

On Certain Special Finsler Spaces of Dimension Four

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Abstract: L. Berwald¹⁻³ developed the theory of two dimensional Finsler spaces using a special orthonormal frame. His study was based on two scalars I and R , which were scalar components of the torsion tensor and the curvature tensor with respect to the orthonormal frame. A. Moór⁴ developed the theory of three dimensional Finsler spaces using a special orthonormal frame whose first vector was the normalized supporting element and the second vector was the normalized torsion tensor. He found three main scalars which are regarded as a generalization of Berwald main scalar I . Makoto Matsumoto⁵ also studied three dimension Finsler spaces using Moór's orthonormal frame. He obtained some results for certain special Finsler spaces. T.N. Pandey and D.K. Diwedi⁶ extended this idea to four dimensional Finsler spaces. In this paper we have discussed certain special Finsler spaces of dimension four and obtained some results for such spaces.

1. Introduction

Consider a four dimensional Finsler space F^4 with the fundamental function $L(x,y)$. T.N. Pandey and D.K. Diwedi⁶ have constructed the orthonormal frame on the basis of Moór's frame. In this frame an arbitrary tensor is expressed in terms of scalar components as follows

$$(1.1) \quad T_{jk}^i = T_{\alpha\beta\gamma} e_{\alpha}^i e_{\beta j} e_{\gamma k}$$

where $e_{1j}^i = \delta_j^i$, $e_{2j}^i = m_j^i$, $e_{3j}^i = n_j^i$, $e_{4j}^i = p_j^i$ and the summation convention is applied to Greek indices also.

The scalar components $T_{\alpha\beta\gamma}$ are given by $T_{\alpha\beta\gamma} = T_{jk}^i e_{\alpha i} e_{\beta j} e_{\gamma k}$

Let $H_{\alpha\beta\gamma}$ and $\frac{1}{L} V_{\alpha\beta\gamma}$ respectively be scalar components of the h - and

v-covariant derivaties $e^i_{\alpha}{}_{|j}$ and $e^i_{\alpha}{}_{|j}$ of the vector e^i_{α} , i.e.

$$(1.2) \quad e^i_{\alpha}{}_{|j} = H_{\alpha) \beta \gamma} e^i_{\beta} e_{\gamma) j}$$

and

$$(1.3) \quad L e^i_{\alpha}{}_{|j} = V_{\alpha) \beta \gamma} e^i_{\beta} e_{\gamma) j}.$$

The orthogonality of Miron frame yields

$$\begin{aligned} H_{1) \beta \gamma} &= 0, & H_{\alpha) \beta \gamma} &= -H_{\beta) \alpha \gamma} \\ V_{1) \beta \gamma} &= \delta_{\beta \gamma} - \delta_{1 \beta} \delta_{1 \gamma}, & V_{\alpha) \beta \gamma} &= -V_{\beta) \alpha \gamma}. \end{aligned}$$

We now define Finsler vector fields

$$\begin{aligned} H_{2) 3 \beta} e_{\beta) j} &= h_j = h_{\beta} e_{\beta) j} \\ (1.4) \quad H_{4) 2 \beta} e_{\beta) j} &= J_j = J_{\beta} e_{\beta) j} \\ H_{3) 4 \beta} e_{\beta) j} &= k_j = k_{\beta} e_{\beta) j} \end{aligned}$$

Also

$$\begin{aligned} V_{2) 3 \gamma} e_{\gamma) j} &= u_j = u_{\gamma} e_{\gamma) j} \\ (1.5) \quad V_{4) 2 \gamma} e_{\gamma) j} &= v_j = v_{\gamma} e_{\gamma) j} \\ V_{3) 4 \gamma} e_{\gamma) j} &= w_j = w_{\gamma} e_{\gamma) j} \end{aligned}$$

Definition: The Finsler vector fields (h_j, j_j, k_j) and (u_j, v_j, w_j) are called *h- and v-connection vectors* respectively.

The skew symmetric matrices $H_{\alpha) \beta \gamma}$ and $V_{\alpha) \beta \gamma}$ for fixed γ are given by

$$H_{\alpha) \beta \gamma} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & h_{\gamma} & -j_{\gamma} \\ 0 & -h_{\gamma} & 0 & k_{\gamma} \\ 0 & j_{\gamma} & -k_{\gamma} & 0 \end{vmatrix}$$

and

$$V_{\alpha)\beta\gamma} = \begin{vmatrix} 0 & \delta_{2\gamma} & \delta_{3\gamma} & \delta_{4\gamma} \\ -\delta_{2\gamma} & 0 & u_{\gamma} & v_{\gamma} \\ -\delta_{3\gamma} & -u_{\gamma} & 0 & w_{\gamma} \\ -\delta_{4\gamma} & -v_{\gamma} & -w_{\gamma} & 0 \end{vmatrix}.$$

Taking h-covariant derivative of (1.1), we get

$$(1.6) \quad \begin{aligned} T_{jk|h}^i &= (\delta_h T_{\alpha\beta\gamma}) e_{\alpha}^i e_{\beta})_j e_{\gamma})_k + T_{\alpha\beta\gamma} e_{\alpha}^i e_{\beta})_{j|h} e_{\gamma})_k \\ &+ T_{\alpha\beta\gamma} e_{\alpha}^i e_{\beta})_{j|h} e_{\gamma})_k + T_{\alpha\beta\gamma} e_{\alpha}^i e_{\beta})_j e_{\gamma})_{k|h}. \end{aligned}$$

Let $T_{\alpha\beta\gamma,\delta}$ be scalar components of $T_{jk|h}^i$ then

$$(1.7) \quad T_{jk|h}^i = T_{\alpha\beta\gamma,\delta} e_{\alpha}^i e_{\beta})_j e_{\gamma})_k e_{\delta})_h.$$

Then we obtain

$$(1.8) \quad T_{\alpha\beta\gamma,\delta} = (\delta_h T_{\alpha\beta\gamma}) e_{\delta}^h + T_{\mu\beta\gamma} H_{\mu)\alpha\delta} + T_{\alpha\mu\gamma} H_{\mu)\beta\delta} + T_{\alpha\beta\mu} H_{\mu)\gamma\delta}.$$

2. Main scalars of a four dimensional Finsler space

Motsumoto⁵ have shown that if $C_{\alpha\beta\gamma}$ are the scalar components of LC_{ijk} with respect to the Miron frame then

- (i) $C_{\alpha\beta\gamma}$ are completely symmetric.
- (ii) $C_{1\beta\gamma} = 0$
- (iii) $C_{2\mu\mu} = LC$, $C_{3\mu\mu} = C_{4\mu\mu} = \dots = C_{n\mu\mu} = 0$ for $n \geq 3$ where C is the length of C^i .

Therefore in four dimensional Finsler space, we have

$$(2.1) \quad \begin{cases} C_{1\beta\gamma} = 0. \\ C_{222} + C_{233} + C_{244} = LC \\ C_{322} + C_{333} + C_{344} = 0 \\ C_{422} + C_{433} + C_{444} = 0 \\ C_{234} \neq 0 \text{ in general} \end{cases}$$

Thus putting

$$(2.2) \quad \begin{aligned} C_{222} &= A, & C_{233} &= B, & C_{244} &= C, & C_{322} &= D, \\ C_{333} &= E, & C_{422} &= F, & C_{433} &= G, & C_{234} &= H, \end{aligned}$$

we have $C_{344} = -(D+E)$, $C_{444} = -(F+G)$.

Definition: Eight scalars $A, B, \dots, G, H$ given by (2.1) are the main scalars of a four dimensional Finsler space.

3. Certain Theorems

If $C_{\alpha\beta\gamma}$ be the scalar components of LC_{ijk} then from (1.7), we have

$$(3.1) \quad LC_{hijk} = C_{\alpha\beta\gamma,\delta} e_{\alpha}{}^h e_{\beta}{}^i e_{\gamma}{}^j e_{\delta}{}^k$$

where

$$(3.2) \quad C_{\alpha\beta\gamma,\delta} = (\delta_k C_{\alpha\beta\gamma}) e_{\delta}^k + C_{\mu\beta\gamma} H_{\mu\alpha\delta} + C_{\alpha\mu\gamma} H_{\mu\beta\delta} + C_{\alpha\beta\mu} H_{\mu\gamma\delta}.$$

The explicit form of $C_{\alpha\beta\gamma,\delta}$ is obtained as follows

$$\begin{aligned} C_{222,\delta} &= (\delta_k C_{222}) e_{\delta}^k + 3C_{\alpha 22} H_{\alpha}{}_{2\delta} \\ &= (\delta_k A) e_{\delta}^k + 3C_{322} H_{3}{}_{2\delta} + 3C_{422} H_{4}{}_{2\delta} + 3C_{222} H_{2}{}_{2\delta} \\ &= A_{,\delta} + 3D(-h_{\delta}) + 3F(J_{\delta}) + 0 \quad \text{where } A_{,\delta} = (\delta_k A) e_{\delta}^k \\ (3.3a) \quad &= A_{,\delta} - 3D h_{\delta} + 3F J_{\delta}. \end{aligned}$$

As we put $C_{222} = A$, we should notice the difference between $A_{,\delta}$ and $C_{222,\delta}$.

Similarly, we get

$$(3.3b) \quad C_{233,\delta} = B_{,\delta} + (2D - E)h_{\delta} + GJ_{\delta} - 2Hk_{\delta}$$

$$(3.3c) \quad C_{244,\delta} = C_{,\delta} + (D + E)h_{\delta} + (-3F - G)J_{\delta} + 2Hk_{\delta}$$

$$(3.3d) \quad C_{322,\delta} = D_{,\delta} + (A - 2B)h_{\delta} + 2HJ_{\delta} - Fk_{\delta}$$

$$(3.3e) \quad C_{333,\delta} = E_{,\delta} + 3Bh_{\delta} - 3Gk_{\delta}$$

$$(3.3f) \quad C_{422,\delta} = F_{,\delta} + (2C - A)J_{\delta} - 2Hh_{\delta} + Dk_{\delta}$$

$$(3.3g) \quad C_{433,\delta} = G_{,\delta} - BJ_{\delta} + 2Hh_{\delta} + (2D + 3E)k_{\delta}$$

$$(3.3h) \quad C_{234,\delta} = H_{,\delta} + (B - C)k_{\delta} - (2D + E)J_{\delta} + (F - G)h_{\delta}$$

$$(3.3i) \quad C_{344,\delta} = -D_{,\delta} - E_{,\delta} + Ch_{\delta} + (F + 3G)k_{\delta} - 2HJ_{\delta}$$

$$(3.3j) \quad C_{444,\delta} = -F_{,\delta} - G_{,\delta} - 3CJ_{\delta} - (3D + 3E)k_{\delta}$$

Adding (3.3d), (3.3e), (3.3i) and using (2.1), we get

$$(3.4) \quad C_{322,\delta} + C_{333,\delta} + C_{344,\delta} = (A + B + C)h_{\delta} = LC.h_{\delta}$$

Adding (3.3f), (3.3g), (3.3j) and using (2.1), we get

$$(3.5) \quad C_{422,\delta} + C_{433,\delta} + C_{444,\delta} = -(A + B + C)J_{\delta} = -LC.h_{\delta}$$

Adding (3.3a), (3.3b), (3.3c) and using (2.1), we get

$$(3.6) \quad C_{222,\delta} + C_{233,\delta} + C_{244,\delta} = A_{,\delta} + B_{,\delta} + C_{,\delta} = (A + B + C)_{,\delta} = (LC)_{,\delta}$$

Definition. A Berwald space is characterised by $C_{hij,k} = 0$, which is given by in terms of scalar $C_{\alpha\beta\gamma,\delta} = 0$

For a Berwald space, from (3.4) we have $h_{\delta} = 0$ and from (3.5) we have $J_{\delta} = 0$ and so from (3.3a) $A_{,\delta} = 0$.

Thus, we have

Theorem (3.1): A non-Riemannian Berwald space of dimension 4 is characterised by the fact that the two components h_i and J_i of the h -connection vector vanish and the main scalar A is h -covariant constant.

Also from (3.6), we have

Theorem (3.2): In a non-Riemannian Berwald space of dimension 4, the unified main scalar LC is h -covariant constant.

Definition. A Landsberg space is characterised by

$$(3.7) \quad C_{hij,0} = 0 \text{ and } C_{hij,k} = C_{hik,j}.$$

ie. in terms of scalars $C_{\alpha\beta\gamma,1} = 0$ and $C_{\alpha\beta\gamma,\delta} = C_{\alpha\beta\delta,\gamma}$.

For a Landsberg space, from (3.4) and (3.5), we get, $h_1 = 0$, $J_1 = 0$ and so $A_{,1} = 0$
 Also from (3.4), we have

$$\begin{aligned} LC.h_2 &= C_{322,2} + C_{333,2} + C_{344,2} \\ &= C_{222,3} + C_{233,3} + C_{244,3} \\ &= (LC)_{,3} \quad \text{from (3.6)} \end{aligned}$$

hence $h_2 = \frac{(LC)_{,3}}{LC}$.

From (3.5), we have

$$\begin{aligned} -LC.J_2 &= C_{422,2} + C_{433,2} + C_{444,2} \\ &= C_{222,4} + C_{233,4} + C_{244,4} \\ &= (LC)_{,4} \quad \text{from (3.4)} \end{aligned}$$

hence $J_2 = -\frac{(LC)_{,4}}{LC}$.

Also from (3.5), we have

$$\begin{aligned} -LC.J_3 &= C_{422,3} + C_{433,3} + C_{444,3} \\ &= C_{322,4} + C_{333,4} + C_{344,4} \\ &= LC.h_4 \end{aligned}$$

hence $J_3 = -h_4$.

Thus, we have

Theorem (3.3): *In a non-Riemannian Landsberg space of dimension 4 the components of h-connection vectors are given by*

$$h_1 = 0 = J_1,$$

$$h_2 = \frac{(LC)_{,3}}{LC},$$

$$J_2 = -\frac{(LC)_{,4}}{LC},$$

$$J_3 = -h_4.$$

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