

The Inverse Problem of Variation Calculus In Three Dimensional Finsler Space

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(Received October 13, 2005)

1. Introduction

To find the Finsler spaces having a given family of curves as the geodesics is an interesting problem for geometers and will be an important problem from the standpoint of application¹. In the papers²⁻⁴ M. Matsumoto gave the complete solutions of this problem in the two-dimensional case.

The purpose of the present paper is to give a geometrical development of the same problem in three-dimensional Finsler space.

2. Differential Equations of Extremals

It is well known that every set of functions $\{x = \phi(t), y = \psi(t), z = \eta(t)\}$ which minimizes or maximizes a definite integral.

$$(2.1) \quad J = \int_{t_1}^{t_2} L(x, y, z, p, q, r) dt; \quad p = dx/dt, q = dy/dt, r = dz/dt$$

must satisfy the differential equation⁵

$$(2.2) \quad (a) \quad L_x - \frac{dL_p}{dt} = 0 = L_x - pL_{xp} - qL_{yp} - rL_{zp} - \dot{p}L_{pp} - \dot{q}L_{pq} - \dot{r}L_{pr},$$

$$(b) \quad L_y - \frac{dL_q}{dt} = 0 = L_y - pL_{xq} - qL_{yq} - rL_{zq} - \dot{p}L_{pq} - \dot{q}L_{qq} - \dot{r}L_{qr},$$

$$(c) \quad L_z - \frac{dL_r}{dt} = 0 = L_z - pL_{xr} - qL_{yr} - rL_{zr} - \dot{p}L_{pr} - \dot{q}L_{qr} - \dot{r}L_{rr},$$

where $L_x = \partial L / \partial x, L_y = \partial L / \partial y, L_z = \partial L / \partial z, L_p = \partial L / \partial p, L_q = \partial L / \partial q, L_r = \partial L / \partial r,$

$L_{xp} = \partial^2 L / \partial x \partial p$ and so on.

The equations (2.2) are referred as Euler's equations and the curve $\{x = \phi(t), y = \psi(t), z = \eta(t)\}$ in R^3 is called an extremal for the integral (1.1). The geodesics of (R^3, L) is nothing but the extremal of the arc length.

$$s = \int_{t_1}^{t_2} L(x, y, z, p, q, r) dt$$

along a curve $x = \phi(t), y = \psi(t), z = \eta(t)$. Thus the differential equations of geodesics of the three-dimensional Finsler space $F^3 = (R^3, L)$ are given by equations (2.2).

Since the fundamental function $L(x, y, z, p, q, r)$ of the F^3 is p -homogeneous⁶ (p. 83), we have from Euler's theorem on homogeneous function:

$$pL_p + qL_q + rL_r = L,$$

which after differentiation with respect to p, q and r , respectively gives

$$(2.3) \quad (a) \quad pL_{pp} + qL_{pq} + rL_{pr} = 0,$$

$$(b) \quad pL_{pq} + qL_{qq} + rL_{qr} = 0,$$

$$(c) \quad pL_{pr} + qL_{qr} + rL_{rr} = 0.$$

Thus if we put

$$(2.4) \quad U = \frac{pL_{pp}}{qr}, V = \frac{qL_{qq}}{pr}, W = \frac{rL_{rr}}{pq},$$

Then we have

$$(2.5) \quad (a) \quad L_{pq} = (r/2)(-U - V + W)$$

$$(b) \quad L_{pr} = (q/2)(-U + V - W)$$

$$(c) \quad L_{qr} = (p/2)(-U - V - W).$$

The functions $U(x, y, z; p, q, r)$, $V(x, y, z; p, q, r)$ and $W(x, y, z; p, q, r)$ are (-2) p -homogeneous in p, q, r .

Since L_x, L_y, L_z are (-1) p -homogeneous in p, q, r , using (2.4) and (2.5), the three equations

of (2.2) may be written as

$$\begin{aligned}
 (2.6) \quad (a) \quad & \frac{L_{xq} - L_{yp}}{r} + \frac{L_{xr} - L_{zp}}{q} - \frac{U}{2} \left(\frac{2\dot{p}}{p} - \frac{\dot{q}}{q} - \frac{\dot{r}}{r} \right) + \frac{1}{2} (V - W) \left(\frac{\dot{q}}{r} - \frac{\dot{r}}{q} \right) = 0, \\
 (b) \quad & \frac{L_{yp} - L_{xq}}{r} + \frac{L_{yr} - L_{zq}}{p} - \frac{V}{2} \left(\frac{2\dot{q}}{q} - \frac{\dot{r}}{r} - \frac{\dot{p}}{p} \right) + \frac{1}{2} (W - U) \left(\frac{\dot{r}}{r} - \frac{\dot{p}}{p} \right) = 0, \\
 (c) \quad & \frac{L_{zp} - L_{xr}}{q} + \frac{L_{zq} - L_{yr}}{p} - \frac{W}{2} \left(\frac{2\dot{r}}{r} - \frac{\dot{p}}{p} - \frac{\dot{q}}{q} \right) + \frac{1}{2} (U - W) \left(\frac{\dot{p}}{p} - \frac{\dot{q}}{q} \right) = 0.
 \end{aligned}$$

It should be remarked here that the three equations given in (2.6) are not independent because (a) + (b) + (c) = 0. Thus we have only two independent differential equations of geodesics. Since the differential equations are of second orders, we have four arbitrary constants in the general solution of these differential equations.

3. The Inverse Problem

We shall deal with the inverse problem in the parametric form. This is just regarded as to find the fundamental function of three-dimensional Finsler space whose geodesic is four-parameter family of curves given by

$$(3.1) \quad x = \phi(t, a, b, c, d), \quad y = \psi(t, a, b, c, d), \quad z = \eta(t, a, b, c, d),$$

and we have

$$p = \dot{\phi}(t, a, b, c, d), \quad q = \dot{\psi}(t, a, b, c, d), \quad r = \dot{\eta}(t, a, b, c, d),$$

where the dot stands for $\partial/\partial t$.

Since we are concerned with a problem of Finsler geometry, the homogeneity of functions in p, q, r is important. In order to get the homogeneity of functions which will appear later on, we take an auxiliary parameter λ such that

$$x = \phi(\lambda t, a, b, c, d), \quad y = \psi(\lambda t, a, b, c, d), \quad z = \eta(\lambda t, a, b, c, d).$$

It is remarked that λ should be restricted to be positive, because the orientation of curve is essential in Finsler geometry.

First of all solve a, b, c, d and λt from

$$\begin{aligned}
 (3.2) \quad & x = \phi(\lambda t, a, b, c, d), \quad y = \psi(\lambda t, a, b, c, d), \quad z = \eta(\lambda t, a, b, c, d), \\
 & p/\lambda = \dot{\phi}(\lambda t, a, b, c, d), \quad q/\lambda = \dot{\psi}(\lambda t, a, b, c, d).
 \end{aligned}$$

The solution of above five equations may be written as

$$\begin{aligned}
 (3.3) \quad a &= \alpha^*(x, y, z, p/\lambda, q/\lambda), & b &= \beta^*(x, y, z, p/\lambda, q/\lambda) \\
 c &= \gamma^*(x, y, z, p/\lambda, q/\lambda), & d &= \delta^*(x, y, z, p/\lambda, q/\lambda) \\
 \lambda t &= \tau^*(x, y, z, p/\lambda, q/\lambda).
 \end{aligned}$$

Then we have

$$(3.4) \quad r/\lambda = \eta(\tau^*, \alpha^*, \beta^*, \gamma^*, \delta^*) := \theta(x, y, z, p/\lambda, q/\lambda, r/\lambda),$$

from which λ will be solved as

$$(3.5) \quad \lambda = \mu(x, y, z, p, q, r).$$

Therefore (3.3) yields

$$\begin{aligned}
 (3.6) \quad a &= \alpha^*(x, y, z, p/\mu, q/\mu) := \alpha(x, y, z, p, q, r), \\
 b &= \beta^*(x, y, z, p/\mu, q/\mu) := \beta(x, y, z, p, q, r), \\
 c &= \gamma^*(x, y, z, p/\mu, q/\mu) := \gamma(x, y, z, p, q, r), \\
 d &= \delta^*(x, y, z, p/\mu, q/\mu) := \delta(x, y, z, p, q, r), \\
 t &= (1/\mu)\tau^*(x, y, z, p/\mu, q/\mu) := \tau(x, y, z, p, q, r),
 \end{aligned}$$

In (3.4), we observe that $\frac{kr}{k\lambda} = \theta(x, y, z, \frac{kp}{k\lambda}, \frac{kq}{k\lambda})$ for any positive k , which implies

$$k\lambda = \mu(x, y, z, kp, kq, kr),$$

from (3.5). Therefore $\mu(x, y, z, p, q, r)$ is (1)p-homogeneous in p, q, r . In consequence $\alpha, \beta, \gamma, \delta$ and τ of (3.6) are (0), (0), (0), (0) and (-1) p-homogeneous.

Next we have

$$\begin{aligned}
 (3.7) \quad \dot{p} &= \ddot{\phi}(\mu\tau, \alpha, \beta, \gamma, \delta)\mu^2 := P(x, y, z, p, q, r), \\
 \dot{q} &= \ddot{\psi}(\mu\tau, \alpha, \beta, \gamma, \delta)\mu^2 := Q(x, y, z, p, q, r), \\
 \dot{r} &= \ddot{\eta}(\mu\tau, \alpha, \beta, \gamma, \delta)\mu^2 := R(x, y, z, p, q, r),
 \end{aligned}$$

where P, Q, R are obviously (2) p-homogeneous. Therefore our problem is to find $L(x, y, z, p, q, r)$ such that (2.6), that is

$$(3.8)(a) \quad \frac{L_{xq} - L_{yp}}{r} + \frac{L_{xr} - L_{zp}}{q} - \frac{U}{2} \left(\frac{2p}{p} - \frac{Q}{q} - \frac{R}{r} \right) + \frac{1}{2} (V - W) \left(\frac{Q}{q} - \frac{R}{r} \right) = 0.$$

$$(b) \quad \frac{L_{yq} - L_{xq}}{r} + \frac{L_{yr} - L_{zq}}{p} - \frac{V}{2} \left(\frac{2Q}{q} - \frac{R}{r} - \frac{P}{p} \right) + \frac{1}{2} (W - U) \left(\frac{R}{r} - \frac{P}{p} \right) = 0,$$

$$(c) \quad \frac{L_{zp} - L_{xr}}{q} + \frac{L_{zq} - L_{yr}}{p} - \frac{W}{2} \left(\frac{2R}{r} - \frac{P}{p} - \frac{Q}{q} \right) + \frac{1}{2} (U - W) \left(\frac{P}{p} - \frac{Q}{q} \right) = 0.$$

be an identity in x, y, z, p, q, r . It is remarked that every term of (3.8) is $(-1)p$ -homogeneous.

Differentiating (3.8) a with respect to p , using (2.4), (2.5) and multiplying by $-p$, we get

$$(3.9) \quad pU_x + qU_y + rU_z + \frac{pU_p}{2} \left(\frac{2p}{p} - \frac{Q}{q} - \frac{R}{r} \right) \\ + \frac{pU}{2} \left(\frac{2p_p}{p} - \frac{Q_p}{q} - \frac{R_p}{r} - \frac{2P}{p^2} \right) - \frac{p(V_p - W_p)}{2} \left(\frac{Q}{q} - \frac{R}{r} \right) \\ - \frac{p(V - W)}{2} \left(\frac{Q_p}{q} - \frac{R_p}{r} \right) = 0.$$

Since V is $(-2)p$ -homogeneous, we have

$$pV_p + qV_q + rV_r = -2V.$$

Therefore differentiating $V = \frac{qL_{qq}}{pr}$ with respect to p , and (2.5) (a) we get

$$pV_p = -qU_q + qW_q + rV_r.$$

Similarly differentiating $W = \frac{rL_{rr}}{pq}$ with respect to p , using homogeneity property of W and (2.5)b we get

$$pW_p = -rU_q + rV_r + qW_q.$$

Therefore

$$(3.10) \quad p(V_p - W_p) = rU_r - qU_q.$$

Substituting (3.10) in (3.9) we get

$$\begin{aligned}
 (3.11)(a) \quad & pU_x + qU_y + rU_z + \frac{pU_p}{2} \left(\frac{2P}{p} - \frac{Q}{q} - \frac{R}{r} \right) + \frac{qU_q}{2} \left(\frac{Q}{q} - \frac{R}{r} \right) \\
 & - \frac{rU_r}{2} \left(\frac{Q}{q} - \frac{R}{r} \right) + \frac{pU}{2} \left(\frac{2P_p}{p} - \frac{Q_p}{q} - \frac{R_p}{r} - \frac{2P}{p^2} \right) \\
 & - \frac{p(V-W)}{2} \left(\frac{Q_p}{q} - \frac{R_p}{r} \right) = 0.
 \end{aligned}$$

Similarly differentiating (3.8) (b) and (3.8) (c) with respect to q and r respectively and using the homogeneity property of U, V and W, we get

$$\begin{aligned}
 (3.11)(b) \quad & pV_x + qV_y + rV_z + \frac{pV_q}{2} \left(\frac{2Q}{q} - \frac{R}{r} - \frac{P}{p} \right) + \frac{rV_r}{2} \left(\frac{R}{r} - \frac{P}{p} \right) \\
 & - \frac{pV_p}{2} \left(\frac{R}{r} - \frac{P}{p} \right) + \frac{qV}{2} \left(\frac{2Q_q}{q} - \frac{R_q}{r} - \frac{P_q}{p} - \frac{2Q}{q^2} \right) \\
 & - \frac{q(W-U)}{2} \left(\frac{R_q}{r} - \frac{P_p}{p} \right) = 0.
 \end{aligned}$$

$$\begin{aligned}
 (3.11)(c) \quad & pW_x + qW_y + rW_z + \frac{pW_q}{2} \left(\frac{2R}{r} - \frac{P}{p} - \frac{Q}{q} \right) + \frac{pW_p}{2} \left(\frac{P}{p} - \frac{Q}{q} \right) \\
 & - \frac{qW_q}{2} \left(\frac{P}{p} - \frac{Q}{q} \right) + \frac{rW}{2} \left(\frac{2R_r}{r} - \frac{P_r}{p} - \frac{Q_r}{q} - \frac{2R}{r^2} \right) \\
 & - \frac{r(U-V)}{2} \left(\frac{P_r}{p} - \frac{Q_r}{q} \right) = 0.
 \end{aligned}$$

It is an open question that whether it is possible to solve the three equations in (3.11) for L or not. We solve these equations under the following conditions.

We choose the given four-parameter family of curves given by (3.1) such that either of any following holds.

Case 1. $\frac{Q}{q} + \frac{R}{r} = 0$, Q and R are independent of p.

Case 2. $\frac{R}{r} + \frac{P}{p} = 0$, P and R are independent of q.

Case 3. $\frac{P}{p} + \frac{Q}{q} = 0$, P and Q are independent of r.

We solve differential equations (3.11) (a), (3.11) (b) and (3.11) (c) one by one under the above three cases respectively.

Case 1. If $\frac{Q}{q} + \frac{R}{r} = 0$, $Q_p = 0$, then $R_p = 0$ and equation (3.11) (a) reduces to

$$(3.12) \quad pU_x + qU_y + rU_z + PU_p + QU_q + RU_r + U(P_p - \frac{P}{p}) = 0$$

Thus we must solve the differential equation (3.12) for U. It is well known that equation (3.12) is equivalent to solving

$$(3.13) \quad \frac{dx}{p} = \frac{dy}{q} = \frac{dz}{r} = \frac{dp}{p} = \frac{dq}{Q} = \frac{dr}{R} = -\frac{dU}{(P_p - P/p)U}.$$

If we quote the common ratio of (3.13) to dt, then we get

$$\frac{dx}{pdt} = p, \quad \frac{dy}{dt} = q, \quad \frac{dz}{dt} = r, \quad \frac{dp}{dt} = p, \quad \frac{dq}{dt} = Q, \quad \frac{dr}{dt} = R,$$

which yield six general integrals

$$(3.14) \quad \begin{aligned} x &= \phi(\lambda(t+t_0); a, b, c, d), & y &= \psi(\lambda(t+t_0); a, b, c, d), \\ z &= \eta(\lambda(t+t_0); a, b, c, d), & p &= \dot{\phi}(\lambda(t+t_0); a, b, c, d), \\ q &= \dot{\psi}(\lambda(t+t_0); a, b, c, d), & r &= \dot{\eta}(\lambda(t+t_0); a, b, c, d), \end{aligned}$$

with sixth integral constant t_0 . Then (3.13) reduces to

$$\frac{dU}{U} = -[P_p(\phi, \psi, \eta, \dot{\phi}, \dot{\psi}, \dot{\eta}) - \frac{1}{\phi} P(\phi, \psi, \eta, \dot{\phi}, \dot{\psi}, \dot{\eta})] dt.$$

Thus putting

$$(3.15) \quad \begin{aligned} X_I(\lambda(t+t_0); a, b, c, d) \\ := \exp \left[\left(P_p(\phi, \psi, \eta, \dot{\phi}, \dot{\psi}, \dot{\eta}) - \frac{1}{\phi} P(\phi, \psi, \eta, \dot{\phi}, \dot{\psi}, \dot{\eta}) \right) dt \right]. \end{aligned}$$

$$\begin{aligned}
 (3.11)(a) \quad & pU_x + qU_y + rU_z + \frac{pU_p}{2} \left(\frac{2P}{p} - \frac{Q}{q} - \frac{R}{r} \right) + \frac{qU_q}{2} \left(\frac{Q}{q} - \frac{R}{r} \right) \\
 & - \frac{rU_r}{2} \left(\frac{Q}{q} - \frac{R}{r} \right) + \frac{pU}{2} \left(\frac{2P_p}{p} - \frac{Q_p}{q} - \frac{R_p}{r} - \frac{2P}{p^2} \right) \\
 & - \frac{p(V-W)}{2} \left(\frac{Q_p}{q} - \frac{R_p}{r} \right) = 0.
 \end{aligned}$$

Similarly differentiating (3.8) (b) and (3.8) (c) with respect to q and r respectively and using the homogeneity property of U, V and W, we get

$$\begin{aligned}
 (3.11)(b) \quad & pV_x + qV_y + rV_z + \frac{pV_q}{2} \left(\frac{2Q}{q} - \frac{R}{r} - \frac{P}{p} \right) + \frac{rV_r}{2} \left(\frac{R}{r} - \frac{P}{p} \right) \\
 & - \frac{pV_p}{2} \left(\frac{R}{r} - \frac{P}{p} \right) + \frac{qV}{2} \left(\frac{2Q_q}{q} - \frac{R_q}{r} - \frac{P_q}{p} - \frac{2Q}{q^2} \right) \\
 & - \frac{q(W-U)}{2} \left(\frac{R_q}{r} - \frac{P_p}{p} \right) = 0.
 \end{aligned}$$

$$\begin{aligned}
 (3.11)(c) \quad & pW_x + qW_y + rW_z + \frac{pW_q}{2} \left(\frac{2R}{r} - \frac{P}{p} - \frac{Q}{q} \right) + \frac{pW_p}{2} \left(\frac{P}{p} - \frac{Q}{q} \right) \\
 & - \frac{qW_q}{2} \left(\frac{P}{p} - \frac{Q}{q} \right) + \frac{rW}{2} \left(\frac{2R_r}{r} - \frac{P_r}{p} - \frac{Q_r}{q} - \frac{2R}{r^2} \right) \\
 & - \frac{r(U-V)}{2} \left(\frac{P_r}{p} - \frac{Q_r}{q} \right) = 0.
 \end{aligned}$$

It is an open question that whether it is possible to solve the three equations in (3.11) for L or not. We solve these equations under the following conditions.

We choose the given four-parameter family of curves given by (3.1) such that either of any following holds.

Case 1. $\frac{Q}{q} + \frac{R}{r} = 0$, Q and R are independent of p.

$$(3.21) \quad pV_x + qV_y + rV_z + PV_p + QV_q + RV_r + V(Q_q - \frac{Q}{q}) = 0.$$

Thus we must solve the differential equation (3.21) for V. It is well known that it is equivalent to solving

$$(3.22) \quad \frac{dx}{p} = \frac{dy}{q} = \frac{dz}{r} = \frac{dp}{P} = \frac{dq}{Q} = \frac{dr}{R} = -\frac{dV}{(Q_q - Q/q)V}.$$

If we quote the common ratio of (3.22) to dt, then the first six equations give the six general integral given by (3.14) and the last equation reduces to.

$$\frac{dV}{V} = -[Q_q(\phi, \psi, \eta, \dot{\phi}, \dot{\psi}, \dot{\eta}) - \frac{1}{\dot{\psi}} Q(\phi, \psi, \eta, \dot{\phi}, \dot{\psi}, \dot{\eta})] dt,$$

Thus putting

$$(3.23) \quad X_2(\lambda(t+t_0); a, b, c, d) \\ = \exp\left[Q_q(\phi, \psi, \eta, \dot{\phi}, \dot{\psi}, \dot{\eta}) - \frac{1}{\dot{\psi}} Q(\phi, \psi, \eta, \dot{\phi}, \dot{\psi}, \dot{\eta})\right] dt.,$$

we get the seventh general integral

$$(3.24) \quad V = \frac{K_2}{X_2}$$

with an integral constant K_2 .

We are now in a position to solve the integral constants a, b, c, λ, t_0 and K_2 from (3.14) and (3.24). First (3.5) and (3.6) show

$$a = \alpha, \quad b = \beta, \quad c = \gamma, \quad d = \delta, \quad \lambda = \mu, \quad t_0 = \tau - t.$$

Thus, if we put

$$(3.25) \quad Y_2(x, y, z, p, q, r) = X_2(\mu\tau; \alpha, \beta, \gamma, \delta),$$

then we get $K_2 = VY_2$ from (3.24). Consequently the differential equation (3.21) shows the existence of a fundamental relation among $\alpha, \beta, \gamma, \delta, \lambda, \tau - t$ and VY_2 .

We must pay attention to the following two requests. First our V should not depend on t explicitly, and so we have a relation of the form

$$VY_2 = H_2^*(\alpha, \beta, \gamma, \delta, \lambda),$$

where H_2^* is an arbitrary function. Secondly our V must be (-2) p-homogeneous.

Since $Y_2(x, y, z, p, q, r)$ is (0) p-homogeneous, the second request is that

$$\frac{V}{k^2} Y_2 = H_2^*(\alpha, \beta, \gamma, \delta, k\lambda),$$

for any positive number k , and so we obtain finally

$$(3.26) \quad V = \frac{H_2(\alpha, \beta, \gamma, \delta)}{Y_2 \lambda^2},$$

where $H_2(\alpha, \beta, \gamma, \delta)$ is an arbitrary function. Thus from (2.4) we get the fundamental function $L(x, y, z, p, q, r)$ as

$$(3.27) \quad L = pr \iint_q^V (dq)^2 + pC_2(x, y, z, p, q, r) + D_2(x, y, z, p, q, r).$$

Substituting from (3.27) in (2.4) we get

$$(3.28) \quad 2C_{2q} + C_{2qq} + D_{lqq} = 0.$$

Case III. If $\frac{P}{p} + \frac{Q}{q} = 0$, $P_r = 0$, then $Q_r = 0$ and equation (3.11) (c) reduces to

$$(3.29) \quad pW_x + qW_y + rW_z + PW_p + QW_q + RW_r + W(R_r - \frac{R}{r}) = 0.$$

Thus we must solve the differential equation (3.29) for W . It is well known that it is equivalent to solving

$$(3.30) \quad \frac{dx}{p} = \frac{dy}{q} = \frac{dz}{r} = \frac{dp}{P} = \frac{dq}{Q} = \frac{dr}{R} = -\frac{dW}{(R_r - R/r)W}.$$

If we quote the common ratio of (3.30) to dt , then the first six equations give the six general integral by (3.14) and the last equation reduces to

$$\frac{dW}{W} = -[R_r(\phi, \psi, \eta, \dot{\phi}, \dot{\psi}, \dot{\eta}) - \frac{1}{\dot{\eta}} R(\phi, \psi, \eta, \dot{\phi}, \dot{\psi}, \dot{\eta})] dt,$$

Thus putting

$$(3.31) \quad \begin{aligned} & X_3(\lambda(t+t_0), a, b, c, d) \\ & = \exp \int [R_r(\phi, \psi, \eta, \dot{\phi}, \dot{\psi}, \dot{\eta}) - \frac{1}{\dot{\eta}} R(\phi, \psi, \eta, \dot{\phi}, \dot{\psi}, \dot{\eta})] dt, \end{aligned}$$

we get the seventh general integral

$$(3.32) \quad W = \frac{K_3}{X_3}$$

with an integral constant K_3 .

We are now in a position to solve the integral constants a, b, c, λ, t_0 and K_3 from (3.14) and (3.32). First (3.5) and (3.6) show.

$$a = \alpha, \quad b = \beta, \quad c = \gamma, \quad d = \delta, \quad \lambda = \mu, \quad t_0 = \tau - t.$$

Thus, if we put

$$(3.33) \quad Y_3(x, y, z, p, q, r) = X_3(\mu\tau; \alpha, \beta, \gamma, \delta),$$

then we get $K_3 = WY_3$ from (3.32). Consequently the differential equation (3.29) shows the existence of a fundamental relation among $\alpha, \beta, \gamma, \delta, \lambda, \tau - t$ and WY_3 .

We must pay attention to the following two requests. First our W should not depend on t explicitly, and so we have a relation of the form.

$$WY_3 = H_3^*(\alpha, \beta, \gamma, \delta, \lambda),$$

where H_3^* is an arbitrary function. Secondly our W must be (-2) p-homogeneous.

Since $Y_3(x, y, z, p, q, r)$ is (0) p-homogeneous, the second request is that

$$\frac{W}{k^2} Y_3 = H_3^*(\alpha, \beta, \gamma, \delta, k\lambda),$$

for any positive number k , and so we obtain finally

$$(3.34) \quad W = \frac{H_3(\alpha, \beta, \gamma, \delta)}{Y_3 \lambda^2},$$

where $H_3(\alpha, \beta, \gamma, \delta)$ is an arbitrary function. Thus from (2.4) we get the fundamental function $L(x, y, z, p, q, r)$ as

$$(3.35) \quad L = pq \int \frac{W}{r} (dr)^2 + rC_3(x, y, z, p, q, r) + D_3(x, y, z, p, q, r).$$

Substituting from (3.35) in (2.4) we get

$$(3.36) \quad 2C_{3r} + C_{3rr} + D_{3rr} = 0.$$

Summarizing these results we have the following:

Theorem (3.1). *All the Finsler metrics $L(x, y, z, p, q, r)$ in R^3 the geodesics of which are*

given by (3.1) with $\frac{Q}{q} + \frac{R}{r} = 0$, $Q_p = 0$ are written in the form (3.19), together with (3.20), where U , Y_1 and X_1 are given by (3.18), (3.17) and (3.15) respectively.

Theorem (3.2). All the Finsler metrics $L(x, y, z, p, q, r)$ in R^3 the geodesics of which are given by (3.1) with $\frac{R}{r} + \frac{P}{p} = 0$, $P_q = 0$ are written in the form (3.27), together with (3.28), where V , Y_2 and X_2 are given by (3.26), (3.25) and (3.23) respectively.

Theorem (3.3). All the Finsler metrics $L(x, y, z, p, q, r)$ in R^3 the geodesics of which are given by (3.1) with $\frac{P}{p} + \frac{Q}{q} = 0$, $P_r = 0$ are written in the form (3.35), together with (3.36), where W , Y_3 and X_3 are given by (3.34), (3.33) and (3.31) respectively.

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