

The Family of Cycloid and Tractrix as Geodesics in a Two Dimensional Finsler Space

T.N. Pandey and B.K. Tripathi

Department of Mathematics & Statistics, D.D.U. Gorakhpur University, Gorakhpur.

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1. Introduction

In 1980 S. Hojo¹ gave five Finsler metrics in $\mathbb{R}^2 - \{0\}$, all the geodesics of which are logarithmic spirals with the pole O and he showed a sufficient condition for a Finsler space to have such special geodesics. M. Matsumoto²⁻⁴ found the Finsler metric function of a two dimensional Finsler space whose geodesics are given as two parameter family of curves. He considered parameteric and non-parameteric equations of curves and found the metric function in both the cases. The metric function thus obtained contains two arbitrary functions of coordinates. The solution thus obtained was theoretical and therefore he gave some concrete examples.

The purpose of the present paper is to find the metric function of a two-dimensional Finsler space whose geodesics are two-parameter family of cycloids and tractrix, to emphasize the theoretical solution obtained by M. Matsumoto.

2. The Metrizable Problem

L.P. Eisenhart, T.Y. Thomas and O. Veblen⁵ were concerned with the metrizable problem of affine paths: given the equations of paths

$$(2.1) \quad \frac{d^2 x^i}{dt^2} + \Gamma_{jk}^i(x) \frac{dx^j}{dt} \frac{dx^k}{dt} = 0$$

in a differentiable manifold M^n of n -dimensions, find the condition for the existence of a Riemannian space whose geodesics coincide with the paths.

This problem may also be stated as follows: given a linear connection $\Gamma = \{\Gamma_{jk}^i(x)\}$ in M^n find a Riemannian metric tensor $g_{ij}(x)$ on M^n such that $\Gamma_{jk}^i(x)$ are the Christoffel

symbols of the second kind constructed from $g_{ij}(x)$. Thus, one is led to the differential equations

$$(2.2) \quad \nabla_k g_{ij} = \partial_k g_{ij} - g_{rj} \Gamma_{ik}^r - g_{ir} \Gamma_{jk}^r = 0,$$

for the Riemannian metrization, where $\Gamma_{jk}^i(x)$ are assumed to be symmetric in j and k .

In 1954, O. Varga⁶ considered the Finslerian metrization problem based on the famous Cartan connection (the "Levi-Civita connection" of Finsler geometry). Given connection $C\Gamma = \{\Gamma_{jk}^i(x, y), C_{jk}^i(x, y)\}$ in the tangent bundle TM^n find a Finsler space $F^n = (M^n, L(x, y))$ such that $C\Gamma$ coincides with the Cartan connection, this being determined from the fundamental tensor $g_{ij}(x, y) = \partial_i \partial_j (L^2 / 2)$ where $\partial_i = \partial / \partial y^i$. Therefore, Varga was led to the system of differential equations similar to (2.2):

$$(2.3) \quad \nabla_k^x g_{ij} = \delta_k g_{ij} - g_{rj} \Gamma_{ik}^{*r} - g_{ir} \Gamma_{jk}^{*r}$$

$$\nabla_k^y g_{ij} = \delta_k g_{ij} - g_{rj} \Gamma_{ik}^{*r} - g_{ir} \Gamma_{jk}^{*r}$$

for the Finslerian metrization, where $\delta_k = \partial_k - (y^s \Gamma_{sk}^{*r}) \partial_r$.

Twenty years later, L. Tamassy⁷ dealt with the problem above with $g_{ij}(x, y)$, together with its reciprocal $g^{ij}(x, y)$, but his result seems to us a little simpler than that of Varga. On the other hand, A. Rapcsak⁸ considered the same problem in another form: given the equations of generalized paths

$$(2.4) \quad \frac{d^2 x^i}{dt^2} + G^i(x, \frac{dx}{dt}) = 0.$$

find a Finsler space F^n such that $G_{jk}^i(x, y) = \partial_j \partial_k G^i(x, y)$ coincide with the connection coefficients of the Berwald connection $B\Gamma = \{G_{jk}^i(x, y)\}$ constructed from the

fundamental function $L(x,y)$. Here G^i is assumed positively homogeneous of degree two in y^i .

It follows from Euler's theorem on homogeneous functions that (2.4) can be written in the form

$$(2.4') \quad \frac{d^2 x^i}{dt^2} + G_{jk}^i(x, \frac{dx}{dt}) \frac{dx^j}{dt} \frac{dx^k}{dt} = 0.$$

If $G_{jk}^i(x,y)$ do not depend on y^i , then the space is called a Berwald space. Riemannian spaces are thus Berwald spaces. Rapcsak dealt, however, with the Finslerizability problem in the wider sense.

Definition (2.1) Two path spaces are called projectively related to each other, if their paths coincide with each other as point sets.

This is equivalent to the time-sequencing invariance mentioned above. As is well known⁹ the necessary and sufficient condition for path space P^n and P^n to be projectively related is the existence of a function $P(x,y)$ called the projective factor satisfying.

$$(2.5) \quad G^{-i}(x,y) = G^i(x,y) + P(x,y)y^i.$$

This defined a new time parameters $ds/dt = \exp\{-\int_{\lambda} p(x,y)dt\}$ along any path γ . Thus the metrizability problem becomes;

Rapcsak's Problem: Given a path space P^n with $\{G_{jk}^i(x,y)\}$, find a Finsler space, which is projectively related to it.

Definition (2.2). Let us consider the set of all Finsler spaces on the same underlying manifold M^n . This set is divided into subsets, in every one of which the member Finsler spaces are projectively related to each other. Each one of such subsets is called a projective equivalence class of Finsler spaces. The totality is denoted by $F_p(M^n)$.

Let us consider a coordinate neighbourhood of a two-dimensional Finsler space. Since our problem is of local character, it is enough to consider a Finsler space $F^2 = (R^2, L(x,y, \dot{x}, \dot{y}))$ defined on the (x,y) -plane R^2 . Throughout the following we use the symbol (p,q) instead of $(\dot{x}, \dot{y})$, generally the quantities $G^i(x,y,p,q)$ in (2.4) are assumed to

be positively homogeneous of degree two in (p, q) . Since it is possible to regard $p > 0$ in our consideration, by suitable orientation of the parameter t , we have $G^i(x, y, p, q) = p^2 G^i(x, y, l, q/p)$. Then (2.4) gives

$$\frac{d^2 y}{dx^2} = y'' = \frac{p\dot{q} - q\dot{p}}{p^3} = 2y' G^1(x, y, l, y') - 2G^2(x, y, l, y')$$

Consequently, we get the equation of paths in the normal form

$$(2.6) \quad y'' = F(x, y, y') = Y_3(y')^3 + Y_2(y')^2 + Y_1(y') + Y_0,$$

$$\text{where} \quad Y_3 = G_{22}^1, \quad Y_2 = 2G_{22}^1 - G_{22}^2, \quad Y_1 = G_{11}^1 - 2G_{12}^2, \quad Y_0 = -G_{11}^2.$$

Thus $F(x, y, y')$ is a polynomial of at most third order in y' in the case of linear connection. Further, the affine parameter t does not appear in the form (2.6).

For a general parameter τ , the equations of paths are written in the form

$$\frac{d^2 x^i}{d\tau^2} + G^i(x, \frac{dx}{d\tau}) = h(\tau) \frac{dx^i}{d\tau}$$

Then the affine or natural parameter t in (2.4) is given by $dt/d\tau = \exp \left\{ \int h(\tau) d\tau \right\}$. Therefore

(2.6), which is our main concern in the present paper, is the equation of paths as a point set. In fact (2.5) shows immediately that we get the same differential equation (2.6) for two-dimensional path spaces, which are projectively related to each other. That is

Lemma (2.1). *In two-dimensional case to each system (2.6) of geodesics as point sets, there corresponds a projective equivalence class $F_p(R^2)$ of Finsler spaces³.*

3. The Inverse Correspondence

Since the differential equation system (2.6) of geodesics is of the normal form, we have the well-known Cauchy-Lipschitz's theorem: a solution of (2.6) is uniquely by the initial conditions, if the function $F(x, y, z)$ satisfies Lipschitz's conditions. Therefore, (2.6) leads to the finite equation

$$(3.1) \quad y = f(x, a, b)$$

of geodesics $C(a,b)$ as a point set with two constant parameters a and b .

It may be difficult to get the finite equation (3.1). But it may be possible to write it down in parametric form

$$(3.2) \quad x = \phi(t, a, b), \quad y = \psi(t, a, b)$$

with a parameter t . In either case the existence of the general solutions is certain.

In the present section, we find the fundamental function $L(x,y,p,q)$ of $F^2=(R^2, L(x,y,p,q))$ if the finite equation (3.2) is a geodesic. In the parametric case (3.2) we are concerned with the equation of geodesics in the Weierstrass form²

$$W(C) = L_{xq} - L_{yp} + (p\dot{q} - q\dot{p})W = 0,$$

where $W = \frac{L_{pp}}{q^2} = \frac{-L_{pq}}{pq} = \frac{L_{qq}}{p^2}$ is the Weierstrass invariant. Since $L(x,y,p,q)$ is (1)P-

homogeneous (abbreviation of positively homogeneous of degree 1 in (p,q)), it is necessary to pay attention to the homogeneity of functions appearing in the following process.

We suppose an auxiliary positive parameter c such that

$$(3.3) \quad x = \phi(ct, a, b), \quad y = \psi(ct, a, b)$$

We use the following notation $\dot{\phi} = \phi_t(ct, a, b)$, $\dot{\psi} = \psi_t(ct, a, b)$. Then (3.3) gives

$$(3.4) \quad p/c = \dot{\phi}(ct, a, b), \quad q/c = \dot{\psi}(ct, a, b)$$

From (3.3) and (3.4) we obtain a, b, c and t as functions of (x,y,p,q) :

$$a = \alpha(x, y, p, q) \quad b = \beta(x, y, p, q) \quad c = \gamma(x, y, p, q) \quad t = \tau(x, y, p, q)$$

These functions α, β, γ and τ are (0),(0),(1) and (-1) p-homogeneous, respectively.

From (3.4) we get

$$\frac{\dot{p}}{c^2} = \ddot{\phi}(ct, a, b), \quad \frac{\dot{q}}{c^2} = \ddot{\psi}(ct, a, b),$$

which implies

$$\dot{p} = \ddot{\phi}(\gamma\tau; \alpha, \beta)\gamma^2 = P(x, y, p, q) \quad \dot{q} = \ddot{\psi}(\gamma\tau; \alpha, \beta)\gamma^2 = Q(x, y, p, q),$$

where P and Q are (2) p-homogeneous functions. Further we consider

$$\pi(x, y, p, q) = P_p + Q_q, \quad \prod(v; a, b) = \pi(\phi(v; a, b), \psi(v; a, b), \ddot{\phi}(v; a, b), \ddot{\psi}(v; a, b))$$

where $v = c(t+t_0)$. We define

$$U(v; a, b) = \exp \int \prod(v; a, b) dv \quad V(x, y, p, q) = U(\gamma\tau; \alpha, \beta).$$

Where, V is (0)p-homogeneous function. Consequently, We get

$$(3.4) \quad W(x, y, p, q) = \frac{H(\alpha, \beta)}{\gamma^3 V(x, y, p, q)}, \quad L^*_1 = q^2 \int \int W(dp)^2, \quad L^*_2 = p^2 \int \int W(dq)^2.$$

Thus we get fundamental function

$$(3.5) \quad L(x, y, p, q) = L^*(x, y, p, q) + C(x, y)p + D(x, y)q,$$

where $L^* = L^*_1$ or L^*_2 and C and D are arbitrary functions of (x, y) satisfying

$$(3.6) \quad C_y - D_x = L^*_{xp} - L^*_{yp} + (pQ - qp)W.$$

In the integrations of (3.5) we have to show that L^* becomes (1)p-homogeneous function. For instance, let us consider (r)p-homogeneous function in (x, y). From $f(x, y) = y^r f(x/y, 1)$ we get

$$g(x, y) = \int f(x, y) dx = y^r \int \{f(t, 1)y\} dt \quad \text{where } t = x/y.$$

If we put $h(t) = \int f(t, 1) dt$, then $h(t)$ is (0) p-homogeneous, hence $g(x, y) = y^{r+1} h(x/y)$ is (r+1)p-homogeneous.

The functions $C(x, y)$ and $D(x, y)$ must satisfy (3.6). Let a pair $(C^0(x, y), D^0(x, y))$ be chosen as to satisfying (3.6). Then we have $(C - C^0)_y = (D - D^0)_x$ for a general (C, D), hence we have such a function $E(x, y)$ that $C - C^0 = E_x$ and $D - D^0 = E_y$. Consequently, $L(x, y, p, q)$ is written in the form

$$(3.6) \quad L(x, y, p, q) = L^*(x, y, p, q) + (C^0 + E_x)_p + (D^0 + E_y)_q.$$

Therefore, the metrics of Finsler spaces belonging to a projectively equivalent class we determined with the two functions $H(\alpha, \beta)$ and $E(x, y)$ being arbitrary.

Theorem (3.1). *In two-dimensional case the solutions of the second order differential*

equations (2.6) of geodesics, as point sets, corresponding to each given projective equivalence class $F_p(R^2)$ of Finsler spaces whose geodesics are given by (2.6) as point set³.

4. Family of cycloids

We consider the family of cycloid $\{C(a,b)\}$ in parametric form of the equations

$$x = a(t + \sin t) + b, \quad y = a(1 + \cos t),$$

which are written in the form

$$(4.1) \quad x = a(ct + \sin ct) + b, \quad y = a(1 + \cos ct),$$

where a, b, c are constant and t is parameter. Differentiating (4.1) with respect to t we get

$$(4.2) \quad p = \dot{x} = ac(1 + \cos ct), \quad q = \dot{y} = -ac \sin ct$$

Solving (4.1) and (4.2) we get

$$(4.3) \quad \begin{cases} a = \alpha(x, y, p, q) = \frac{(p^2 + q^2)y}{2p^2} \lambda, \\ b = \beta(x, y, p, q) = (1/p^2) \{ xp^2 + ypq + y(p^2 + q^2) \tan^{-1}(q/p) \} \lambda, \\ c = \gamma(x, y, p, q) = p/y, \quad t = \tau(x, y, p, q) = -(2y/p) \tan^{-1}(q/p), \end{cases}$$

Thus functions α, β, γ and τ are $(0), (0), (1)$ and $(-1)p$ -homogeneous, respectively.

Again differentiating (4.2) we get

$$(4.4) \quad \dot{p} = -ac^2 \sin ct, \quad \dot{q} = -ac^2 \cos ct.$$

From (4.2) and (4.3) we get

$$(4.5) \quad \dot{p} = \frac{pq}{y} = P(x, y, p, q), \quad \dot{q} = \frac{1}{2y}(q^2 - p^2) = Q(x, y, p, q).$$

From (4.5) we have

$$(4.6) \quad \pi(x, y, p, q) = P_p + Q_q = 2q/y.$$

Using (4.1), (4.2) and (4.3) in (4.6) we get

$$(4.7) \quad \Pi(v; a, b) = -\frac{2 \sin v}{1 + \cos v}.$$

Where $v = c(t + t_0)$. Consequently, we have

$$(4.8) \quad U(v; a, b) = (1 + \cos v)^2, \quad V(x, y, p, q) = \frac{4p^4}{(p^2 + q^2)^2}$$

From (3.4), (4.3) and (4.8) we have

$$(4.9) \quad W(x, y, p, q) = \alpha^2 H(\alpha, \beta) \frac{y^3}{p^3} = \frac{y^3}{p^3} \bar{H}(\alpha, \beta)$$

$$\text{where } \bar{H}(\alpha, \beta) = \alpha^2 H(\alpha, \beta) \lambda.$$

Since $H(\alpha, \beta)$ is an arbitrary function, we chose $\bar{H}(\alpha, \beta) = 1$. Then (4.9) becomes

$$(4.10) \quad W(x, y, p, q) = \frac{y^3}{p^3}.$$

Now from (3.4) and (4.10) we get

$$(4.11) \quad L^* = L^*_1 = L^*_2 = \frac{q^2 y^3}{2p} y^2.$$

From (3.6) we get

$$(4.12) \quad C_y - D_x = \frac{(2p^2 - q^2)}{2p^2} y^2.$$

Since $C(x, y)$ and $D(x, y)$ are arbitrary functions, we choose $C = 0$, then (4.12) gives

$$D = \frac{(p^2 - 2q^2)}{2p^2} xy^2.$$

Using (4.11) and (4.12) in (3.5) we get fundamental function

$$(4.13) \quad L(x, y, p, q) = \frac{q^2 y^3}{2p} + \frac{(p^2 - 2q^2)}{2p^2} qxy^2.$$

Theorem (4.1). *The fundamental function $L(x, y, p, q)$ of a Finsler space $(R^2, L(x, y, p, q))$ having the family of cycloid (4.1) as the geodesics is given by (4.13) and such path spaces are projectively related to Finsler spaces.*

5. The Family Of Tractrix

We consider the family of tractrix $\{C(a, b)\}$ in parametric form of the equation

$$x = a \cos t + (a/2) \log \tan^2 (t/2) + b, \quad y = a \sin t.$$

which are written in the form

$$(5.1) \quad x = a \cos ct + (a/2) \log \tan^2 (ct/2) + b, \quad y = a \sin ct.$$

Differentiating (5.1) with respect to t get

$$(5.2) \quad p = \dot{x} = \frac{ac \cos^2 ct}{\sin ct} \quad q = \dot{y} = ac \cos ct.$$

Solving (5.1) and (5.2) we get

$$(5.3) \quad \left[\begin{aligned} a &= \alpha(x, y, p, q) = \frac{\sqrt{p^2 + q^2}}{q} y, \\ b &= \beta(x, y, p, q) = x - \frac{yp}{q} - \frac{\sqrt{p^2 + q^2}}{2q} y \log \tan^2 \{(1/2) \tan^{-1}(q/p)\} \\ c &= \gamma(x, y, p, q) = q^2 / (yp), \quad t = \tau(x, y, p, q) = (yp / p^2) \tan^{-1}(q/p), \end{aligned} \right.$$

Thus functions α, β, γ and τ are $(0), (0), (1)$ and (-1) -homogeneous, respectively. Again differentiating (5.2) we get

$$(5.4) \quad \dot{p} = \frac{-ac^2(1 + \sin^2 ct) \cos ct}{\sin^2 ct} \quad \dot{q} = -ac^2 \sin ct.$$

From (5.5) and (5.3) we get

$$(5.5) \quad \dot{p} = \frac{-(p^2 + 2q^2)q}{yp} = p(x, y, p, q), \quad \dot{q} = \frac{-q^4}{yp^2} = Q(x, y, p, q),$$

From (5.5) we have

$$(5.6) \quad \pi(x, y, p, q) = p_p + Q_q = \frac{-(qp^2 + 2q^3)}{yp^2}.$$

Using (5.1), (5.2) and (5.3) in (5.6) we get

$$(5.7) \quad \Pi(v; a, b) = -\frac{1}{\sin v \cos v},$$

where $v = c(t + t_0)$. Consequently, we have

$$(5.8) \quad U(v; a, b) = \cot v, \quad V(x, y, p, q) = p/q.$$

From (3.4), (5.3) and (5.8) we have

$$(5.9) \quad W(x, y, p, q) = \frac{y^3 p^2}{q^5} H(\alpha, \beta).$$

Since $H(\alpha, \beta)$ is an arbitrary function, we chose $H(\alpha, \beta) = 1$. Then (5.9) becomes

$$(5.10) \quad W(x, y, p, q) = \frac{y^3 p^2}{q^5}$$

From (3.4) and (5.10), we get

$$(5.11) \quad L^* = L_1^* = L_2^* = \frac{p^4 y^3}{12q^3}.$$

from (3.6), we get

$$(5.12) \quad C_y - D_x = \frac{y^2 p}{q}.$$

Since $C(x,y)$ and $D(x,y)$ are arbitrary functions, we choose $C = 0$, then (5.12) gives $D = -xy^2 p/q$

Using (5.11) and (5.12) in (3.5) we get fundamental function

$$(5.13) \quad L(x, y, p, q) = \frac{p^4 y^3}{12q^3} - xy^2 p.$$

Theorem (5.1). *The fundamental function $L(x,y,p,q)$ of a Finsler space $(R^2, L(x,y,p,q))$ having the family of tractrix (5.1) as the geodesics is given by (5.13) and such path spaces are projectively related to Finsler spaces.*

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