

Fundamental Function of two Dimensional Finsler Space whose Geodesics Constitute with Family of Semi-ellipses and Family Pair of Straight Lines

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Abstract: To find the Finsler spaces having a given family of curves as geodesics is an interesting problem for differential geometers. This is an important problem from the standpoint of application also¹. In his paper M. Mastsumoto² treated semi-circle, parabola and hyperbola respectively as the given family of curves and find fundamental function $L(x,y)$ in each cases. The purpose of the present paper is to find the fundamental function $L(x,y)$ in two-dimensional Finsler space whose geodesics are the family of semi-ellipses and family of pair of straight lines.

1. Preliminaries

Let us consider an n -dimensional Finsler space $F^n = (M^n, L(x,y))$ on an underlying manifold M^n with the fundamental function $L(x,y)$. The fundamental tensor g_{ij} , the angular metric tensor h_{ij} and the normalized supporting element $l_i(x,y)$ are defined by

$$g_{ij} = h_{ij} + l_i l_j, \quad h_{ij} = L \frac{\partial^2 L}{\partial y^i \partial y^j}, \quad l_i = \frac{\partial L}{\partial y^i},$$

respectively. The extremal of the length integral, $s = \int_{t_0}^t L(x,y) dt$, where $y^i = \frac{dx^i}{dt}$, along an oriented curve $C: x^i = x^i(t)$ from point $P = x^i(t_0)$ to point $Q = x^i(t)$, $t > 0$ is given by Euler's equation

$$(1.1) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial y^i} \right) - L_i = 0$$

where $L_i = \frac{\partial L}{\partial x^i}$. In terms of $F(x,y) = L^2(x,y)/2$, (1.1) is written in the form

$$(1.2) \quad \frac{d^2 x^i}{dt^2} + 2G^i(x,y) = h(t) \frac{dx^i}{dt}$$

where we put

$$(1.3) \quad 2g_{ij}G^i(x,y) = y^r \frac{\partial^2 F}{\partial x^r \partial y^i} - \frac{\partial F}{\partial x^j} \text{ and } h(t) = \frac{d^2 s / dt^2}{ds / dt}$$

Consider two dimensional Finsler space and use the notation (x,y) and (p,q) respectively, instead of (x^1, x^2) and (y^1, y^2) . The fundamental function $L(x,y;p,q)$ L_x and L_y as well are positively homogeneous of degree one in p and q .

Therefore, we have

$$(1.3 a) \quad L_x = pL_{xp} + qL_{xq}, \quad L_y = pL_{yp} + qL_{yq}$$

Also from homogeneity of L_p and L_q we have

$$(1.3 b) \quad \frac{L_{pp}}{q^2} = \frac{-L_{pq}}{pq} = \frac{L_{qq}}{p^2} = W(say),$$

where W is called Weirstrass invariant. The two equations of (1.1) are given by

$$\frac{d}{dt} L_p - L_x = 0, \quad \frac{d}{dt} L_q - L_y = 0.$$

using (1.3a) and (1.3b), these two equations reduced to

$$(1.4) \quad L_{xq} - L_{yp} + (p\dot{q} - q\dot{p})W = 0$$

which is called Weirstrass form of geodesic equation.

Now consider the associate fundamental function $A(x,y,z)$, $z = y' = dy/dx$ defined as follows

$$(1.5) \quad A(x,y,z) = L(x,y;1,z), \quad L(x,y;p,q) = A(x,y, \frac{q}{p})p.$$

$$\text{Therefore } L_p = A - zA_z, \quad L_p = A_z, \quad L_{yp} = A_y - zA_{yz}, \quad L_{xq} = A_{xz},$$

$$L_{pp} = (z^2 / p)A_{zz}, \quad L_{pq} = (-z / p)A_{zz}, \quad L_{qq} = (1 / p)A_{zz}.$$

On using these values in (1.4), we have

$$(1.6) \quad A_{zz}Y'' + A_{yz}y' + A_{xz} - A_y = 0, z = y'$$

which is called the Rashevsky form of geodesic equation, but $y' = (q/p)$ gives

$$y'' = \frac{p\dot{q} - q\dot{p}}{p^3}.$$

Hence from (1.2) we have another form of geodesic

$$(1.7) \quad y'' = (2/p^3)(qG^1 - pG^2)$$

Next consider the function $G^i(x, y)$ in detail. In F^n we have Berwald connection $B\Gamma = (G_{jk}^i, G_j^i)$ defined by $G_j^i = \partial G^i / \partial y^j$ and $G_{jk}^i = \partial G_j^i / \partial y^k$ and two kinds of covariant differentiation of a Finslerian vector field $V^i(x, y)$ given by

$$V_{(j)}^i = \partial V^i / \partial x^j - (\partial V^i / \partial y^r) G_j^r + V^r G_{rj}^i, \quad V_{\cdot j}^i = \partial V^i / \partial y^j,$$

called the h- and v-covariant derivatives of V^i respectively. Since $B\Gamma$ is L-metrical i.e. $L_{(i)} = 0$, we have

$$(1.8) \quad L_i = l_r G_i^r.$$

Furthermore from (1.8) we have

$$(1.9) \quad L_{i,j} = (1/L) h_{ij} G_i^r + l_r G_{ij}^r.$$

Next, from (1.3) we have $2g_{ij} G^i y^j = (\partial F / \partial x^r) y^r$, that is

$$(1.10) \quad 2G^i l_i = L_r y^r.$$

Now we shall return to two-dimensional case. Since the matrix (h_{ij}) is of rank one, we get $\epsilon = \pm 1$ and the vector field $m_i(x, y)$ satisfies

$$(1.11) \quad h_{ij} = \epsilon m_i m_j,$$

so we have

$$(1.12) \quad g_{ij} = l_i l_j + \epsilon m_i m_j$$

Therefore, we get easily

$$(1.13) \quad l^i l_i = 1, \quad m^i m_i = \epsilon, \quad l^i m_i = 0.$$

therefore we obtain the orthonormal frame field (l^i, m^i) , called the Berwald frame.

From (1.13) we have

$$\frac{-l_1}{l_2} = \frac{m^2}{m^1}, \quad \frac{-l^1}{l^2} = \frac{m_2}{m_1}.$$

Therefore, we have scalar field $h(x,y;p,q)$ and $k(x,y;p,q)$ such that

$$(1.14) \quad \begin{cases} (m_1, m_2) = h(-l^2, l^1), & (m^1, m^2) = k(-l_2, l_1), \\ (hl^1)(kl_1) + (-hl^2)(-kl_2) = m_1 m^1 + m_2 m^2 & \text{and } hk = \epsilon. \end{cases}$$

Then (1.12) and (1.14) gives,

$$(1.15) \quad g = \det.(g_{ij}) = g_{11}g_{22} - g_{12}^2 = \epsilon(l_1 m_2 - l_2 m_1)^2$$

$$(1.16) \quad g = \epsilon h^2 \text{ and } (1/h)(m^1, m^2) = (hk/h^2)(-l_2, l_1) = (1/g)(-l_2, l_1).$$

Next consider W . We have $h_{11} = LL_{pp} = LWq^2$ and $h_{11} = \epsilon(m_1)^2 = \epsilon h^2(l^2)^2$

$$= \epsilon h^2(q/L)^2,$$

So, we have

$$(1.17) \quad L^3 W = \epsilon h^2 = g$$

Now consider the expression G^i in Berwald frame⁴. From (1.9)

$$\begin{aligned} L_{xq} - L_{yp} &= (1/L)(h_{rj}G_i^r - h_{ri}G_j^r) \\ &= \frac{\epsilon m_r}{L}(m_j G_i^r - m_i G_j^r) \\ &= \frac{\epsilon m_r}{L}(m_2 G_1^r - m_1 G_2^r) \\ &= \frac{\epsilon m_r}{L}h(l^1 G_1^r + l^2 G_2^r) \\ &= \frac{\epsilon m_r}{L^2}h(G_i^r y^i) \end{aligned}$$

From homogeneity of G^r , we have $G_i^r y^i = 2G^r$, therefore $2G^i m_i = (L_{xq} - L_{yp})L^2 / \epsilon h$.

This equation and (1.10) give

$$(1.18) \quad \begin{aligned} 2G^i(l_i l^j + \epsilon m_i m^j) &= (L_r y^r)l^j + m^j(L_{xq} - L_{yp})L^2/h \\ 2G^i &= (L_r y^r)l^i + (L^2/h)(L_{xq} - L_{yp})m^i. \end{aligned}$$

If we put

$$(1.19) \quad L_0 = L_r y^r, \quad M = L_{xq} - L_{yp},$$

then it follows from (1.16) and (1.17) that (1.18) is written in the more convenient form

$$(1.20) \quad \begin{cases} 2G^1 = \frac{1}{L} (L_0 p - \frac{M}{W} L_q), \\ 2G^2 = \frac{1}{L} (L_0 q - \frac{M}{W} L_p). \end{cases}$$

2. From geodesics to the Finsler metrics

Consider the family of curves $\{C(a,b)\}$ on the (x,y) -plane R^2 , given by the equation

$$(2.1) \quad y = f(x; a, b).$$

with the parametrs (a,b) . From (2.1) we have

$$(2.2) \quad z (= y') = f_x(x; a, b).$$

From (2.1) and (2.2) we have,

$$(2.3) \quad a = \alpha(x, y, z), \quad b = \beta(x, y, z).$$

Next, $y'' = f_{xx}(x; a, b)$ and (2.3) gives the function $u(x, y, z)$

$$(2.4) \quad z' = f_{xx}(x; \alpha, \beta) = u(x, y, z),$$

which is second order differential equation of y characterizing $\{C(a,b)\}$,

Now from Rashevsky form (1.6) of geodesic

$$L_{qq} = (1/p) A_{zz} \text{ and } A_{zz} = Wp^3$$

so from (1.5) and (1.17) we have

$$(2.5) \quad g = L^3 W = A^3 A_{zz}.$$

Thus it is suitable to call A_{zz} the associated Weirstrass invariant. If we put $B = A_{zz}$ then differentiation of (1.6) with respect to z gives

$$(2.6) \quad B_x + B_y z + B_z u + B u_z = 0,$$

which is first order quasilinear partial differential equation. Its auxiliary equation is given by

$$(2.7) \quad \frac{dx}{I} = \frac{dy}{z} = \frac{dz}{u} = \frac{dB}{-Bu_z}.$$

Consider $\frac{dx}{I} = \frac{dB}{-Bu_z}$ which gives $\frac{B}{H(\alpha, \beta)} = e^{-\int u_z dx}$ or $B = \frac{H(\alpha, \beta)}{e^{\int u_z dx}}$. therefore

$$(2.8) \quad B = \frac{H(\alpha, \beta)}{V(x, y, z)},$$

where H is an arbitrary non-zero function of line arguments. From $A_{zz} = B$ we get A in the form

$$(2.9) \quad A(x, y, z) = A^*(x, y, z) + C(x, y) + zD(x, y),$$

where $A^* = \int \int B(x, y, z) dz \} dz^*$, $A^* = \int_{z_0}^z (z-t)B(x, y, t)dt$, C and D are arbitrary but must be chosen

so that A may satisfy (1.6) that is

$$(2.10) \quad C_y - D_x = A^*_{zz} u + zA^*_{yz} + A^*_{xz} - A^*_y.$$

If (C_0, D_0) be chosen such that they satisfy (2.10) then $(C-C_0)_y = (D-D_0)_x$. So that we have locally function $E(x, y)$ satisfying $E_x = C-C_0$ and $E_y = D-D_0$. Thus (2.9) is written as

$$(2.11) \quad A = A^* + C_0 + D_0 z + E_x + E_y z.$$

Therefore (1.5) leads to the fundamental function

$$(2.12) \quad \begin{aligned} L(x, y; p, q) &= L_0(x, y; p, q) + e(x, y; p, q) \\ L_0 &= A^*(x, y, q/p)p + C_0(x, y)p + D_0(x, y)q \\ e(x, y; dx, dy) &= E_x dx + E_y dy \end{aligned}$$

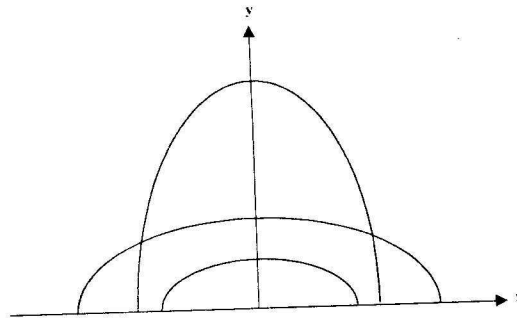
Here e is derived form.

3. Family of semi-ellipses

Consider the family of semi-ellipses $\{C(a, b)\}$ given by the equation

$$(3.1) \quad x^2 + ay^2 = b, \quad y, b, a > 0.$$

on the semi-plane R^2 having center (0,0) on x-axis, and find the Finsler metrics in this section



From (3.1), we have

$$(3.2) \quad x + ayz = 0, \quad z = y'.$$

Thus, equations (3.1) and (3.2) gives, the functions $\alpha(x, y, z)$ and $\beta(x, y, z)$:

$$(3.3) \quad a = -\frac{x}{yz} = \alpha(x, y, z) \quad b = x^2 - \frac{xy}{z} = \beta(x, y, z).$$

Therefore the function $u(x, y, z)$, as in the last section is given by,

$$(3.4) \quad z' (= y'') = \frac{z}{x} - \frac{z^2}{y} = u(x, y, z)$$

or

$$(3.4)' \quad y'' - \frac{y'}{x} + \frac{(y')^2}{y} = 0$$

which is second order differential equation in y , characterizing $\{C(a, b)\}$.

Next, we find the function $U(x; a, b)$ and $V(x, y, z)$ defined by (2.7);

$$u_z = \frac{1}{x} - \frac{2z}{y}, \quad u_x = \frac{1}{x} + \frac{2x}{b - x^2}$$

$$\text{Therefore } U(x; a, b) = \exp \int \left(\frac{1}{x} + \frac{2x}{b - x^2} \right) dx = \frac{x}{b - x^2}.$$

and

$$V(x, y, z) = -\frac{z}{y}$$

Consequently we have from (2.8) and (2.9),

$$(3.5) \quad B(x, y, z) = -\frac{y}{z} H(\alpha, \beta) = \frac{y}{z} H(\alpha, \beta)$$

$$\text{and} \quad A^*(x, y, z) = \int_1^z (z-t) \frac{y}{t} H(\alpha, \beta) dt$$

or

$$(3.6) \quad A^*(x, y, z) = y \int_1^z \frac{z-t}{t} F(t) dt$$

$$\text{where} \quad F(t) = H\left(-\frac{x}{yt}, x^2 - \frac{xy}{t}\right).$$

Here in (3.5), on account of arbitrariness of H , we may observe negative sign in H . and

it is noted that $\int_1^z$ applied in (3.6), instead of $\int_0^z$, because t is in the denominator of $F(t)$.

The associate fundamental function $A(x, y, z)$ is written as,

$$(3.7) \quad A(x, y, z) = A^*(x, y, z) + C(x, y) + zD(x, y)$$

$$\text{Now if we put } F_1(t) = H_\alpha\left(-\frac{x}{yt}, x^2 - \frac{xy}{t}\right), \quad F_2(t) = H_\beta\left(-\frac{x}{yt}, x^2 - \frac{xy}{t}\right),$$

then from (3.6) we have

$$A^*_y = \int_1^z \frac{z-t}{t} F(t) dt + \frac{x}{y} \int_1^z \frac{z-t}{t^2} F_1(t) dt - xy \int_1^z \frac{z-t}{t^2} F_2(t) dt,$$

$$A^*_z = \int_1^z \frac{F(t)}{t} dt,$$

$$A^*_{yz} = \int_1^z \frac{F(t)}{t} dt + \frac{x}{y} \int_1^z \frac{F_1(t)}{t^2} dt - xy \int_1^z \frac{F_2(t)}{t^2} dt,$$

$$A^*_{xz} = y \int_1^z \left\{ -\frac{1}{yt} F_1(t) + \left(2x - \frac{y}{t}\right) F_2(t) \right\} dt,$$

$$A^*_{zz} = B = \frac{y}{z} H(\alpha, \beta).$$

Thus (2.10) implies

$$(3.8) \quad C_y - D_x = \left(\frac{y}{x} - z\right) H(\alpha, \beta) + \int_1^z F(t) dt - \int_1^z \frac{F_1(t)}{t^2} dt \\ + \frac{x}{y} \int_1^z \frac{F_1(t)}{t} dt + xy \int_1^z \frac{F_2(t)}{t} dt - y^2 \int_1^z \frac{F_2(t)}{t^2} dt.$$

Therefore we obtain,

Theorem 1. Every associate fundamental function $A(x, y, z)$ of a Finsler space $(R^2, L(x, y; p, q))$ having the semi-ellipse (3.1) as the geodesics is given by,

$$A = A^*(x, y, z) + C(x, y) + zD(x, y)$$

where A^* is defined by (3.6), H is an arbitrary function of (α, β) given by (3.3) and function (C, D) must be chosen so as to satisfy (3.8).

Example 1. In particular, we first put $H(\alpha, \beta) = \alpha = -\frac{x}{yz}$ in the above result.

Then (3.6) and (3.8) give

$$A^*(x, y, z) = y \int_1^z \frac{z-t}{t} \left(-\frac{x}{yt}\right) dt = x(1-z + \log z).$$

$$C_y - D_x = \frac{x}{y} - 1$$

Thus choosing $C = 0$ and $D = x - \frac{x^2}{2y}$, from (3.7), we have

$$A(x, y, z) = x(1 + \log z - \frac{xz}{2y}).$$

$$A(x, y, z) = x(1 + \log z - \frac{xz}{2y}).$$

Therefore it follows from (1.5) that, the fundamental function $L(x,y;p,q)$ is given by

$$(3.9) \quad L(x, y; p, q) = x(p + p \log(q/p) - \frac{xq}{2y}).$$

Therefore $L_{yp}=0$, $L_{xq} = (p/q) - (x/y)$, $L_{qq} = -(px)/q^2$. From (1.3) (b) we have $W = -(x/(pq^2))$. So using these values in (1.4), where we use $(pq - qp) = y'' p^3$ and $q/p = z = y'$. Then it leads to (3.4)' immediately.

Example 2. If we take $H(\alpha, \beta) = \beta = x^2 - xy/z$ then, we have from (3.6),

$$A^*(x, y, z) = y \int_1^z \frac{z-t}{t} (x^2 - \frac{xy}{t}) dt$$

$$\text{or} \quad A^*(x, y, z) = xy \{ (xz + y) \log z + (x + y)(1 - z) \}$$

and from (3.8),

$$C_y - D_x = 2xy - x - y^2.$$

If we chose $C = 0$ and $D = \frac{x^2}{2} + xy^2 - x^2 y$, from (3.7), we have

$$A(x, y, z) = xy \{ (xz + y) \log z + (x + y)(1 - z) \} + xz \left(\frac{x}{2} + y^2 - xy \right)$$

$$\text{or} \quad A(x, y, z) = xy \{ (xz + y) \log z + 2zy + (x + y)p \} + \frac{qx^2}{2}$$

Therefore, it follows from (1.5) that the fundamental function $L(x,y;p,q)$ is written as

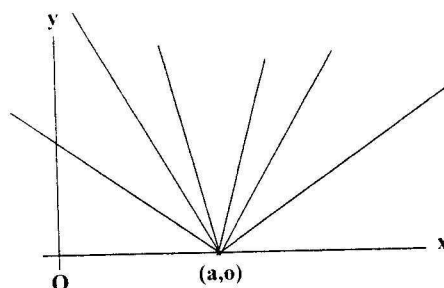
$$L(x, y; p, q) = xy \{ (xq + yp) \log(q/p) + 2qy + (x + y)p \} + \frac{qx^2}{2}.$$

4. Family of pair of straight lines

Consider the family of straight lines $\{C(a,b)\}$ given by the equation

$$(4.1) \quad (x - a)^2 - by^2 = 0, \quad b, y > 0,$$

on the plane R_+^2 , passes through $(a, 0)$ on x-axis.



From (4.1), we have

$$(4.2) \quad (x-a) - byz = 0, \quad z = y'.$$

From (4.1) and (4.2) the function $\alpha(x, y, z)$ and $\beta(x, y, z)$ in preceding section are given by

$$(4.3) \quad a = x - \frac{y}{z} = \alpha(x, y, z), \quad b = 1/z^2 = \beta(x, y, z)$$

Next, from (4.2),

$$(4.4) \quad z' (= y'') = \frac{1}{by} - \frac{z^2}{y} = 0 = u(x, y, z)$$

which is a differential equation

$$(4.5) \quad y'' = 0$$

characterizing $\{C(a, b)\}$. Therefore, the function $U(x; a, b)$ and $V(x, y, z)$ defined by (2.7) are given by,

$$u_z = 0, \quad U = \exp \int u_z dx = 1, \quad V(x, y, z) = 1.$$

Consequently we have from (2.8),

$$(4.6) \quad B(x, y, z) = H(\alpha, \beta), \quad A^*(x, y, z) = \int \left\{ \int H(\alpha, \beta) dz \right\} dz$$

The associated fundamental function $A(x, y, z)$ is given by

$$(4.7) \quad A(x, y, z) = A^*(x, y, z) + C(x, y) + zD(x, y)$$

where C and D must be chosen so that they satisfy (2.10). Now

$$(4.8) \quad A^* = \int_1^z (z-t) F(t) dt$$

where $F(t) = H(x - \frac{y}{t}, \frac{1}{t^2})$. Further, if we put

$$F_1(t) = H_\alpha(x - \frac{y}{t}, \frac{1}{t^2}), \quad F_2(t) = H_\beta(x - \frac{y}{t}, \frac{1}{t^2}),$$

Then we have from (4.8)

$$A^*_{xy} = - \int_1^z \frac{z-t}{t} F_1(t) dt, \quad A^*_z = \int_1^z F(t) dt, \quad A^*_{yz} = - \int_1^z \frac{F_1(t)}{t} dt,$$

$$A^*_{xz} = \int_1^z F_1(t) dt \quad \text{and} \quad A^*_{zz} = B = H$$

Thus (2.10) implies

$$(4.9) \quad C_y - D_x = 0$$

Therefore from (4.9), we have a locally function $J(x,y)$ such that,

$$(4.9)' \quad C = J_x, \quad D = J_y,$$

Therefore from (4.6), we have

$$(4.10) \quad A(x, y, z) = A^*(x, y, z) + J_x + zJ_y.$$

Therefore, we have the following:

Theorem 2. Every associate fundamental function $A(x,y,z)$ of Finsler space $\{R^2, L(x,y;p,q)\}$ having a family of pair of straight lines (4.1), as the geodesics is given by (4.10), where A^* defined by (4.8), H is an arbitrary function of (α, β) given by (4.3) and function $J(x,y)$ satisfy (4.9)'.

Example 3. If we take $H = \alpha = x - (y/z)$ then from (4.8)

$$A^* = \int_1^z (z-t)(x - \frac{y}{t}) dt \quad \text{or} \quad A^* = (xz + y)(z-1) - (x/2)(z^2 - 1) - zy \log z$$

and from (4.9)

$$C_y - D_x = 0$$

If we choosen $J(x,y) = xy - y^2/2$, from (4.10) we have

$$A = (x/2)(z^2 + 1) - zy \log z.$$

Therefore from (1.5) it follows that the fundamental function $L(x,y;p,q)$ is written as

$$(4.11) \quad L(x, y; p, q) = \frac{x}{2} \left(\frac{q^2}{p^2} + 1 \right) p - qy \log \left(\frac{q}{p} \right).$$

Example 4. If we take $H = \beta = 1/z^2$ then from (4.8), we have

$$A^*(x, y, z) = (z-1) - \log z.$$

Choosing $J(x,y) = 0$, from (4.10), we have $A(x,y,z) = (z-1) - \log z$. So from (1.5), it follows that fundamental function is given by

$$(4.12) \quad L(x,y;p,q) = (q-p) - p \log(q/p).$$

From (4.12) it is clear that, the fundamental function $L(x,y;p,q)$ is independent of positional coordinates (x,y) of the space. Thus we have the following:

Theorem 3. A two-dimensional Finsler space $\{R^2, L(x,y;p,q)\}$ whose geodesics are family of pair of straight lines is a locally Minkowskian space provided arbitrary function $H(\alpha,\beta)$ is independent from α .

Further from (4.12)

$$L_{xq} = 0, L_{yp} = 0, L_{qq} = p/q^2.$$

$$\text{so} \quad W = L_{qq}/p^2 = 1/pq^2, \quad M = L_{xq} - L_{yp} = 0, \quad L_0 = L_r y^r = 0.$$

If we use these values in (1.4), it leads to (4.5) immediately. Further, from (1.20), $G^i = 0, i=1, 2$.

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