

# The Family of Astroids as Geodesics in Two Dimensional Finsler Space

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**Abstract:** In this paper we obtained metric Function of two dimensional Finsler space whose geodesics are two parameter family of astroids.

## 1. Introduction

S. Hojo<sup>1</sup> studied five Finsler metrics in  $R^2 - \{0\}$ , all the geodesics of which are logarithmic spirals with the pole 0 and he obtained certain sufficient conditions for a Finsler space to have such special geodesics. Finsler metric function of two dimensional Finsler space, whose geodesics are two parameter family of curves have been obtained by M. Matsumoto<sup>2,3,4</sup>. He considered parametric and non-parametric equations of curves and obtained metric function in both cases. Such metric function contains two arbitrary function of coordinates. T.N. Pandey<sup>5</sup> found the metric function of two dimensional Finsler space whose geodesics are two parameter family of cycloids and tractrices. The aim of this paper is to obtain the metric function of a two dimensional Finsler space whose geodesics are two parameter family of astroids.

## 2. The Metrizable Problem

Usually we find a linear connection  $\Gamma = \{\Gamma_{jk}^i(x)\}$  in an n-dimensional manifold  $M^n$  form a Riemannian metric  $g_{ij}(x)$  on  $M^n$ , and we obtain the equation of parths.

L.P. Eisenhart, T.Y. Thomas and O. Veblen<sup>6</sup>, considered the reverse problem i.e. to find a Riemannian metric from the given equations of parths. He also found the condition for the existence of a Riemannian space whose geodesics are te given paths.

O.Varga<sup>7</sup> discussed this problem for a Finslerian metric. His consideration was based on the Cartan connection. He formulated the problem in the following manner. Given a connection  $CG = \{\Gamma_{jk}^i(x, y), C_{jk}^i(x, y)\}$  in the tangent bundle  $TM^n$ , find a Finsler space  $F^n = (M^n, L(x, y))$  such that  $CG$  coincides with the Cartan connection.

Later on, after a long gap of twenty years L. Tamassy<sup>8</sup> discussed the similar problem and obtained his result simpler than that of O. Varga<sup>7</sup>.

The same problem was discussed by A.Rapcsak<sup>9</sup>. He considered the equations of generalized paths:

$$(2.1) \quad \frac{d^2 x^i}{dt^2} + G^i \left( x, \frac{dx}{dt} \right) = 0$$

and found a Finsler space  $F^n$  whose Berwald connection  $B\Gamma = G_{jk}^i(x, y)$  constructed from fundamental function  $L(x, y)$  is given by  $G_{jk}^i(x, y) = \partial_j \partial_k G^i(x, y)$ . He assumed that  $G^i$  is positively homogeneous of degree two in  $y^i$ .

In view of Euler's theorem on homogeneous functions and degree of homogeneity of  $G^i$  in  $y^i$ , the equation (2.1) can be written as:

$$(2.2) \quad \frac{d^2 x^i}{dt^2} + G_{jk}^i \left( x, \frac{dx}{dt} \right) \frac{dx^j}{dt} \frac{dx^k}{dt} = 0.$$

Now we shall give some definitions:

**Definition (2.1)** If  $G_{jk}^i(x, y)$  are independent of  $y^i$ , the space is called Berwald space.

**Definition (2.2)** Two path spaces are called projectively related to each other, if their paths coincide with each other.

We know that the necessary and sufficient condition for path spaces  $P^n$  and  $\bar{P}^n$  to be projectively related is that there exist a function  $P(x, y)$  called projective factor satisfying.

$$(2.3) \quad \bar{G}^i(x, y) = G^i(x, y) + P(x, y)y^i.$$

In fact, Rapcsak<sup>9</sup> discussed the problem to find a Finsler space which is projectively related to it, from a given path space  $P^n$  with  $\{G_{jk}^i(x, y)\}$ .

Makato Mastumoto<sup>3</sup> considered the set of all Finsler spaces defined on the same underlying manifold  $M^n$ . He defined an equivalence relation in this set by the consideration that a Finsler space of the above set is related to another if the first one Finsler space of the set is projectively related to the later. This equivalence relation partitions the above set into equivalence classes of Finsler spaces. Obviously one equivalence class contains all those Finsler space which are projectively related to each other. He called such equivalence classes as projective equivalence classes of Finsler spaces.

Let us consider a co-ordinate neighbourhood of a two dimensional Finsler space. Since our problem is of local character. It is enough to consider a Finsler space  $F^2 = (R^2, L(x, y, \dot{x}, \dot{y}))$  defined on  $(x, y)$  plane  $R^2$ . Henceforth, we use the symbols  $(p, q)$  for  $(\dot{x}, \dot{y})$ . Since it is possible to regard  $p > 0$  in our consideration, by suitable orientation of the

parameter  $t$ , we may write  $G^i(x, y, p, q) = p^2 G^i(x, y, l, q/p)$  then (2.1) gives.

$$\frac{d^2 y}{dx^2} = y'' = \frac{p\dot{q} - q\dot{p}}{p^3} = 2y' G^1(x, y, l, y') - 2G^2(x, y, l, y')$$

therefore equation of paths in the normal form is given by:

$$(2.4) \quad y'' = F(x, y, y') = Y_3(y')^3 + Y_2(y')^2 + Y_1(y') + Y_0$$

where  $Y_3 = G_{22}^1$ ,  $Y_2 = 2G_{22}^1 - G_{22}^2$ ,  $Y_1 = G_{11}^1 - 2G_{22}^2$ ,  $Y_0 = -G_{11}^2$ .

Thus  $F(x, y, y')$  is a polynomial of at most third order in  $y'$  in the case of linear connection.

The equation of paths for a general parameter  $\tau$  are rewritten in the form

$$\frac{d^2 x^i}{d\tau^2} + G^i\left(x, \frac{dx}{d\tau}\right) = h(\tau) \frac{dx^i}{d\tau}$$

then the affine or natural parameter  $t$  in (2.1) is given by  $\frac{dt}{d\tau} = \exp\left\{\int h(\tau) d\tau\right\}$ . Therefore

(2.4) is the required of paths as a point set. In fact from (2.3) we find the same differential equation (2.4) for two dimensional path spaces, which are projectively related to each other.

**Lemma (2.1)** *To each system (2.4) of geodesics as point set, there corresponds a projective equivalent class  $(F_p(R^2))$  of two dimensional Finsler space.*

### 3. The Inverse Correspondence

Since the differential equation system (2.4) of geodesics is of the normal form, so solution of (2.4) is uniquely determined by the initial conditions, if the function  $F(x, y, z)$  satisfied Lipschitz conditions. Therefore from (2.4) we get finite equation.

$$(3.1) \quad y = f(x, a, b)$$

of geodesics  $C(a, b)$  as a point set with two constant parameter  $a$  and  $b$ .

Therefore equation (3.1) may also written in the parametric form

$$(3.2) \quad x = \Phi(t, a, b) \quad y = \Psi(t, a, b) \quad \text{where } t \text{ is a parameter.}$$

If the finite equation (3.2) is a geodesic, then we obtained the fundamental function  $L(x, y, p, q)$  of  $F^2 = (R^2, L(x, y, p, q))$ . In the parametric case (3.2) we are concerned with the equation of geodesics in the Weirstrass form

$$W(c) = L_{xq} - L_{yp} + (p\dot{q} - q\dot{p})W = 0$$

where  $W = \frac{L_{pp}}{q^2} = \frac{-L_{pq}}{pq} = \frac{L_{qq}}{p^2}$  is the Weirstrass invariant.

Now we consider an auxiliary parameter  $c$  such that

$$(3.3) \quad x = \varphi(ct, a, b), \quad y = \psi(ct, a, b).$$

We use the following notation  $\dot{\varphi} = \varphi_t(ct, a, b)$  and  $\dot{\psi} = \psi_t(ct, a, b)$ .

$$(3.4) \quad p/c = \dot{\varphi}(ct, a, b) \quad q/c = \dot{\psi}(ct, a, b),$$

From (3.3) and (3.4) we obtain  $a, b, c$  and  $t$  as function of  $(x, y, p, q)$ :

$$a = \alpha(x, y, p, q) \quad b = \beta(x, y, p, q) \quad c = \gamma(x, y, p, q) \quad t = \tau(x, y, p, q).$$

These function  $\alpha, \beta, \gamma$  and  $\tau$  are  $(0), (0), (1)$  and  $(-1)p$  homogeneous respectively.

From (3.4) we get

$$\dot{p}/c^2 = \ddot{\varphi}(ct, a, b) \quad \dot{q}/c^2 = \ddot{\psi}(ct, a, b)$$

which implies

$$(3.5) \quad \dot{p} = \ddot{\varphi}(\gamma\tau, \alpha, \beta)\gamma^2 = P(x, y, p, q), \quad \dot{q} = \ddot{\psi}(\gamma\tau, \alpha, \beta)\gamma^2 = Q(x, y, p, q),$$

where  $P$  and  $Q$  are  $(2)$ - $p$  homogenous functions. Again we consider

$$(3.6) \quad \pi(x, y, p, q) = P_p + Q_q \quad \text{and}$$

$$\Pi(v; a, b) = \pi(\varphi(v; a, b), \psi(v; a, b), \dot{\varphi}(v; a, b), \dot{\psi}(v; a, b))$$

where  $v = c(t+t_0)$ .

From (3.6) We get

$$(3.7) \quad U(v; a, b) = \exp \int \Pi(v; a, b) dv, \quad v(x, y, p, q) = U(\gamma\tau, \alpha, \beta)$$

where  $v$  is  $(0)$ - $p$  homogenous function. Consequently we get,

$$(3.8) \quad W(x, y, p, q) = \frac{H(\alpha, \beta)}{\gamma^3 v(x, y, p, q)}, \quad L_1^* = q^2 \iint W(dp)^2, \quad L_2^* = p^2 \iint W(dq)^2.$$

Therefore we get fundamental function

$$(3.9) \quad L(x, y, p, q) = L^*(x, y, p, q) + C(x, y).p + D(x, y).q$$

where  $L^* = L_1^*$  or  $L_2^*$  and  $C$  and  $D$  are arbitrary function of  $(x, y)$  satisfying

$$(3.10) \quad C_v - D_x = L_{xa}^* - L_{vb}^* + (pQ - qP)W.$$

The function  $C(x,y)$  and  $D(x,y)$  must satisfy (3.10). Let us choose a pair  $(C^0(x,y), D^0(x,y))$  such that which satisfy (3.10). Then we have  $(C-C^0)_y = (D-D^0)_x$  for a general  $(C,D)$  hence we have a function  $E(x,y)$  such that  $(C-C^0) = E_x$  and  $D-D^0 = E_y$ .

consequently  $L(x,y,p,q)$  is written in the form

$$(3.11) \quad L(x,y,p,q) = L^*(x,y,p,q) + (C^0 + E_x)_p + (D^0 + E_y)_q.$$

Therefore the metrics of Finsler spaces with the two arbitrary function  $H(\alpha,\beta)$  and  $E(x,y)$  belonging to a projectively equivalent class.

**Theorem (3.1):** For each given projective equivalence class  $F_p(\mathbb{R}^2)$  of two dimensional Finsler space whose geodesics are given by (2.4) as point set are the solution of the second order different equation (2.4) geodesics.

#### 4. Family of Astroids

We consider the family of astroids  $C(a,b)$  in parametric form of the equation

$$x = a \cos^3 t, \quad y = a \sin^3 t + b,$$

which are written in the form

$$(4.1) \quad x = a \cos^3 ct, \quad y = a \sin^3 ct + b$$

where  $a,b,c$  are constants and  $t$  is parameter.

Differentiating (4.1) with respect to  $t$  we get

$$(4.2) \quad p = \dot{x} = -3acsinct.cos^2 ct, \quad q = \dot{y} = 3acsint^2 ct.cosct$$

solving (4.1) and (4.2) we get

$$(4.3) \quad \left[ \begin{aligned} a &= \alpha(x, y, p, q) = \frac{x(p^2 + q^2)^{3/2}}{p^3} \\ b &= \beta(x, y, p, q) = \frac{(p^3 y + q^3 x)}{p^3} \\ c &= \gamma(x, y, p, q) = \frac{p^2}{3xq} \\ t &= \tau(x, y, p, q) = \frac{-3xq \tan^{-1}(q/p)}{p^2} \end{aligned} \right]$$

This function  $\alpha, \beta, \gamma$ , and  $\tau$  are  $(0), (0), (1)$  and  $(-1)$   $p$  homogeneous respectively.

Again differentiating (4.2) we get

$$(4.4) \quad \begin{aligned} \dot{p} &= 6ac^2 \sin^2 ct \cos ct - 3ac^2 \cos^3 ct, \\ \dot{q} &= 6ac^2 \cos^2 ct \sin ct - 3ac^2 \sin^3 ct \end{aligned}$$

From (4.2) and (4.3), we get

$$\dot{p} = \frac{2p^2}{3x} - \frac{p^4}{3xq^2} \equiv P(x, y, p, q),$$

$$\dot{q} = \frac{pq}{3x} - \frac{2p^3}{3qx} \equiv Q(x, y, p, q).$$

From (4.5) we have

$$(4.6) \quad \pi(x, y, p, q) = p_p + Q_q = 5p/3x - 2p^3/3xq^2.$$

Using (4.1), (4.2) and (4.3) in (4.6) we get

$$(4.7) \quad \Pi(v, a, b) = -5 \tan v + 2 \cot v$$

where  $v = c(t + t_0)$  consequently, we have

$$(4.8) \quad \begin{aligned} U(v, a, b) &= \cos^5 v \sin^2 v, \\ V(x, y, p, q) &= \frac{q}{3x} \left( \frac{p^2}{p^2 + q^2} \right)^{7/2}. \end{aligned}$$

From (3.4), (4.3) and (4.8), we have

$$(4.9) \quad W(x, y, p, q) = \alpha^{7/3} H(\alpha, \beta) x^{5/3} q^2 / p^6 = x^{5/3} q^2 / p^6 \bar{H}(\alpha, \beta)$$

where

$$\bar{H}(\alpha, \beta) = \alpha^{7/3} H(\alpha, \beta).$$

Since  $H(\alpha, \beta)$  is an arbitrary function, we choose  $\bar{H}(\alpha, \beta) = 1$  then (4.9) becomes

$$(4.10) \quad W(x, y, p, q) = x^{5/3} \frac{q^2}{p^6}.$$

Now from (3.4) and (4.10) we get

$$(4.11) \quad L^* = L_I = \left( \frac{q^4}{p^4} \right) \frac{x^{5/3}}{20}.$$

$$L_2^* = \left( \frac{q^4}{12p^4} \right) \cdot x^{5/3}.$$

From (3.6) we have

$$(4.12) \quad C_y - D_x = \left( \frac{-q}{3p^2} \right) \cdot x^{2/3}.$$

Since  $C(x,y)$  and  $D(x,y)$  are arbitrary function, we choose  $D = 0$ , then (4.12) gives

$$C = \left( \frac{-q}{3p^2} \right) \cdot x^{2/3} y.$$

Using (4.11) and (4.12) in (3.5), we get fundamental function

$$L(x, y, p, q) = \left( \frac{q^4}{p^4} \right) \cdot \frac{x^{5/3}}{20} - \left( \frac{q}{3p} \right) \cdot x^{2/3} y.$$

**Theorem (4.1)** The fundamental function  $L(x,y,p,q)$  of a Finsler space  $(\mathbb{R}^2, L(x,y,p,q))$  having the family of astroids (4.1) as the geodesics is given by (4.13) and such path spaces are projectively related to Finsler spaces.

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