

Three Dimensional Landsberg Space with Constant Unified Main Scalar

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Abstract: Certain properties of a three dimensional Landsberg space with constant unified main scalar have been discussed in the present paper. A new scalar K has been introduced and its importance has been shown.

1. Introduction

In 1984 F. Ikeda¹ studied the properties of Finsler Spaces satisfying the condition $L^2C^2 = f(x)$, where L is the fundamental function and C is the length of the torsion vector C_i . In his paper² he considered the condition $L^2C^2 = \text{non-zero constant}$, which is stronger than the corresponding condition considered in the paper¹. A two dimensional Berwald space is an example of such a Finsler space with constant function LC . However in three dimensional Landsberg space the function LC is not always constant. Some properties of three dimensional Finsler space with non-zero constant function LC have been studied by Ikeda³. The function LC has been called the unified main scalar in the case of three dimensional Finsler space. By introducing a scalar T , F. Ikeda has shown that every three dimensional Landsberg space is a Berwald space provided unified main scalar and T -scalar are non-zero constants.

The purpose of the present paper is to discuss some more properties of three dimensional Landsberg space with constant unified main scalar. We have introduced a scalar K which has important role in this paper.

In the three dimensional Finsler space we have three essential scalar fields called main scalars H , I , J and h - and v -connection vectors h_i and v_i . The orthonormal frame field called Moór's frame (l^i, m^i, n^i) plays important role in three dimensional Finsler space⁴.

2. Scalar Components in Moór's Frame

We consider a three dimensional Finsler space F^3 with the fundamental function $L(x, y)$ and the frame (l^i, m^i, n^i) called Moór's frame of F^3 where l^i is the normalized supporting element

i.e. $l^i = \frac{y^i}{L}$, m^i is the normalized torsion vector i.e. $m^i = c^i/c$, n^i is constructed by $g_{ij}l^in^j = 0$, $g_{ij}m^in^j = 0$, $g_{ij}n^in^j = 1$ and g_{ij} is the fundamental tensor of F^3 defined as .
 $g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j L^2$, $\dot{\partial}_i = \frac{\partial}{\partial y^i}$.

In the Moór's frame an arbitrary tensor be expressed in terms of scalar components, for instance a tensor $T (=T^i_{jk})$ of (1.2) type can be written as

$$T^i_{jk} = T_{\alpha\beta\gamma} e_{\alpha}^i e_{\beta j} e_{\gamma k}$$

where $e_{1j}^i = l^i$, $e_{2j}^i = m^i$, $e_{3j}^i = n^i$ and the summation convention is applied to greek indices also. The scalar components $T_{\alpha\beta\gamma}$ are given by

$$T_{\alpha\beta\gamma} = T^i_{jk} e_{\alpha i} e_{\beta j} e_{\gamma k}$$

From the equation $g_{ij} e_{\alpha}^i e_{\beta}^j = \delta_{\alpha\beta}$ we have

$$(2.1) \quad g_{ij} = l_i l_j + m_i m_j + n_i n_j$$

Next the C-tensor $C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}$ satisfies $C_{ijk} l^i = 0$ and symmetric in i,j,k, therefore if

$C_{\alpha\beta\gamma}$ are scalar components of $L C_{ijk}$ i.e.

$$L C_{ijk} = C_{\alpha\beta\gamma} e_{\alpha i} e_{\beta j} e_{\gamma k},$$

then we have

$$(2.2) \quad L C_{ijk} = H m_i m_j m_k - J s_{(ijk)} \{ m_i m_j m_k \} + I s_{(ijk)} \{ n_i n_j n_k \} + j n_i n_j n_k,$$

where, $s_{(ijk)} \{ \}$ denote the cyclic interchange of i,j,k and summation and

$$(2.3) \quad C_{l\beta\gamma} = 0, C_{222} = H, C_{333} = -C_{223} = J, C_{233} = I.$$

We shall be using Cartan's connection $CG = (F^i_{jk}, G^i_j, C^i_{jk})$ in the following section of this paper. The h- and v- covariant derivatives of a tensor field with respect to CG are indicated by the short and long lines respectively.

First we have ⁴

$$(2.4) \quad l_{i j} = 0, m_{i j} = n_i h_j, n_{i j} = -m_i h_j,$$

$$(2.5) \quad Ll_{i j} = h_{ij}, Lm_{i j} = -l_i m_j + n_i v_j, Ln_{i j} = -l_i n_j - m_i v_j,$$

where h_{ij} are components of angular metric tensor, h_i and v_i are component of h and v -connection vectors respectively.

The equation (2.4) and (2.5) may be written as

$$(2.6) \quad e_{\alpha\mu j} = H_{\alpha\beta\gamma} e_{\beta\mu} e_{\gamma j},$$

$$(2.7) \quad L e_{\alpha\mu j} = V_{\alpha\beta\gamma} e_{\beta\mu} e_{\gamma j},$$

where $H_{\alpha\beta\gamma}$ and $V_{\alpha\beta\gamma}$ being fixed are given by.

$$(2.8) \quad (H_{\alpha\beta\gamma}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & h_\gamma \\ 0 & -h_\gamma & 0 \end{pmatrix}, \quad (V_{\alpha\beta\gamma}) = \begin{pmatrix} 0 & \delta_\gamma^2 & \delta_\gamma^3 \\ -\delta_\gamma^2 & 0 & v_\gamma \\ -\delta_\gamma^3 & -v_\gamma & 0 \end{pmatrix}.$$

where h_γ and v_γ are scalar components of h - and v -connection vectors h_i and v_i respectively, i.e.

$$(2.9) \quad h_i = h_\gamma e_{\gamma i}, \quad v_i = v_\gamma e_{\gamma i}$$

The first scalar component v_1 of v -connection vector v_i vanishes identically⁴.

If $T_{\alpha\beta\gamma,\delta}$ be scalar components of $T_{jk|ih}^i$ then we have

$$(2.10) \quad T_{\alpha\beta\gamma,\delta} = (\delta_k T_{\alpha\beta\gamma}) e_{\delta}^k + T_{\mu\beta\gamma} H_{\mu\alpha\delta} + T_{\alpha\mu\gamma} H_{\mu\beta\delta} + T_{\alpha\beta\mu} H_{\mu\gamma\delta},$$

where, $\delta_k = \partial_k - G_k^r \partial_r$

Similarly if $T_{\alpha\beta\gamma,\delta}$ be scalar component of $L T_{jk|ih}^i$ then

$$(2.11) \quad T_{\alpha\beta\gamma,\delta} = L(\partial_k T_{\alpha\beta\gamma}) e_{\delta}^k + T_{\mu\beta\gamma} V_{\mu\alpha\delta} + T_{\alpha\mu\gamma} V_{\mu\beta\delta} + T_{\alpha\beta\mu} V_{\mu\gamma\delta}.$$

The scalar components $C_{\alpha\beta\gamma\delta}$ of $L C_{hijk}$ are given by⁴

$$(2.12) \quad \begin{cases} C_{1\beta\gamma,\delta} = 0, & C_{222\delta} = H_{,\delta} + 3Jh_\delta, \\ C_{223\delta} = -J_{,\delta} + (H - 2I)h_\delta, \\ C_{233\delta} = I_{,\delta} - 3Jh_\delta, & C_{333\delta} = J_{,\delta} + 3Ih_\delta. \end{cases}$$

where $H_{,\delta}$ for instance is the h -scalar derivative of H i.e. $H_{,\delta} = (\delta_i H) e_{\delta}^i$.

On the other hand, the scalar components $C_{\alpha\beta\gamma\delta}$ of LC_{hijk} is completely symmetric therefore it gives

$$(2.13) \quad \begin{cases} (H-2I)v_2 - 3Jv_3 = J_{,2} + H_{,3}, \\ 3Jv_2 + (H-2I)v_3 = I_{,2} = J_{,3}, \\ 3Iv_2 + 3Jv_3 = -J_{,2} + I_{,3}. \end{cases}$$

We quote the following theorem which has been proved by M. Matsumoto⁴ in page 197.

Theorem (2.1). *The h-curvature tensor R_{hijk} of CF in any three dimensional Finsler space is written in the form*

$$(2.14) \quad R_{hijk} = Q_{(jk)} \{ g_{ij} L_{ik} + g_{ik} L_{hj} \}$$

where L_{ij} is defined as

$$(2.15) \quad L_{ij} = R_{ij} - \frac{R}{4} g_{ij}.$$

R_{ij} is the h-Ricci tensor defined by $R_{ij} = R_{ihj}^h$ and $R = R_{ij} g^{ij}$.

Here $Q_{(jk)}$ denote the interchange of indices j, k and subtraction.

The scalar component $L_{\alpha\beta}$ of L_{ij} are given by

$$(2.16) \quad L_{\alpha\beta} = \frac{I}{2} \delta_{\alpha\beta} * R_{\mu\mu} - * R_{\beta\alpha}.$$

where $*R_{\alpha\beta}$ are scalar componets of $*R^{rs}$ and $*R^{rs}$ is given by $R_{hijk} \cdot = \epsilon_{hir} \epsilon_{jks} *R^{rs}$.

3. Constant Unified Main Scalar

Contracting (2.2) with g^{jk} , we get

$$(3.1) \quad LC_i = (H+1)m_i,$$

which shows that the unified main scalar LC is equal to $H+1$.

We now assume that the unified main scalar LC is non-zero constant then by addition of first and third equations of (2.13) we get, $(H+1)v_2 = 0$, which gives $v_2 = 0$. Hence we have the following:

Proposition (3.1). *In a three dimensional Finsler space with non-zero constant unified main scalar the v -connection vector v_i may be written as $v_i = v_3 n_i$.*

A Finsler space is called a Landsberg space, if the Berwald connection $B\Gamma$ is h -metrical which is equivalent to $P_{ijk} = 0$ ⁴.

Since

$$(3.2) \quad P_{hijk} = Q_{(ih)} \{ C_{ijk|h} + C_{hj}^r P_{rik} \},$$

$$(3.3) \quad P_{hikj} = Q_{(hi)} \{ \dot{\partial}_h P_{ijk} + P_{khr} C_{ij}^r \},$$

we have the following.

Proposition (3.2). *A finsler space is a Landsberg space if and only if $C_{ijk|h}$ is completely symmetric tensor.*

If F^3 is a three dimensional Landsberg space then $C_{\alpha\beta\gamma,1} = 0$ and in view of proposition (3.2) it follows that $C_{\alpha\beta\gamma,\delta} = C_{\delta\beta\gamma,\alpha}$ which gives

$$(3.4) \quad \begin{cases} (a) & H_{,1} + 3Jh_1 = 0 \\ (b) & -J_{,1} + (H-2I)h_1 = 0 \\ (c) & I_{,1} - 3Jh_1 = 0 \\ (d) & J_{,1} + 3Ih_1 = 0 \end{cases}$$

$$(3.5) \quad \begin{cases} (a) & (H-2I)h_2 - 3Jh_3 = J_{,2} + H_{,3} \\ (b) & 3Jh_2 + (H-2I)h_3 = I_{,2} + J_{,3} \\ (c) & 3Ih_2 + 3Jh_3 = -J_{,2} + I_{,3}. \end{cases}$$

Now, if unified main scalar $LC(=H+1)$ is non-zero constant then addition of (3.4) b and (3.4) d gives $h_1=0$ and hence $H_{,1} = I_{,1} = J_{,1} = 0$ further more addition of (3.5) a and (3.5) c gives $h_2=0$. Hence from (3.5) it follows that

$$(3.6) \quad -3Jh_3 = J_{,2} + H_{,3}, \quad (H-2I)h_3 = I_{,2} + J_{,3}.$$

If all the main scalar H, I, J are constant then from (3.6) it follows that either h_3 is equal to zero or $J=0$ and $H=2I$. Summarizing these results we get,

Theorem (3.1). *In a three dimensional Landsberg space with non-zero constant unified main scalars we have $h_1 = I_{,1} = J_{,1} = H_{,1} = 0$.*

Theorem (3.2). *In a three dimensional Landsberg space with non-zero constant main scalars either h -connection vector vanishes or $J = 0$ and $H = 2I$.*

The h-covariant derivative of (3.1) gives $LC_{ij} = (H+I)n_i h_j$, where we have used the equation (2.4) and the fact that $L_{ih} = 0$ and $H+I$ is constant. Since in a Landsberg space C_{ij} is symmetric tensor, the above equation gives that the h-connection vector h_i is parallel to n_i and we have the following:

Theorem (3.3). *If the unified main scalar of three dimensional Landsberg space F^3 is non-zero constant then C_{ij} and h-connection vector h_i are represented by*

$$C_{ij} = L^{-1} K n_i n_j, \quad h_i = (H+I)^{-1} K n_i$$

respectively.

The function K which occurs in theorem (3.3) will be called K -scalar and is given by $K = (H+I)h_3$.

Next we consider the Ricci identity⁴ $C_{ijk} - C_{ikj} = -C_{ij} R_{jk}^r - C_{ir} R_{jk}^r$ substituting equation (2.14) into this identity and paying attention to equation (2.4) and theorem (3.3) we get,

$$(3.7) \quad L^{-1} [K_k n_j - K_j n_k - K^2 (H+I)^{-1} (m_j n_k - m_k n_j)] n_i \\ = -C_r (g_{ij} L_k^r + \delta_k^r L_{ij} - g_{ik} L_j^r - \delta_j^r L_{ik}) - C_{ir} (y_j L_k^r + \delta_k^r L_{oj} - y_k L_j^r - \delta_j^r L_{ok}),$$

where we have used the fact that $R_{jk}^r = g^{ri} R_{hijk} y^h = R_{hijk}^r y^h$ and $L_k^r = g^{jr} L_{jk}$.

Since $C_i = C m_i$, in view of (2.5) and proposition (3.1) its v-covariant derivatives gives

$$(3.8) \quad C_{ir} = CL^{-1} [v_3 n_i n_r - m_i l_r - l_i m_r].$$

Since $L_{\alpha\beta}$ are scalar components of L_{ij} and $L_{\alpha\beta}$ are given by equation (2.16) we may write

$$(3.9) \quad L_{ij} = \frac{1}{2} (*R_{22} + *R_{33} - *R_{11}) l_i l_j + \frac{1}{2} (*R_{11} + *R_{33} - *R_{22}) m_i m_j \\ + \frac{1}{2} (*R_{11} + *R_{22} - *R_{33}) n_i n_j - *R_{21} l_i m_j - *R_{12} m_i l_j \\ - *R_{31} l_i n_j - *R_{13} n_i l_j - *R_{32} m_i n_j - *R_{23} n_i m_j.$$

Now we assume that K is constant then from (3.7), (3.8) and (3.9) we get

$$(3.10) \quad \begin{cases} (a) & K^2 = -(H+I)^2 (*R_{11} + v_3 *R_{21}), \\ (b) & *R_{12} + *R_{22} v_3 = 0, \\ (c) & *R_{13} + *R_{23} v_3 = 0. \end{cases}$$

From these results, we have the following:

Theorem (3.4). *Let F^3 be a three dimensional Landsberg space. If the unified main scalar of F^3 is non-zero constant and the K-scalar of F^3 is constant, then $(*R_{11} + v_3 *R_{21})$ is constant and the equation (3.10) holds.*

Now we consider the Ricci identity⁴:

$$C_{ijk} - C_{ikj} = -C_r P_{ijk}^r - C_{ir} C_{jk}^r - C_{ir} P_{jk}^r.$$

If F^3 is a Landsberg space then $P_{ijk}^r = 0 = P_{jk}^r$. Therefore, in view of equation (2.2) and theorem (3.3), the above Ricci identity reduces to

$$(3.11) \quad C_{ijk} - C_{ikj} = L^{-1} K \{ J n_i m_j m_k - I n_i (n_j m_k + m_j n_k) - J n_i n_j n_k \}.$$

Now assuming the K-scalar to be a constant, in view of equations (2.4), (2.5), (3.8) and theorem (3.3), the L.H.S. of equation (3.11) may be written as

$$(3.12) \quad C_{ijk} - C_{ikj} = -L^{-2} K [n_i n_j l_k + l_i n_j n_k + n_i l_j n_k + (m_i n_j + n_i m_j) v_k] \\ + L^{-2} K [m_i n_j l_k v_3 + n_i n_j l_k + l_i n_j l_k] - (H + I) L^{-2} n_i v_{kj}.$$

Comparing equations (3.11) and (3.12), we get

$$(3.13) \quad \begin{cases} (H+I)v_{2,2} = -LKJ \\ (H+I)v_{2,3} = LKI \\ (H+I)v_{3,2} = LKJ - KJ \\ (H+I)v_{3,3} = LKJ \\ (H+I)v_{3,I} = -K \end{cases}$$

Since $v_{2,2} = 0$ and $h_3 = (H+I)^{-1}K$ equation (3.13) gives

$$J = 0, \quad v_{3,3} = 0, \quad v_{3,2} - v_{2,3} = h_3 v_3.$$

Hence we have the following.

Theorem (3.5). *Let F^3 be a three dimensional Landsberg space. If the unified main scalar of F^3 is non-zero constant and the K-scalar of F^3 is constant then $J = 0$, $v_{3,3} = 0$, $v_{3,2} - v_{2,3} = h_3 v_3$.*

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