

## Ricci Parallel Trans-Sasakian Manifolds

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**Abstract:** S. Tanno<sup>1</sup> classified connected almost contact metric manifolds with automorphism groups possessing the maximum dimension. The sectional curvature of a plane section of such a manifold containing  $U$  is constant, say  $c$ . He divided these manifolds into three classes: (1) homogeneous normal contact Riemannian manifolds with  $c > 0$ , (2) global Riemannian products of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature if  $c = 0$  and (3) a warped product space  $R \times_f C^n$  if  $c < 0$ . We know that a manifold belonging to class (1) is characterized by admitting a Sasakian structure. Kenmotsu<sup>2</sup> paid much attention to the manifolds belonging to class (3) and showed that the structure of such a manifold is in general non-Sasakian. In this paper we have studied some particular cases of a Ricci-parallel trans-Sasakian manifold.

This is a trans-Sasakian manifold whose Ricci tensor is parallel, i.e.  $\nabla_X Ric = 0$  along any arbitrary vector field  $X$ . An expression for  $Ric(X, Y)$  has been derived. We have also obtained the expression for Weyl, Conformal, Concurcular and Conharmonic curvature tensors on a Ricci parallel trans-Sasakian manifold.

### 1. Introduction

Let  $M_n$  be an almost contact manifold with an almost contact metric structure  $(F, U, u, g)$ , where  $F$  is a  $(1,1)$  tensor field,  $U$  is a vector field,  $u$  is 1-form and  $g$  is a compatible Riemannian metric such that,

$$(1.1) \quad (i) \quad F^2 = -I + u \otimes U, \quad (ii) \quad u(U) = 1,$$

$$(iii) \quad F(U) = 0, \quad (iv) \quad u \circ F = 0.$$

$$(1.2) \quad g(FX, FY) = g(X, Y) - u(X)u(Y)$$

and

$$(1.3) \quad g(X, FY) = -g(FX, Y), \quad g(X, U) = u(X),$$

for all  $X, Y \in T_M$ .

An almost contact metric structure  $(F, U, u, g)$  on  $M_n$  is called a trans-Sasakian

structure if  $(M_n \times R, J, G)$  belongs to the class  $W_a$  of Gray-Hervella classification of almost Hermitian manifolds, where  $J$  is the almost complex structure of  $M_n \times R$  defined by

$$(1.4) \quad J\left(X, a \frac{d}{dt}\right) = \left(FX - aU, u(X) \frac{d}{dt}\right)$$

for all vector fields  $X$  on  $M_n$  and smooth functions  $a$  on  $M_n \times R$  and  $G$  is the product metric on  $M_n \times R$ . This may be expressed by the condition

$$(1.5) \quad (\nabla_X F)(Y) = \alpha\{g(X, Y)U - u(Y)X\} + \beta\{g(FX, Y)U - u(Y)FX\}$$

for some smooth functions  $\alpha$  and  $\beta$  on  $M_n$ , and we say that the trans-Sasakian structure is of type  $(\alpha, \beta)$ . From (1.5), it follows that

$$(1.6) \quad \nabla_X U = -\alpha(FX) + \beta\{X - u(X)U\}$$

We note that the trans-Sasakian structures of type  $(0, 0)$  are cosymplectic<sup>3</sup>, trans-Sasakian structures of type  $(0, \beta)$  are  $\beta$ -Kenmotsu<sup>2</sup> and trans-Sasakian structures of type  $(\alpha, 0)$  are  $\alpha$ -Sasakian. Thus, in trans-Sasakian structure of type  $(0, 0)$ , the equations (1.5) and (1.6) reduce to

$$(1.7) \quad \nabla_X F = 0, \quad \nabla_X U = 0.$$

For  $\beta$ -Kenmotsu structure (1.5) and (1.6) reduce to

$$(1.8) \text{ (i)} \quad (\nabla_X F)(Y) = \beta\{g(FX, Y)U - u(Y)FX\},$$

$$\text{(ii)} \quad \nabla_X U = \beta\{X - u(X)U\},$$

while for  $\alpha$ -Sasakian structure, we have

$$(1.9) \text{ (i)} \quad (\nabla_X F)(Y) = \alpha\{g(X, Y)U - u(Y)X\}$$

$$\text{(ii)} \quad \nabla_X U = -\alpha(FX).$$

If  $\alpha=1$ ,  $\alpha$ -Sasakian structure reduces to Sasakian.

We have obtained Ricci curvature tensor as<sup>3</sup>

$$Ric(Y, U) = -(n-2)\omega(Y) - \omega(U)u(Y) + (n-1)(\alpha^2 - \beta^2)u(Y) - v(\bar{Y}).$$

$$\text{where} \quad X\alpha = (d\alpha)(X) = v(X), \quad Y\alpha = (d\alpha)(Y) = v(Y),$$

$$X\beta = (d\beta)(X) = \omega(X), \quad Y\beta = (d\beta)(Y) = \omega(Y).$$

Let  $\alpha, \beta$  are constants, then  $\nu = 0$ ,  $\omega = 0$  and the Ricci tensor on trans-Sasakian manifold is given by

$$(1.10) \quad Ric(Y, U) = (n-1)(\alpha^2 - \beta^2)u(Y).$$

## 2. Ricci tensor of a Ricci-parallel trans-Sasakian manifold

Let the trans-Sasakian manifold is Ricci parallel i.e.  $\nabla_X Ric = 0$  for an arbitrary vector field  $X$ . Taking covariant derivative of (1.10), we get

$$(2.1) \quad Ric(\nabla_X Y, U) + Ric(Y, \nabla_X U) = (n-1)(\alpha^2 - \beta^2)\{(\nabla_X u)(Y) + u(\nabla_X Y)\}.$$

We know that

$$(2.2) \quad \nabla_X U = -\alpha\bar{X} + \beta\{X - u(X)U\}.$$

and

$$(2.3) \quad (\nabla_X u)(Y) = -\alpha g(Y, \bar{X}) + \beta g(X, Y) - \beta u(X)u(Y).$$

Replacing  $Y$  in (1.1) by  $\nabla_X Y$ , we have

$$(2.4) \quad Ric(\nabla_X Y, U) = (n-1)(\alpha^2 - \beta^2)u(\nabla_X Y).$$

In view of (2.2), (2.3) and (2.4), (2.1) yields

$$(2.5) \quad -\alpha Ric(\bar{X}, Y) + \beta Ric(X, Y) = (n-1)(\alpha^2 - \beta^2)\{-\alpha g(\bar{X}, Y) + \beta g(X, Y)\}.$$

Thus, we see that

$$(2.6) \quad Ric(X, Y) = (n-1)(\alpha^2 - \beta^2)g(X, Y)$$

is a solution of the equation (2.5).

Hence we have:

**Theorem 2.1:** *If  $\alpha, \beta$  are constants, then Ricci-parallel trans-Sasakian manifold is an Einstein Manifold.*

**Theorem 2.2:** *If  $\alpha, \beta$  are constants and equal, then Ricci-parallel trans-Sasakian manifold is Ricci flat.*

**Proof:** Since  $\alpha, \beta$  are constant and  $\alpha = \beta$  from (2.6) we have

$$\text{Ric}(X, Y) = 0$$

i.e. Sasakian manifold is Ricci flat.

### 3. Weyl Projective Curvature Tensors

If  $\alpha, \beta$  are constants, i.e.  $v=0$ ,  $\omega=0$ , then the associate curvature tensor  $'K(X, Y, Z, U)$  in trans-Sasakian manifold is given by

$$(3.1) \quad \begin{aligned} 'K(X, Y, Z, U) = & (\alpha^2 - \beta^2) \{ u(X)g(Y, Z) - u(Y)g(X, Z) \} \\ & + 2\alpha\beta \{ u(X)g(\bar{Y}, Z) - u(Y)g(\bar{X}, Z) \} \end{aligned}$$

Now we propose the following:

**Theorem 3.1:** *If  $\alpha, \beta$  are constants, the projective curvature tensor on Ricci-parallel trans-Sasakian manifold is given by*

$$(3.2) \quad 'W(X, Y, Z, U) = 2\alpha\beta \{ u(X)g(\bar{Y}, Z) - u(Y)g(\bar{X}, Z) \}$$

**Proof:** Projective curvature tensor is defined as

$$(3.3) \quad \begin{aligned} 'W(X, Y, Z, U) = & 'K(X, Y, Z, U) + \frac{1}{n-1} \{ g(Y, U) \text{Ric}(X, Z) \\ & - g(X, U) \text{Ric}(Y, Z) \} \end{aligned}$$

Using (3.1) and (2.6) in (3.3), we have (3.2).

**Theorem 3.2:** *If  $\alpha \neq 0$ ,  $\beta \neq 0$  and  $\alpha, \beta$  are constants, the Ricci-parallel trans-Sasakian manifold is not projectively flat.*

**Proof:** The projective curvature tensor in trans-Sasakian manifold is given by (3.2).

Let it be projectively flat, then  $'W = 0$ , which implies

$$u(X)g(\bar{Y}, Z) = u(Y)g(\bar{X}, Z).$$

Putting  $X = U$ , we get  $g(\bar{Y}, Z) = 0$ , which is a contradiction. Hence  $'W \neq 0$ .

**Theorem 3.3:** *If  $\alpha \neq 0$ ,  $\beta \neq 0$  and  $\alpha, \beta$  are constants, on Ricci-parallel trans-Sasakian manifold, we have*

$$(3.4) \text{ (i)} \quad 'W(\bar{X}, \bar{Y}, Z, U) = 0$$

$$(ii) \quad 'W(\bar{X}, Y, Z, U) + 'W(X, \bar{Y}, Z, U) + 'W(X, Y, \bar{Z}, U) = 0$$

**Proof:** Barring X, Y in (3.2) yields (3.4) and barring X in (3.2), we get

$$(3.5) \quad 'W(\bar{X}, Y, Z, U) = 2\alpha\beta\{u(Y)g(X, Z) - u(Y)u(X)u(Z)\},$$

Barring Y in (3.2) yields

$$(3.6) \quad 'W(X, \bar{Y}, Z, U) = -2\alpha\beta\{u(X)g(Y, Z) - u(X)u(Y)u(Z)\},$$

and barring Z in (3.2) yields

$$(3.7) \quad 'W(X, Y, \bar{Z}, U) = 2\alpha\beta\{u(X)g(Y, Z) - u(Y)g(X, Z)\}.$$

Adding (3.5), (3.6) and (3.7), we get (3.4ii).

#### 4. Conformal Curvature Tensor

**Theorem 4.1:** The conformal curvature tensor on a Ricci-parallel trans-Sasakian manifold is given by

$$(4.1) \quad 'V(X, Y, Z, U) = \frac{\{r - n(n-1)(\alpha^2 - \beta^2)\}}{(n-1)(n-2)} \{u(X)g(Y, Z) - u(Y)g(X, Z)\} \\ + 2\alpha\beta\{u(X)g(\bar{Y}, Z) - u(Y)g(\bar{X}, Z)\}.$$

**Proof:** Conformal curvature tensor is given by

$$(4.2) \quad 'V(X, Y, Z, U) = 'K(X, Y, Z, U) - \frac{1}{(n-2)} \{u(X)Ric(Y, Z) - u(Y)Ric(X, Z) \\ - g(X, Z)Ric(Y, U) + g(Y, Z)Ric(X, U)\} \\ + \frac{r}{(n-1)(n-2)} \{u(X)g(Y, Z) - u(Y)g(X, Z)\}.$$

Using (2.6) and (3.1) in (4.2), we get

$$'V(X, Y, Z, U) = \left[ (\alpha^2 - \beta^2) - \frac{2(n-1)(\alpha^2 - \beta^2)}{(n-2)} + \frac{r}{(n-1)(n-2)} \right]$$

$$\{u(X)g(Y, Z) - u(Y)g(X, Z)\} + 2\alpha\beta\{u(X)g(\bar{Y}, Z) - u(Y)g(\bar{X}, Z)\}$$

which yields

$$'V(X, Y, Z, U) = \left[ \frac{r - n(n-1)(\alpha^2 - \beta^2)}{(n-1)(n-2)} \right] \{u(X)g(Y, Z) - u(Y)g(X, Z)\} \\ + 2\alpha\beta\{u(X)g(\bar{Y}, Z) - u(Y)g(\bar{X}, Z)\}$$

**Theorem 4.2:** On Ricci-parallel trans-Sasakian manifold, we have

$$(4.3) \text{ (i)} \quad 'V(\bar{X}, \bar{Y}, Z, U) = 0$$

$$(ii) \quad 'V(\bar{X}, Y, Z, U) + 'V(X, \bar{Y}, Z, U) + 'V(X, Y, \bar{Z}, U) = 0$$

**Proof:** From (4.1), we get (4.3i) and (4.3ii).

**Theorem 4.3:** Ricci-parallel trans-Sasakian manifold is not conformally flat.

**Theorem 4.4:** Concircular curvature tensor on a Ricci-parallel trans-Sasakian manifold is given by

$$(4.4) \quad 'C(X, Y, Z, U) = \left[ (\alpha^2 - \beta^2) - \frac{r}{n(n-1)} \right] \{u(X)g(Y, Z) - u(Y)g(X, Z)\} \\ + 2\alpha\beta\{u(X)g(\bar{Y}, Z) - u(Y)g(\bar{X}, Z)\},$$

where  $\alpha, \beta$  are constants.

**Proof:** Concircular curvature tensor is given by

$$(4.5) \quad 'C(X, Y, Z, U) = 'K(X, Y, Z, U) - \frac{r}{n(n-1)} \{u(X)g(Y, Z) - u(Y)g(X, Z)\}.$$

Using (3.1) in (4.5), we get (4.4).

**Theorem 4.5:** *On a Ricci-parallel trans-Sasakian manifold, we have*

$$(4.6) \quad (i) \quad 'C(\bar{X}, \bar{Y}, Z, U) = 0,$$

$$(ii) \quad 'C(\bar{X}, Y, Z, U) + 'C(X, \bar{Y}, Z, U) + 'C(X, Y, \bar{Z}, U) = 0.$$

**Theorem 4.6:** *On a Ricci-parallel trans-Sasakian manifold,  $r = n(n-1)(\alpha^2 - \beta^2)$  implies*

$$(4.7) \quad 'C(X, Y, Z, U) = 'V(X, Y, Z, U) = 'W(X, Y, Z, U)$$

where  $\alpha, \beta$  are constants.

**Theorem 4.7:** *Conhormonic curvature tensor on Ricci-parallel trans-Sasakian manifold is given by*

$$(4.8) \quad 'L(X, Y, Z, U) = -\frac{n(\alpha^2 - \beta^2)}{(n-2)} \{u(X)g(Y, Z) - u(Y)g(X, Z)\} \\ + 2\alpha\beta \{u(X)g(\bar{Y}, Z) - u(Y)g(\bar{X}, Z)\},$$

where  $\alpha, \beta$  are constants.

**Proof:** Conhormonic curvature tensor in a Riemannian manifold is given by

$$(4.9) \quad 'L(X, Y, Z, U) = 'K(X, Y, Z, U) - \frac{1}{(n-2)} \{u(X)Ric(Y, Z) - u(Y)Ric(X, Z)\} \\ - g(X, Z)Ric(Y, U) + g(Y, Z)Ric(X, U)\}.$$

Using (2.6) and (3.1) in (4.9), we get (4.8).

**Theorem 4.8:** *On Ricci-parallel trans-Sasakian manifold, we have*

$$(4.10) \quad (i) \quad 'L(\bar{X}, \bar{Y}, Z, U) = 0,$$

$$(ii) \quad 'L(\bar{X}, Y, Z, U) + 'L(X, \bar{Y}, Z, U) + 'L(X, Y, \bar{Z}, U) = 0$$

**Proof:** Barring  $X$  and  $Y$  in (4.8), we get (4.10i).

Barring  $X$  in (4.8), we get

$$(4.11) \quad 'L(\bar{X}, Y, Z, U) = \frac{n(\alpha^2 - \beta^2)}{(n-2)} u(Y)g(\bar{X}, Y) - 2\alpha\beta \{u(X)u(Y)u(Z) - u(Y)g(X, Z)\}$$

Barring  $Y$  in (4.8), we get

$$(4.12) \quad 'l(X, \bar{Y}, Z, U) = -\frac{n(\alpha^2 - \beta^2)}{(n-2)} u(X)g(\bar{Y}, Z) + 2\alpha\beta\{u(X)u(Y)u(Z) - u(X)g(Y, Z)\}.$$

Barring  $Z$  in (4.8), we get

$$(4.13) \quad 'L(X, Y, \bar{Z}, U) = -\frac{n(\alpha^2 - \beta^2)}{(n-2)} \{u(X)g(Y, \bar{Z}) - u(Y)g(X, \bar{Z})\} + 2\alpha\beta\{u(X)g(Y, Z) - u(Y)g(X, Z)\}.$$

Adding (4.11), (4.12) and (4.13), we get (4.10ii).

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