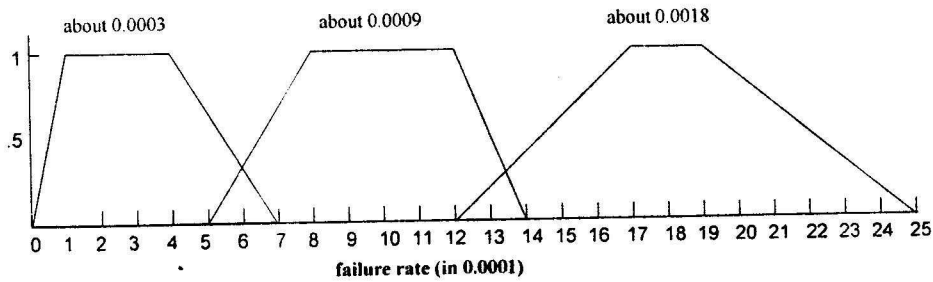


Fig. 1

In a similar manner the fuzzy failure rate $\tilde{\lambda}_2$ of components of the unit B can be characterized by the fuzzy sets about (0.0004), about (0.0010) and about (0.0020) given as below and show in fig.-2 for the value of failure rate λ_2 ranging from [0, 0.0025] by the statistical data available to us for a specified time.

$$\lambda_2' = (\text{about } 0.0004) = \frac{1}{1 + 5(\lambda_2 - 0.0004)^2} \text{ for all } \lambda_2 \in [0, 0.0025]$$

$$\lambda_2'' = (\text{about } 0.0010) = \frac{1}{1 + (\lambda_2 - 0.0010)^2} \text{ for all } \lambda_2 \in [0, 0.0025]$$

$$\lambda_2''' = (\text{about } 0.0020) = \frac{1}{1 + (\lambda_2 - 0.0020)^2} \text{ for all } \lambda_2 \in [0, 0.0025]$$

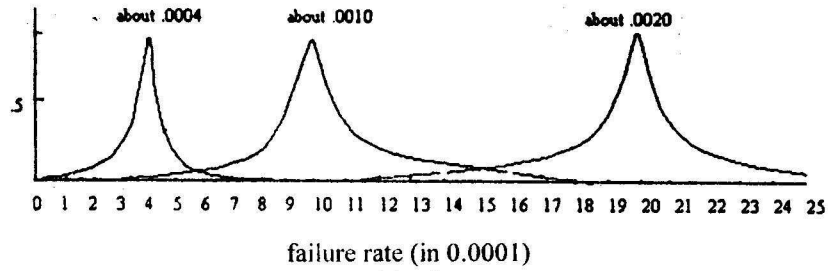


Fig. 2

Using the centroid method of defuzzification we have the crisp value 0.00112 and 0.0013 for $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ respectively. Which may be used for profust reliability evaluation of the system.

Fuzzy Reliability Evaluation: Let A(n) and B(m) be two parallel redundant units having n and m components, connected in series in a system. In conventional reliability where we do not allow any imprecision, we say that system has two possible states only i.e. working state and failed state. But here allowing all imprecision we evaluate the reliability of the system by using fuzzy set theory and probability theory.

The system is fully working if all the n components of unit A and m components of the unit B are working, and failure of each components of the unit A and B causes the degradation of the system and the system works until it reaches to the desirable threshold.

Let K and L are the minimum number of working components of the unit A and B respectively which are required to be the system "working". Then according to conventional reliability theory, the reliability of such a system may be given as

$$(1) \quad R_{KL}(t) = \left(\sum_{k=K}^n \binom{n}{k} [P(t)]^k [1 - P(t)]^{n-k} \right) \times \left(\sum_{l=L}^m \binom{m}{l} [P(t)]^l [1 - P(t)]^{m-l} \right)$$

Where $P(t)$ is the probability that a component is still working. Now if λ_1 and λ_2 are constant failure rates for the components of the unit A and B respectively, then the above expression given by equation (1) may be written as

$$(2) \quad R_{KL}(t) = \left(\sum_{k=K}^n \binom{n}{k} [e^{-\lambda_1 t}]^k [1 - e^{-\lambda_1 t}]^{n-k} \right) \times \left(\sum_{l=L}^m \binom{m}{l} [e^{-\lambda_2 t}]^l [1 - e^{-\lambda_2 t}]^{m-l} \right)$$

But the degradable system of such type may be working and failed to different degrees between a “fully working” and “fully failed” state. Thus the conventional reliability theory fails to describe the reliability characteristics of such type of systems. So we have to look at fuzzy set theory to overcome such problems i.e. fuzzy set theory is a better tool to give the reliability of a degradable system which describes the system reliability in a natural way.

Here we suppose that S_{ij} be the state of the system in which i components of the unit A and j components of unit B are working where $i=0, 1, \dots, n$ and $j=0, 1, \dots, m$. Such types of degradable systems are working to some degrees between their “fully working” and “fully not working” states. This idea of fuzzy reliability was firstly introduced by Cai and Wen⁴. Let $A = \{A_1, A_2, \dots, A_n\}$ and $B = \{B_1, B_2, \dots, B_m\}$ be the set of all possible states of unit A and B respectively. The fuzzy event working ($\tilde{W}_1$) and failed ($\tilde{F}_1$) can be defined by the following fuzzy sets.

$$\begin{aligned} \tilde{W}_1 &= \{ (A_i, \mu_{\tilde{W}_1}(A_i)), i=1, 2, \dots, n \} \\ \tilde{F}_1 &= \{ (A_i, \mu_{\tilde{F}_1}(A_i)), i=1, 2, \dots, n \} \end{aligned}$$

Where A_i s are the all-possible states of the unit A and corresponding degree of membership of all these states may be given as $\mu_{\tilde{W}_1}(A_i)$ and $\mu_{\tilde{F}_1}(A_i)$, in the fuzzy sets ($\tilde{W}_1$) and ($\tilde{F}_1$) respectively. Now what we find that each state is a member of both, the set of working state and the set of failed state with different degrees.

Here also we have assumed that the event “working” and “failed” are complementary fuzzy i.e. $\mu_{\tilde{W}_1} = 1 - \mu_{\tilde{F}_1}$

Now according as conventional reliability theory the reliability of a system is defined as

$$(3) \quad R(t_0, t_0+t) = P(T \text{ does not occur in the time interval } [t_0, t_0+t])$$

Where T denotes the transition from working to failed state during the time interval $[t_0, t_0+t]$.

Since the “working” and “failed” both are fuzzy events for a degradable system and hence the transition from a working state to a failed state will also be a fuzzy event, Let $\tilde{T}_1$ be the transition from a working state ($\tilde{W}_1$) to a failed state ($\tilde{F}_1$) for a unit A and likewise $\tilde{T}_2$ be the transition from a working state ($\tilde{W}_2$) to a failed state ($\tilde{F}_2$) for the unit B.

Hence $\tilde{T}_1$ may be defined on a universe $\{r_{i,i'}, i, i' = 1, 2, \dots, n\}$

$$\tilde{T}_1 = \{(r_{i,i'}, \mu_{\tilde{T}_1}(r_{i,i'})) \mid i, i' = 1, 2, \dots, n\}$$

Where, $\mu_{\tilde{T}_1}(r_{i,i'})$ represents the membership function for the fuzzy set $\tilde{T}_1$ and $r_{i,i'}$ is the transition from state A_i to $A_{i'}$.

In a similar manner we can define a fuzzy event $\tilde{T}_2$ i.e. the transition from a working state ($\tilde{W}_2$) to a failed state ($\tilde{F}_2$) for the unit B and time may be defined on the universe $\{s_{j,j'}, j, j' = 1, 2, \dots, m\}$ by the fuzzy set.

$$\tilde{T}_2 = \{(s_{j,j'}, \mu_{\tilde{T}_2}(s_{j,j'})) \mid j, j' = 1, 2, \dots, m\}$$

Where $s_{j,j'}$ represents the transition from state B_j to $B_{j'}$.

Now let us denote relative degree of “failed” for the state A_i by $\beta_{\tilde{W}_1\tilde{F}_1}(A_i)$

And define as

$$\beta_{\tilde{W}_1\tilde{F}_1}(A_i) = \frac{\mu_{\tilde{F}_1}(A_i)}{\mu_{\tilde{F}_1}(A_i) + \mu_{\tilde{W}_1}(A_i)} \quad i = 1, 2, \dots, n$$

If ($\tilde{W}_1$) and ($\tilde{F}_1$) are fuzzy complement i.e. $\tilde{W}_1 = \tilde{F}_1^c$. Then

$$\mu_{\tilde{W}_1} = 1 - \mu_{\tilde{F}_1} \text{ and it gives } \beta_{\tilde{W}_1\tilde{F}_1}(A_i) = \mu_{\tilde{F}_1}(A_i)$$

Now we can say that transition from state A_i to A_i' will only occur if relative degree of failed for state A_i is less than that of A_i' i.e. $\beta_{\tilde{W}_1 \tilde{F}_1}(A_i) < \beta_{\tilde{W}_1 \tilde{F}_1}(A_i')$

Hence the membership funtion $\mu_{\tilde{T}_1}$ for the fuzzy set $\tilde{T}_1$ is defined as

$$(4) \quad \mu_{\tilde{T}_1}(r_{i,i'}) = \begin{cases} \beta_{\tilde{W}_1 \tilde{F}_1}(A_{i'}) - \beta_{\tilde{W}_1 \tilde{F}_1}(A_i) & \text{if } \beta_{\tilde{W}_1 \tilde{F}_1}(A_{i'}) > \beta_{\tilde{W}_1 \tilde{F}_1}(A_i) \\ 0 & \text{otherwise} \end{cases}$$

Also we can find the membership function $\mu_{\tilde{T}_2}$ for the fuzzy set $\tilde{T}_2$ as below

$$(5) \quad \mu_{\tilde{T}_2}(s_{j,j'}) = \begin{cases} \beta_{\tilde{W}_2 \tilde{F}_2}(A_{j'}) - \beta_{\tilde{W}_1 \tilde{F}_1}(A_j) & \text{if } \beta_{\tilde{W}_2 \tilde{F}_2}(A_{j'}) > \beta_{\tilde{W}_1 \tilde{F}_1}(A_j) \\ 0 & \text{otherwise} \end{cases}$$

During the interval $[t_0, t_0+t]$, the fuzzy system reliability of a system having two units A and B connected in series is the probability that no transition from a less failed state to more failed state occurs i.e.

$$(6) \quad \tilde{R}(t_0, t_0+t) = P(\text{Neither } \tilde{T}_1 \text{ nor } \tilde{T}_2 \text{ occur during the time interval } [t_0, t_0+t])$$

Also the transition $\tilde{T}_1$ and $\tilde{T}_2$ both are restricted in such a way that state A_i and B_j can go to the lower without by passing any intermediate state.

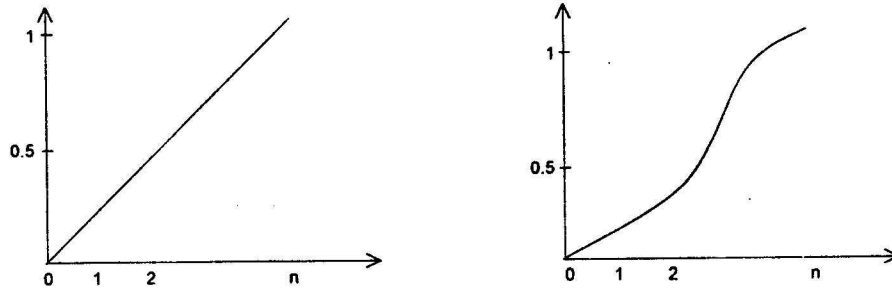


Fig. (3)

For the sake of simplicity we here consider only the linear and quadratic degradation of the given system.

Firstly let we consider the linear degradation of the unit A and also of unit B shown in fig. 3(a) i.e.

$$\mu_{\tilde{W}_1}(A_i) = \frac{i}{n} \quad \text{and} \quad \mu_{\tilde{W}_2}(B_j) = \frac{j}{m}$$

Also if we assume that initially unit A is in state A_n and unit B is in the state B_m then from the definition of $\mu_{\tilde{T}_1}(r_{i,i'})$ and $\mu_{\tilde{T}_2}(s_{j,j'})$ as given by the equation (4) and (5), the membership of the transition $\tilde{T}_1$ from state A_n to A_i and $\tilde{T}_2$ from B_m to B_j at time t is given below.

$$\mu_{\tilde{T}_1}(r_{n,i}) = \frac{n-i}{n}, \quad \mu_{\tilde{T}_2} = \frac{m-j}{m}$$

Now if the degradation of the unit A and B is quadratic as shown in fig. 3(b) i.e.

$$\mu_{\omega_1}(A_i) = \begin{cases} \frac{2i^2}{n^2} & 0 \leq i \leq \frac{n}{2} \\ 1 - \frac{2(n-i)^2}{n^2} & \frac{n}{2} < i \leq n \end{cases}$$

$$\mu_{\omega_2}(B_j) = \begin{cases} \frac{2j^2}{m^2} & 0 \leq j \leq \frac{m}{2} \\ 1 - \frac{2(m-j)^2}{m^2} & \frac{m}{2} < j \leq m \end{cases}$$

Therefore from the definition of $\mu_{\tilde{T}_1}(r_{n,i})$ and $\mu_{\tilde{T}_2}(s_{m,j})$, we get

$$\mu_{\tilde{T}_1}(r_{n,i}) = \begin{cases} 1 - \frac{2i^2}{n^2} & 0 \leq i \leq \frac{n}{2} \\ \frac{2(n-i)^2}{n^2} & \frac{n}{2} < i \leq n \end{cases}$$

$$\mu_{\tilde{T}_2}(s_{m,j}) = \begin{cases} 1 - \frac{2j^2}{m^2} & 0 \leq j \leq \frac{m}{2} \\ \frac{2(m-j)^2}{m^2} & \frac{m}{2} < j \leq m \end{cases}$$

Thus from the equation (6) the fuzzy system reliability.

$$\begin{aligned} \tilde{R}(t_0, t_0 + t) = & \left(1 - \sum_{k=1}^{n-1} \mu_{\tilde{T}_1}(r_{n,k}) \times P(\text{the unit A is in state } A_k \text{ at time } t_0 + t) \right) \times \\ & \left(1 - \sum_{l=1}^{m-1} \mu_{\tilde{T}_2}(s_{m,l}) \times P(\text{the unit B is in state } B_l \text{ at time } t_0 + t) \right) \end{aligned}$$

If we consider that initially i.e. at time $t = 0$ the system was in state S_{nm} where both the unit A and B were in state A_n and B_m respectively. Then we get

$$\begin{aligned} \tilde{R}(t) = & \left(1 - \sum_{k=1}^{n-1} \mu_{\tilde{T}_1}(r_{n,k}) \times P(\text{the unit A is in state } A_k \text{ at time } t) \right) \times \\ (7) \quad & \left(1 - \sum_{l=1}^{m-1} \mu_{\tilde{T}_2}(s_{m,l}) \times P(\text{the unit B is in state } B_l \text{ at time } t) \right) \end{aligned}$$

Since during the failure rate estimation we have obtained the crisp value λ_1 and λ_2 of the failure rate of the components of unit A and B of the given system considering them as a fuzzy set $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ for the unit A and B respectively.

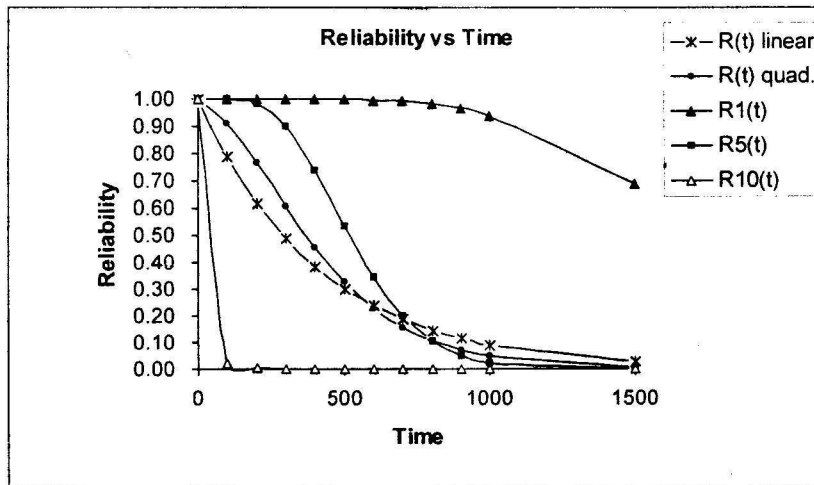
Then equation (7) becomes

$$\begin{aligned} \tilde{R}(t) = & \left(1 - \sum_{k=1}^{n-1} \mu_{\tilde{T}_1}(r_{n,k}) \times \binom{n}{k} [e^{-\lambda_1 t}]^k [1 - e^{-\lambda_1 t}]^{n-k} \right) \times \\ (8) \quad & \left(1 - \sum_{l=1}^{m-1} \mu_{\tilde{T}_2}(s_{m,l}) \times \binom{m}{l} [e^{-\lambda_2 t}]^l [1 - e^{-\lambda_2 t}]^{m-l} \right) \end{aligned}$$

Table 1. Comparison of fuzzy reliability $\tilde{R}(t)$ and conventional reliability $R_{KL}(t)$ for $n=m=10$, $\lambda_1 = 0.00114$ and $\lambda_2 = 0.0013$.

Time	$\tilde{R}(t)$		$R_{KL}(t)$		
	Linear	Quadratic	K=1, L=1 (Parallel)	K=5, L=5	K=10, L=10 (series)
0	1	1	1	1	1
100	0.7866	0.9106	1	0.9994	0.0228
200	0.6188	0.7666	0.9999	0.9814	0.0040
300	0.4868	0.6054	0.9999	0.9011	0.0007
400	0.3829	0.4543	0.9998	0.7402	6.7E-05
500	0.3012	0.3278	0.9992	0.5358	6.1E-06
600	0.2369	0.2298	0.9971	0.3434	5.6E-07
700	0.1864	0.1579	0.9922	0.1976	5.1E-08
800	0.1466	0.1071	0.9826	0.1036	4.6E-09
900	0.1153	0.0721	0.9663	0.0501	4.4E-10
1000	0.0907	0.0484	0.9412	0.0227	3.9E-11
1500	0.0273	0.0068	0.6915	0.0004	1.4E-14

Tabel-1 gives the result of $\tilde{R}(t)$ for both linear and quadratic degradation of the unit and also gives the value for K and L=1,5,10 by the equation (2) and (8). The comparison of the fuzzy reliability and classical reliability has been shown in fig. 4.



Conclusion

Due to the complete monopoly of statistical tools for uncertainty measurements. We could not dare to avoid these tools till the introduction of fuzzy set theory in 1965. But due to a gap between the mathematics and this imprecision world, we can not give a realistic description of the system failure. In this paper we have opted the fuzzy set theory rather than crisp set theory to characterize the system failure i.e. the failure rate of the system has treated as a fuzzy set instead of a real number. Also the system degradation is considered to be gradual not random that meets to the actual degradation of the system and it is characterized by the fuzzy set theory.

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