

A Note on Subtractive Abelian Loops

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Abstract: Kazim and Husain¹ studied subtractive abelian loops by means of postulates in terms of the 'left difference' and the 'right difference', while M.K. Sen² studied characterization of abelian groups by means of postulates in terms of the two binary operations. In this paper, various sets of two-postulates systems in terms of the two binary operations have been developed, each set characterizing subtractive abelian loop individually as well as established equivalences of some of these sets.

1. Introduction

According to Bruck³, a loop is defined as a groupoid with cancellation laws and an identity. Sen² studied characterization of abelian groups by means of postulates in terms of the two operations. Kazim and Husain¹ studied subtractive abelian loops by means of two postulates in terms of the 'left difference' and the 'right difference'. In this direction, Agrawal and Kumar⁴ have also formulated various sets of two-postulates in terms of the two binary operations characterizing subtractive abelian loops individually.

In this paper, we have formulated new sets of two-postulates, each set characterizing subtractive abelian loop individually as well as established equivalences of some of these sets.

2. Preliminaries

Let G be a system of elements $a, b, c, \dots$ with three binary operations ' $\cdot$ ', ' $*$ ' and ' o ' defined on its elements.

For any $a, b, c \in G$, consider the following list of postulates:

- | | |
|---|--|
| (A ₁) $(a * b) o c = c * (a * b)$ | (A ₉) $a o (b o c) = a o (c * b)$ |
| (A ₂) $(a * b) o c = c * (b o a)$ | (A ₁₀) $(a o b) * c = (b * a) * c$ |
| (A ₃) $a * (b o c) = a * (c * b)$ | (A ₁₁) $(a o b) * c = (b * a) * c$ |
| (A ₄) $(a * b) o c = (b o a) o c$ | (A ₁₂) $(a o b) * a = b$ |
| (A ₅) $a * (b o c) = (b o c) o a$ | (A ₁₃) $a o (b * a) = b$ |

$$(A_6) \quad a * (b \circ c) = (c * b) \circ a$$

$$(A_{14}) \quad (b * a) * a = b$$

$$(A_7) \quad (a \circ b) * c = c \circ (a \circ b)$$

$$(A_{15}) \quad a \circ b = c \Leftrightarrow c * a = b$$

$$(A_8) \quad (b * a) * c = c \circ (b * a)$$

We also consider the definition of Kazim and Hussain¹ expressed here as

Definition (2.1). For any $a, b \in G$,

$$a \quad b = (e \circ a) * b,$$

where $e = a \circ a = a * a$ etc.

Proposition (2.1). According to Kazim and Hussain¹, a groupoid $(G, \circ)$ with the conditions

$$(2.1) \quad a \circ b = b * a$$

and

$$(2.2) \quad (a \circ b) * a = b, \quad a \circ (b * a) = b$$

is a subtractive abelian loop under the Definition (2.1).

Proposition (2.2). It has been established by Agrawal and Kumar⁴ that the system $(G, \circ)$ with the postulates $\{A_9, A_{12}\}$ characterizes a subtractive abelian loop.

Proposition (2.3). It has also been proved by Agrawal and Kumar⁴ that the system $(G, \circ)$ alongwith the postulates $\{A_{10}, A_{12}\}$ characterizes a subtractive abelian loop.

Proposition (2.4). $(A_{15}) \Rightarrow (A_{12})$

Proof. Consider $a \circ b = c$

$$\begin{aligned} &\Rightarrow (a \circ b) * a = c * a \\ &= b \text{ in view of } (A_{15}) \end{aligned}$$

Thus, we get (A_{12}) .

Proposition(2.5). $(A_{15}) \Rightarrow (A_{13})$.

Proof. Consider $c * a = b$

Pre-operating by a w.r.t. ' $\circ$ ', we have

$$\begin{aligned} &a \circ (c * a) = a \circ b \\ &= c \text{ in view of } (A_{15}) \end{aligned}$$

which is (A_{13}) .

3. Some Results

In this section, we consider some new combinations of two-postulates from the list (A_1) to (A_{15}) which lead to the characterization of subtractive abelian loops individually as follows:

Results (3.1). *The system $(G, \cdot)$ with the postulates (A_{11}, A_{12}) characterizes a subtractive abelian loop.*

Proof. Setting $a \circ b$ in place of b in (A_{12}) , we have

$$(a \circ (a \circ b)) * a = a \circ b$$

Applying (A_{11}) , we get

$$b * a = a \circ b$$

which is (2.1).

Again, (A_{11}) with (2.1) yields

$$a \circ (b * a) = b$$

Thus, we obtain (2.2). Consequently, the result follows in view of the Proposition (2.1).

Result (3.2). *The system $(G, \cdot)$ with the postulates $\{A_{11}, A_{13}\}$ characterizes a subtractive abelian loop.*

Proof. Pre-operating (A_{13}) on both sides by a w.r.t. ' $\circ$ ', we have

$$a \circ (a \circ (b * a)) = a \circ b$$

or, by virtue of (A_{11})

$$b * a = a \circ b$$

which is (2.1).

Again, (A_{11}) alongwith (2.1) leads to

$$(a \circ b) * a = b$$

which is (2.2).

Hence the result.

Remark (3.1). *The system $(G, \cdot)$ with the postulates $\{A_{11}, A_{13}\}$ characterizes a subtractive abelian loop in view of the Proposition (2.4) and the Result (3.1).*

Result (3.3). *The system $(G, \cdot)$ with the postulates $\{A_{12}, A_{14}\}$ characterizes a*

subtractive abelian loop.

Proof. Post-operating (A_{12}) by a w.r.t. $*$ on both sides, we have

$$((a \circ b) * a) * a = b * a$$

Applying (A_{14}) , we obtain

$$a \circ b = b * a$$

which is (2.1).

Again, (A_{12}) with (2.1) yields $a \circ (b * a) = b$ i.e. (2.2) is satisfied. Hence the result.

Result (3.4). *The system $(G, \circ)$ with the postulates $\{A_{13}, A_{14}\}$ characterizes a subtractive abelian loop.*

Proof. Pre-operating (A_{14}) by a on both sides w.r.t. $\circ$, we have

$$a \circ ((b * a) * a) = a \circ b$$

In view of (A_{13}) , we get

$$b * a = a \circ b$$

which is (2.1).

It is simple to obtain (2.2) from (A_{13}) with (2.1). Thus, the result follows.

Remark (3.2). *The system $(G, \circ)$ with the postulates $\{A_{14}, A_{15}\}$ characterizes a subtractive abelian loop in view of the Proposition (2.4) and the Result (3.3) or alternatively, in view of the Proposition (2.5) and the Result (3.4).*

Result (3.5). *The system $(G, \circ)$ with the postulates $\{A_7, A_{13}\}$ characterizes a subtractive abelian loop.*

Proof. Since $(A_{15}) \Rightarrow (A_{12})$ in view of the Proposition (2.4), put $c = a$ in (A_7) and applying (A_{12}) , we get

$$\begin{aligned} a \circ (a \circ b) &= (a \circ b) * a \\ &= b \end{aligned}$$

which is (A_{11}) . Consequently, the system $\{A_7, A_{15}\}$ imply $\{A_{11}, A_{12}\}$ and hence characterizes a subtractive abelian loop in view of the Result (3.1).

Result (3.6). *The system $(G, \circ)$ alongwith the postulates $\{A_8, A_{15}\}$ characterizes a subtractive abelian loop.*

Proof. Put $c = a$ in (A_8) , we have

$$(b * a) * a = a o (b * a)$$

or, in view of the Proposition (2.5)

$$= b$$

which is (A_{14}) . Thus, $\{A_8, A_{15}\} \Rightarrow \{A_{14}, A_{15}\}$ and hence the given system characterizes a subtractive abelian loop in view of the Remark (3.2).

Result (3.7). *The system $(G, \cdot)$ alongwith the postulates $\{A_9, A_{15}\}$ forms a subtractive abelian loop.*

Proof. The proof follows in view of the Proposition (2.4) and the Proposition (2.2).

Result (3.8). *The system $(G, \cdot)$ alongwith the postulates $\{A_{10}, A_{15}\}$ forms a subtractive abelian loop.*

Proof. The proof is evident in view of the Proposition (2.4) and the Proposition (2.3).

4. Some Equivalences

In this section, we establish equivalences among various sets of postulates formulated from the list (A_7) to (A_{15}) .

Result (4.1). $\{A_7, A_{11}\} \Leftrightarrow \{A_9, A_{11}\}$

Proof. Putting $c = a$ and applying (A_{11}) , we obtain (A_{12}) so that, vide the Result (3.1), we can infer (2.1) i.e.

$$b * a = a o b$$

Applying (2.1) in (A_7) , we get (A_9) .

Conversely, pre-operating (A_9) by a w.t. 'o', we have

$$a o (a o (b o c)) = a o (a o (c * b))$$

which, with (A_{11}) , yields

$$b o c = c * b$$

Thus,

$$(a o b) * c = c o (a o b)$$

which is (A_7) and hence the equivalence.

Result (4.2). $\{A_{10}, A_{12}\} \Leftrightarrow \{A_{10}, A_{15}\}$

Proof. Since (A_{15}) imply (A_{12}) in view of the Proposition (2.4), it follows that

$$\{A_{10}, A_{15}\} \Rightarrow \{A_{10}, A_{12}\}$$

Conversely, we can infer (2.1) from $\{A_{11}, A_{12}\}$. In other words, we can write

$$(4.1) \quad c * a = a o c$$

$$\text{Let} \quad c * a = b$$

$$\text{or, in view of (4.1),} \quad a o c = b$$

$$\Rightarrow a o (a o c) = a o b$$

$$\text{Applying } (A_{11}), \text{ we get} \quad c = a o b$$

$$\text{Again, whenever} \quad a o b = c$$

$$\Rightarrow (a o b) * a = c * a$$

$$\text{or, in view of } (A_{12})$$

$$b = c * a$$

Thus, we arrive at (A_{15}) .

Result (4.3). $\{A_{11}, A_{12}\} \Leftrightarrow \{A_{11}, A_{13}\}$

Proof. From $\{A_{11}, A_{12}\}$, we infer (2.1).

Now, (A_{11}) with (2.1) yields

$$\begin{aligned} b &= a o (a o b) \\ &= a o (b * a) \end{aligned}$$

which is (A_{13}) .

Conversely, (A_{13}) with (A_{11}) imply (2.1) vide the Result (3.2).

Again, (A_{11}) with (2.1) yields

$$\begin{aligned} b &= a o (a o b) \\ &= (a o b) * a \end{aligned}$$

which is (A_{12}) . Hence the result.

Result (4.4). $\{A_{12}, A_{14}\} \Leftrightarrow \{A_{13}, A_{14}\}$

Proof. From $\{A_{12}, A_{14}\}$, we infer (2.1) vide the Result (3.3).

Now, (A_{14}) with (2.1) yields

$$\begin{aligned} b &= (b * a) * a \\ &= a o (b * a) \end{aligned}$$

which is (A_{13}) .

Conversely, $\{A_{13}, A_{14}\}$ imply (2.1) vide the Result (3.4). Then (A_{14}) with (2.1) yields

$$\begin{aligned} b &= (b * a) * a \\ &= (a o b) * a \end{aligned}$$

which is (A_{12}) . Hence the result.

Result (4.5). $\{A_{12}, A_{14}\} \Leftrightarrow \{A_{14}, A_{15}\}$

Proof. Since (A_{15}) imply (A_{12}) , it follows that

$$\{A_{14}, A_{15}\} \Rightarrow \{A_{12}, A_{14}\}$$

Conversely, we infer from the Result (3.3) that $\{A_{12}, A_{14}\}$ imply (2.1).

Then (A_{12}) with (2.1) yields

$$\begin{aligned} b &= (a o b) * a \\ &= a o (a o b) \end{aligned}$$

which is (A_{11}) .

Consequently,

$$\begin{aligned} \{A_{12}, A_{14}\} &\Rightarrow \{A_{14}, A_{12}\} \\ &\Rightarrow (A_{15}) \text{ vide the Result (4.2).} \end{aligned}$$

Therefore, $\{A_{12}, A_{14}\}$ imply $\{A_{14}, A_{15}\}$ and hence the equality.

Remark (4.1). $\{A_{13}, A_{14}\} \Leftrightarrow \{A_{14}, A_{15}\}$

holds true in view of the Result (4.4) and the Result (4.5).

Result (4.6). $\{A_7, A_{13}\} \Leftrightarrow \{A_8, A_{13}\}$

Proof. In view of the Result (3.5) and the Result (3.1), we have

$$b * a = a o b$$

which, with (A_7) , yields (A_8) .

Conversely, in view of the Result (3.6) and the Proposition (2.4)

$$\{A_8, A_{13}\} \Rightarrow \{A_{12}, A_{14}\}$$

which, vide the Result (3.3), yields.

$$a o b = b * a$$

Henceforth, we obtain (A_7) .

Result (4.7). $\{A_9, A_{13}\} \Leftrightarrow \{A_{10}, A_{13}\}$

Proof. Since (A_{15}) imply (A_{12}) , from (A_9) we get

$$(a o (b o c)) * a = (a o (c * b)) * a$$

Applying (A_{12}) ,

$$b \circ c = c * b$$

Thus,

$$(a \circ b) * c = (b * a) * c$$

Conversely, put $c = a$ in (A_{10}) , we have

$$(a \circ b) * a = (b * a) * a$$

which, in view of (A_{12}) , yields

$$b = (b * a) * a$$

Also, since $(A_{15}) \Rightarrow (A_{12})$, setting $a \circ b$ in place of b and applying (A_{12}) , we obtain

$$b * a = a \circ b$$

Henceforth

$$a \circ (b \circ c) = a \circ (c * b)$$

which is (A_9) .

Remark (4.2). The system $(G, \circ)$ characterizes a subtractive abelian loop if any one of the following hold :

- | | | |
|----------------------------|-----------------------------|---------------------------|
| (i) $\{A_7, A_{15}\}$ | (ii) $\{A_8, A_{15}\}$ | (iii) $\{A_9, A_{15}\}$ |
| (iv) $\{A_{10}, A_{15}\}$ | (v) $\{A_{11}, A_{12}\}$ | (vi) $\{A_{11}, A_{13}\}$ |
| (vii) $\{A_{11}, A_{15}\}$ | (viii) $\{A_{12}, A_{14}\}$ | (ix) $\{A_{13}, A_{14}\}$ |
| (x) $\{A_{14}, A_{15}\}$ | | |

Proof. The proof follows in view of the Results (3.5) to (3.8), the Results (3.1) to (3.2), the Result (4.2), the Result (3.3) and the Results (4.4) to (4.5) respectively.

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