

Bayesian Estimation of Reliability of Generalized Burr Life Time Distribution under Precautionary Loss Function with Censoring

Narayan Kumar Joshi

Department of Mathematics & Statistics, Dr. R.M.L. Avadh University, Faizabad. 224001.

(Received February 9, 2004)

Abstract. In this paper we have obtained the Bayesian estimation of reliability of generalized Burr distribution with the help of a censored sample. The loss functions used are precautionary as well as squared error loss functions. These estimators are derived considering non-informative and beta prior distributions.

1. Introduction

Reliability and life testing is very fast growing and very important field in consumer and capital goods industry, in space and defence industries. Reliability and life testing provide the theoretical and practical tools whereby it can be ascertained that the probability and capability of part, component, products, and item to perform their required functions in pacified environments for the desired function period without failure, or the desire or specified reliability and life have indeed been designed and manufactured in them. The life time distribution used to model the situation is of prime importance. In recent past the generalized Burr distribution has been found to be an excellent model to represent a life time distribution in many situations.

The two parameter generalized Burr (type XII) distribution with probability density function (pdf)

$$(1.1) \quad f(x; c, k) = c k x^{c-1} (1 + X^c)^{-(k+1)} ; \quad (c, k) > 0, 0 < x < \infty$$

and its distribution function (cdf) is

$$(1.2) \quad F(x; c, k) = 1 - (1 + X^c)^{-k} ; \quad (c, k) > 0, 0 < x < \infty$$

This was first introduced in literature by Burr¹. The Burr distribution is widely used in areas such as business, engineering, reliability, hydrology and mineralogy as a failure model and its properties have been studied by Burr and Cislak². The Burr (c,k) distribution was proposed as a life time model by Dubey^{3,4}. Rodriguez⁵ and Tadikamalla⁶. They show that the Burr distribution covers the curve shape characteristics for the Normal, Logistic

and exponential (Pearson type X) distribution as well as a significant portion of the curve shape characteristic for Pearson type I (beta), II, III (gamma), V, VII, IX and XII distributions. Lewis⁷ noted that the Weibull and exponential distribution are special limiting cases of the parameter values of the Burr distribution.

Wingo^{8,9} have considered the M.L. method for inferential problems in Burr XII distribution. Hossain and Shyamal¹⁰ studied the estimation procedures in presence of outliers. Popadopoulos¹¹, Evans and Ragab¹², Lingappaiah¹³, Al-Hussaini and Jaheen¹⁴ have used Bayesian approach for obtaining the inference about this distribution.

Let us reparameterize the c.d.f. (1.2) to get the c.d.f. of generalized Burr distribution in the following form

$$(1.3) \quad F(x) = 1 - (1 + x^c)^{-\frac{1}{\theta}} \quad ; \quad (c, \theta) > 0, \quad 0 < x < \infty.$$

whose probability density function (pdf) comes out to be

$$(1.4) \quad f(x; c, \theta) = \frac{c}{\theta} x^{c-1} (1 + x^c)^{-\left(\frac{1}{\theta} + 1\right)} \quad ; \quad c, \theta > 0, \quad x > 0$$

where c and θ are the shape parameters. This reparametrization leads to mathematical tractability in calculations. One can easily verify that

$$(1.5) \quad \text{Mean}(\mu_1) = \frac{1}{\theta} B \left[\frac{1}{\theta} - \frac{1}{c}, \left(1 + \frac{1}{c}\right) \right],$$

and

$$(1.6) \quad \text{Variance}(\mu_2) = \frac{1}{\theta} B \left[\frac{1}{\theta} - \frac{2}{c}, \left(1 + \frac{2}{c}\right) \right] - \left\{ \frac{1}{\theta} B \left[\frac{1}{\theta} - \frac{1}{c}, \left(1 + \frac{1}{c}\right) \right] \right\}^2,$$

and it exists for $(c/\theta) > 2$

For generalized Burr distribution, the reliability function is

$$(1.7) \quad R(t) = (1 + t^c)^{-\frac{1}{\theta}} \quad ; \quad \theta > 0, \quad t > 0,$$

and the instantaneous failure rate or hazard rate is given by

$$(1.8) \quad h(t) = \frac{c}{\theta} t^{c-1} (1+t^c)^{-1} \quad ; \quad t > 0.$$

The Bayesian estimation procedures for reliability function have been developed generally under squared error loss function

$$(1.9) \quad L(\hat{R}(t), R(t)) = (\hat{R}(t) - R(t))^2$$

which is a symmetrical loss function and assigns equal losses to over estimation and underestimation. Canfield¹⁵ points out that the use of symmetric loss function may be inappropriate in the estimation of reliability function. Over estimation of reliability function or average lifetime is usually much more serious than underestimation of reliability function or mean failure time. Also, an underestimate of the failure rate results in more serious consequences than an overestimation of the failure rate. This led to statisticians to think about asymmetrical loss functions which have been proposed in statistical literature.

Norstrom¹⁶ introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss function with quadratic loss function as a special case. We have used this loss function for the purpose of Bayesian estimation of reliability in this paper.

A very useful and simple asymmetric precautionary loss function is

$$(1.10) \quad L(\hat{R}(t), \hat{R}(t)) = \frac{(R(t) - \hat{R}(t))^2}{\hat{R}(t)}$$

Let E_g and E_π denote the prior and posterior expectations respectively. The posterior expectation of loss function in (1.10) is

$$(1.11) \quad E_\pi [L(R(t), \hat{R}(t))] = E_\pi \left(\frac{(R(t))^2}{\hat{R}(t)} \right) + E_\pi(\hat{R}(t)) - 2E_\pi(R(t)).$$

The value of $\hat{R}(t)$ that minimises (1.11), denoted by $\hat{R}(t)_p$, Bayes estimator of $R(t)$ under precautionary loss function is obtained by solving the following equation

$$\frac{d}{d\hat{R}(t)} E_\pi \left[\frac{(R(t))^2}{\hat{R}(t)} \right] = 0$$

which implies

$$(1.12) \quad \hat{R}(t)_p = \left[E_\pi(R(t))^2 \right]^{1/2}.$$

It may be noted that $E_{\pi}(R(t))$ is the Bayes estimator under squared error loss function.

Let us consider two prior distributions of $R(t)$ to obtain the Bayes estimators of reliability which are given by

(i) **Non-informative Prior:** For the situation where the experimenter has no information about $R(t)$, he may use the non-informative prior distribution

$$(1.13) \quad h_1(R(t)) = \frac{1}{R(t) \log R(t)} \quad ; \quad 0 < R(t) \leq 1.$$

(ii) **Beta Prior:** The most widely used prior distribution for $R(t)$ is a beta distribution with parameters $m, n > 0$, with density function given by

$$(1.14) \quad h_2(R(t)) = \frac{1}{B(\mu, \nu)} [R(t)]^{\mu-1} [1-R(t)]^{\nu-1} \quad ; \quad 0 < R(t) \leq 1.$$

The object of the present paper is to obtain Bayesian estimator of Reliability $R(t)$ using two prior distributions under Precautionary loss function with the help of item censored (type II) sample.

2. Bayesian estimation of the Reliability function of the generalized Burr distribution

Let $R(t)$, the reliability function at time 't', that is the probability that a system will survive a specified time 't',

$$R(t) = (1+t^c)^{-\frac{1}{\theta}} \quad ; \quad \theta > 0, t > 0,$$

which implies

$$(2.1) \quad \frac{1}{\theta} = \frac{-\log R(t)}{\log(1+t^c)}$$

Substituting the value of $1/\theta$ from equation (2.1) in equation (1.4), we get

$$(2.2) \quad f(x; c, R(t)) = c \left\{ \frac{-\log R(t)}{\log(1+t^c)} \right\} x^{c-1} (1+x^c)^{-\left(\frac{-\log R(t)}{\log(1+t^c)} + 1 \right)} \quad ; \quad 0 < R(t) < 1$$

Let us suppose that n items are put to life test and terminate the experiment when $m (< n)$ items have failed. If $x_1, \dots, x_m$ denote the first m observations having a

common density function as given in (1.4) with c the shape parameter is known. The joint probability density function is given by

$$f(\underline{x} | R(t)) = \frac{n!}{(n-m)!} \left\{ \frac{-c \log R(t)}{\log(1+t^c)} \right\}^m \left(\prod_{i=1}^m x_i^{c-1} \right) e^{\left(\frac{\log R(t)}{\log(1+t^c)} - 1 \right) z_m};$$

where

$$(2.3) \quad z_m = \left[\sum_{i=1}^m \log(1+x_i^c) + (n-r) \log(1+x_m^c) \right]$$

Thus, the maximum likelihood estimator of $R(t)$ comes out to be

$$(2.4) \quad \hat{R}(t) = \left(1 + t^c \right)^{-\frac{m}{z_m}}.$$

2.1 Bayes Estimator of $R(t)$ under $h_1(R(t))$: Under $h_1(R(t))$, the posterior distribution becomes

$$(2.5) \quad f(R(t) | \underline{x}) = \frac{u^m}{\Gamma(m)} [R(t)]^{u-1} [-\log R(t)]^{m-1}$$

where

$$u = \frac{Z_m}{\log(1+t^c)}.$$

The Bayes estimator of $R(t)$ under squared error loss function is the posterior mean

$$(2.6) \quad \hat{R}(t)_s = \left(1 + \frac{1}{u} \right)^{-m}$$

The Bayes estimator of $R(t)$ under precautionary loss function $\hat{R}(t)_p$ using the value of $f(R(t) | \underline{x})$ from equation (2.3), is the solution of equation (1.12) given by

$$(2.7) \quad \hat{R}(t)_p = \left(1 + \frac{l}{u}\right)^{-m/2}$$

In this case the risk functions and Bayes risks are not easy to calculate. A simulation study is recommended in order to obtain the risk functions.

2.2 Bayes Estimator under $h_2(R(t))$: Under $h_2(R(t))$, the posterior distribution is

$$(2.8) \quad f(R(t)|\underline{x}) = \frac{[R(t)]^{u+\mu-l} [-\log R(t)]^m [1-R(t)]^{v-l}}{\Gamma(m+1) \left[\sum_{i=0}^{v-l} (-1)^i \binom{v-l}{i} \left(\frac{l}{u+\mu+i} \right)^{m+l} \right]} ; \quad 0 < R(t) \leq 1.$$

The Bayes estimator of $R(t)$ under squared error loss function is the posterior mean

$$(2.9) \quad \hat{R}(t)_s = \frac{\left[\sum_{j=0}^{v-l} (-1)^j \binom{v-l}{j} \left(\frac{l}{u+\mu+j+1} \right)^{m+l} \right]}{\left[\sum_{i=0}^{v-l} (-1)^i \binom{v-l}{i} \left(\frac{l}{u+\mu+i} \right)^{m+l} \right]}$$

They Bayes estimator of $R(t)$ under precautionary loss function $\hat{R}(t)_p$ using the value of $f(R(t)|\underline{x})$ from equation (2.8), is the solution of equation (1.12) given by

$$(2.10) \quad \hat{R}(t)_p = \left[\frac{\sum_{j=0}^{v-l} (-1)^j \binom{v-l}{j} \left(\frac{l}{u+\mu+j+2} \right)^{m+l}}{\sum_{i=0}^{v-l} (-1)^i \binom{v-l}{i} \left(\frac{l}{u+\mu+i} \right)^{m+l}} \right]^{\frac{l}{2}}$$

In this case the risk functions and corresponding Bayes risks are not obtainable in closed forms.

References

1. I.W. Burr: Cumulative frequency functions. *Ann. Math. Statist.*, **13** (1942) 215-222.
2. I.W. Burr and P.J. Cislak: On the general system of distributions; its curve-shape characteristic: II. The sample median, *J. Amer. Statist. Assoc.*, **63** (1968) 627-638.
3. S.D. Dubey: Statistical contributions to reliability engineering, ARLTR 72-0120. AD 751261 (1972).
4. S.D. Dubey: Statistical treatment of certain life testing and reliability problems, ARLTR 73-0155. AD751261 (1973).
5. R.N. Rodriguez: A guide of the Burr type XII distribution. *Biometrika*, **64** (1977) 129-134.
6. P.R. Tadikamalla: A look at the Burr and related distributions. *International Statistical Rev.*, **48** (1980) 337-340.
7. A.W. Lewis. : The Burr distribution as a general parametric family in survivorship and reliability theory applications. Ph.D. thesis: Department of Biostatistics, University North Carolina. 1981.
8. D.R. Wingo: Maximum likelihood methods for fitting the Burr type XII distribution to life test data, *Biometricsl. J.*, **25** (1983) 77-84.
9. D.R. Wingo: Maximum likelihood estimation of Burr XII type distribution parameters under type II, *Censoring. Microelectron. Reliab.*, **33** (1993) 1251-1257.
10. A. Hossain and K.N. Shyamal: Estimation of parameter in the presence of outliers for a Burr XII, *Distributions Comm. Statist. Theo. & Meth.*, **26(3)** (1997) 637-652.
11. A.S. Papadopoulos: The Burr distribution as a failure model from Bayesian approach. *IEEE Trans. on Rel. R.*, **27** (1978) 369-371.
12. I.G. Evans and A.S. Ragab: Bayesian inference given a type II censored sample from a Burr distribution, *Comm. Statist. Theo. & Meth.*, **12** (1983) 1569-1580.
13. G.S. Lingappaiah: Bayesian approach to the estimation of the parameters in the Burr's distribution with outliers. *J. Orissa Math. Soc.*, **1** (1983) 55-59.
14. E.K. Al-Hussaini and Z.F. Jaheen: Bayesian estimator of the parameters. reliability and failure rate functions of the Burr type XII failure model. *J. Statist Comput. Simul.*, **42** (1992) 31-40.
15. R.V. Canfield: A Bayesian approach to reliability estimation using a loss function, *IEEE Trans. Rel.*, **R-19** (1970) 13-16.
16. J.G. Norstrom: The use of precautionary loss functions in risk analysis, *IEEE Trans. Reliab.*, **45(3)** (1996) 400-403.

