

Some Tauberian Theorems Based on (D, α, β) (C, ℓ) Summability Methods of Functions

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Abstract: Some Tauberian theorems for functions based on Kwee-Cesàro product summability method have been obtained in this paper.

1. Introduction

Ishiguro¹ introduced the summability method (D, α) for sequences and raised the question of what are the main properties of (D, α) . Kuttner² dealt with this problem and introduced the summability method (D, α) for functions. Pathak³ defined the summability method (D, α) (C, ℓ) for functions. Kwee⁴ introduced a summability method (D, α, β) for series and functions by generalizing the (D, α) summability method. In this paper, we obtain some Tauberian theorems based on (D, α, β) (C, ℓ) summability methods of functions.

2. Some Relations and Definitions

Let $s(y)$ be any function which is Lebesgue integrable in $(0, X)$ for all finite $x > 0$, and which is bounded in some right hand neighbourhood of the origin. Integrals of the form $\int_0^\infty$

are throughout to be taken as $\lim_{x \rightarrow \infty} \int_0^x, \int_0^x$ being a Lebesgue integral.

For any $\ell > 0$, we write $s_\ell(x)$ for the ' ℓ^{th} integral',

$$s_\ell(x) = \frac{1}{\Gamma(\ell)} \int_0^x (x-y)^{\ell-1} s(y) dy, \text{ and } s_0(x) = s(x).$$

The (C, ℓ) transform of $s(x)$, which we denote by $C_\ell(x)$, is given by

$$(2.1) \quad C_\ell(x) = \begin{cases} s(x), & \ell = 0. \\ \frac{\ell}{x^\ell} \int_0^x (x-y)^{\ell-1} s(y) dy, & \ell > 0. \end{cases}$$

If this exists and $C_\ell(x)$ tends to a limit s as $x \rightarrow \infty$, then we say that $s(x)$ is summable (C, ℓ) to s , and we write $s(x) \rightarrow s (C, \ell)$.

We write

$$(2.2) \quad g(y) = g(\alpha, \beta, y) = \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(\alpha)\Gamma(\beta + 1)} y^{\beta+1} \int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}} s(x) dx,$$

if this exists. We also write

$$(2.3) \quad U_{\alpha, \beta, \ell}(y) = \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(\alpha)\Gamma(\beta + 1)} y^{\beta+1} \int_0^\infty \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}} C_\ell(x) dx,$$

if this exists where $y > 0, \alpha > 0, \beta > -1, \ell > 0$.

Suppose that $s(x)$ is of the form

$$(2.4) \quad s(x) = \int_0^x a(y) dy,$$

where $a(y)$ is Lebesgue integrable in $(0, x)$ for any finite x . Then it is easy to show, under the assumption that the integral defining $U_{\alpha, \beta, \ell-1}(y)$ converges, that, for all $y > 0$

$$(2.5) \quad \begin{aligned} y \frac{d}{dy} U_{\alpha, \beta, \ell}(y) &= -\ell [U_{\alpha, \beta, \ell}(y) - U_{\alpha, \beta, \ell-1}(y)] \\ &= B_{\alpha, \beta, \ell}(y), \quad (\ell \geq 1), \end{aligned}$$

$$\text{where } B_{\alpha, \beta, \ell}(y) = \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(\alpha)\Gamma(\beta + 1)} y^{\beta+1} \int_0^\infty \tau_\ell(y) \frac{x^{\alpha-1}}{(x+y)^{\alpha+\beta+1}} dy.$$

Also, we know that

$$(2.6) \quad y \frac{d}{dy} C_\ell(y) = \ell [C_{\ell-1}(y) - C_\ell(y)] = \tau_\ell(y) = \frac{\ell}{y^\ell} \int_0^y (y-u)^{\ell-1} u \alpha(u) du.$$

If we do not assume (2.4), then the first equation of (2.6) still holds if $\ell \geq 1$. Further, we take the second equation as defining $\tau_\ell(y)$ when $\ell \geq 1$.

Srivastava⁵ defined the strong and absolute summabilities to index p , denoted as $[(D, \alpha, \beta)(C, \ell)]_p$ and $|(D, \alpha, \beta)(C, \ell)|_p$ respectively as follows:

Suppose that $\alpha > 0$, $\beta > -1$, $\ell \geq 1$, $p \geq 1$. Then we say that the function $s(x)$ is

(i) summable $[(D, \alpha, \beta)(C, \ell)]_p$ or strongly summable $(D, \alpha, \beta)(C, \ell)$ with index p to the sum s , if

$$\int_1^x |U_{\alpha, \beta, \ell-1}(y) - s|^p dy = o(x) \text{ as } x \rightarrow \infty,$$

(ii) summable $|(D, \alpha, \beta)(C, \ell)|_p$, if

$$\int_1^\infty y^{-1} \left| y \frac{d}{dy} U_{\alpha, \beta, \ell}(y) \right|^p dy < \infty.$$

According to Mishra⁶, the function $s(x)$ is summable $[C, \ell]_p$, where $\ell \geq 1$, $p \geq 1$, if

$$\int_1^x |C_{\ell-1}(y) - s|^p dy = o(x) \text{ as } x \rightarrow \infty.$$

3. Known Results

Theorem A. (Mishra and Srivastava⁷). If $\ell > m \geq 1$, $\alpha > 0$, $\beta > -1$ and $\int_1^\infty \frac{C_{m-1}(x)}{x^{\beta+2}} dx$ converges, then for $x > 0$,

$$(3.1) \quad (I) \quad U_{\alpha, \beta, \ell-1}(x) = \frac{\Gamma(\ell)}{\Gamma(\ell-m)\Gamma(m)} X^{-(\ell-1)} \int_0^x y^{m-1} (x-y)^{\ell-m-1} U_{\alpha, \beta, m-1}(y) dy;$$

$$(II) \lim_{x \rightarrow \infty} |U_{\alpha, \beta, \ell-1}(x)| \leq \lim_{x \rightarrow \infty} |U_{\alpha, \beta, m-1}(x)|,$$

where (3.1) is to be taken as including the assertion that both sides exist.

Theorem B. Let $0 < \mu < 1$,

$$S(\omega) = \frac{1}{\omega} \int_0^\omega \phi(x) dx \quad (\phi(x) \geq 0),$$

$$\text{and} \quad S = \sup_{0 < \omega < \infty} s(\omega).$$

$$\text{Then } \int_0^\omega x^{-\mu} \phi(x) dx \leq C(\mu) S \omega^{1-\mu}.$$

If we take $S(\omega) = o(1)$ as $\omega \rightarrow \infty$, then

$$\int_0^\omega x^{-\mu} \phi(x) dx = o(\omega^{1-\mu}).$$

This is given by Mishra⁸.

4. Some Definitions

Suppose that $p \geq 1$, $\alpha > 0$, $\beta > -1$, $\ell \geq 1$. Suppose that the function $s(x)$ satisfies the condition that $s(x)$ is Lebesgue integrable in $(0, x)$ for all finite $x > 0$ and which is bounded in some right hand neighbourhood of the origin, and that (2.4) holds.

We say that the function $s(x)$ is summable

$$(I) \quad |C, \ell|_p, \text{ if}$$

$$(4.1) \quad \int_T^\infty \frac{|\tau_\ell(t)|^p}{t} dt < \infty, \text{ for some } T > 0,$$

(II) $|(D, \alpha, \beta)(C, \ell)|_p$, or absolutely summable $(D, \alpha, \beta)(C, \ell)$ with index p , if the integral defining $U_{\alpha, \beta, \ell}(t)$ is convergent for all $t > 0$ and

$$(4.2) \quad \int_1^\infty \frac{|B_{\alpha, \beta, \ell}(t)|^p}{t} dt < \infty,$$

where $B_{\alpha, \beta, \ell}(t)$ is defined in (2.5)

We also say that the function $s(x)$ is summable

(i) $[C, \ell]_p$ to the sum s , if

$$(4.3) \quad \int_T^\infty \frac{|C_{\ell-l}(t) - s|^p}{t} dt < \infty, \quad (T > 0),$$

(ii) $[(D, \alpha, \beta)(C, \ell)]_p$ to the sum s , if

$$(4.4) \quad \int_T^\infty \frac{|U_{\alpha, \beta, \ell-l}(t) - s|^p}{t} dt < \infty$$

It is worth mentioning that (as may easily be proved by integration by parts $[C^*, \ell]_p$ implies $[C, \ell]_p$, but not conversely.

It is easily seen that, for $\ell > 0$, $[C^*, \ell]_p$ implies $[C^*, \ell+1]_p$. We thus see with the aid of (2.6) and a straightforward application of Minkowski's inequality that summability $[C^*, \ell]_p$ is equivalent to summability $[C^*, \ell+1]_p$ together with summability $[C, \ell]_p$. In analogy with $[C^*, \ell]_p$, this enables us to extend the definition to the case $\ell=0$. We say that the function $s(x)$ is summable $[C^*, 0]_p$ if it is summable $[C^*, 1]_p$ and is also summable $[C, 0]_p$.

In fact, the proof of the above statement is given below: First we note that

$$|y \frac{d}{dy} C_\ell(y)|^p = \ell^p [C_{\ell-l}(y) - C_\ell(y)]^p \leq 2^p \ell^p [|C_{\ell-l}(y) - s|^p + |C_\ell(y) - s|^p]$$

Hence from (2.6) and Minkowski's inequality, we have if $p \geq 1$

$$\begin{aligned} & \left\{ \int_T^\infty 2^p \ell^p \frac{|C_{\ell-l}(y) - s|^p}{y} dy \right\}^{\frac{1}{p}} - \left\{ \int_T^\infty 2^p \ell^p \frac{|C_\ell(y) - s|^p}{y} dy \right\}^{\frac{1}{p}} \leq \\ & \left\{ \int_T^\infty y^{-1} |y \frac{d}{dy} C_\ell(y)|^p dy \right\}^{\frac{1}{p}}. \\ & \leq \left\{ \int_T^\infty 2^p \ell^p \frac{|C_{\ell-l}(y) - s|^p}{y} dy \right\}^{\frac{1}{p}} + \left\{ \int_T^\infty 2^p \ell^p \frac{|C_\ell(y) - s|^p}{y} dy \right\}^{\frac{1}{p}}. \end{aligned}$$

Since summability $[C^*, \ell]_p$ implies $[C^*, \ell+1]_p$, the necessary and sufficient conditions that

$\int_T^{\infty} \frac{|C_{\ell-I}(y) - s|^p}{y} dy < \infty$ are that

$$\int_T^{\infty} y^{-I} \left| y \frac{d}{dy} C_{\ell}(y) \right|^p dy < \infty \quad \text{and} \quad \int_T^{\infty} \frac{|C_{\ell-I}(y) - s|^p}{y} dy < \infty$$

5. Tauberian Theorems On $(D, \alpha, \beta) (C, \ell)$ Summability Methods

In this section, we prove the following theorems.

Theorem 5.1. Suppose that (2.4) holds and if $\alpha > 0, \beta > -1, \ell \geq 1$ and $\ell > m \geq 0$. Then for any s ,

$$(5.1) \quad \sup_{0 < t < \infty} |C_{\ell-I}(y) - s| \leq C \sup_{0 < t < \infty} |\tau_{\ell}(t)| + \sup_t |U_{\alpha, \beta, m}(t) - s|$$

Proof. Let $T = \sup_t |\tau_{\ell}(t)|$.

By Theorem A, we have

$$(5.2) \quad \begin{aligned} U_{\alpha, \beta, \ell}(t) - s &= \frac{\Gamma(\ell+1)}{\Gamma(\ell-m)\Gamma(m+1)} t^{-\ell} \int_0^t (t-x)^{\ell-m-1} x^m \{U_{\alpha, \beta, m}(x) - s\} dx \\ &= \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} t^{\beta+1} \int_0^{\infty} \frac{x^{\alpha-1}}{(x+t)^{\alpha+\beta+1}} \{C_{\ell}(x) - s\} dx \end{aligned}$$

The expression on the right of (5.2) is equal to

$$\begin{aligned} &\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} t^{\beta+1} \int_0^{\infty} \frac{x^{\alpha-1}}{(x+t)^{\alpha+\beta+1}} \{C_{\ell}(x) - C_{\ell}(t)\} dx \\ &+ \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} t^{\beta+1} \int_0^{\infty} \frac{x^{\alpha-1}}{(x+t)^{\alpha+\beta+1}} \{C_{\ell}(t) - s\} dx \end{aligned}$$

Hence

$$(5.3) \quad \begin{aligned} |C_{\ell}(t) - s| &\leq \frac{\Gamma(\ell+1)}{\Gamma(\ell-m)\Gamma(m+1)} t^{-\ell} \int_0^t (t-x)^{\ell-m-1} x^m |U_{\alpha, \beta, m}(x) - s| dx \\ &+ \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} t^{\beta+1} \left| \int_0^{\infty} \frac{x^{\alpha-1}}{(x+t)^{\alpha+\beta+1}} \{C_{\ell}(t) - C_{\ell}(x)\} dx \right|. \end{aligned}$$

Now, we have, for $x < t$

$$\begin{aligned} |C_\ell(t) - C_\ell(x)| &= \left| \int_x^t \frac{\tau_\ell(u)}{u} du \right| \\ &\leq T \left| \int_x^t u^{-1} du \right| = T \left| \log \frac{t}{x} \right|. \end{aligned}$$

There is a similar result with x, t interchanged when $x > t$ and thus in any case

$$(5.4) \quad |C_\ell(t) - C_\ell(x)| \leq T \left| \log \frac{t}{x} \right|.$$

Hence

$$\begin{aligned} (5.5) \quad & \left| \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(\alpha)\Gamma(\beta + 1)} t^{\beta+1} \int_0^\infty \frac{x^{\alpha-1}}{(x+t)^{\alpha+\beta+1}} [C_\ell(t) - C_\ell(x)] dx \right| \\ & \leq \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(\alpha)\Gamma(\beta + 1)} t^{\beta+1} T \int_0^\infty \frac{x^{\alpha-1}}{(x+t)^{\alpha+\beta+1}} \left| \log \frac{t}{x} \right| dx. \end{aligned}$$

On making the substitution $x = vt$, the expression on the right hand side of (5.5) becomes

$$\begin{aligned} & \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(\alpha)\Gamma(\beta + 1)} T \int_0^\infty \frac{v^{\alpha-1}}{(1+v)^{\alpha+\beta+1}} \left| \log \frac{1}{v} \right| dv \\ & \leq CT. \end{aligned}$$

Also, the first part of right hand side of (5.3) is

$$\begin{aligned} (5.6) \quad & \leq \frac{\Gamma(\ell + 1)}{\Gamma(\ell - m)\Gamma(m + 1)} t^{-\ell} \sup_t |U_{\alpha, \beta, m}(t) - s| \left| \int_0^t (t-x)^{\ell-m-1} x^m dx \right| \\ & \leq \sup_t |U_{\alpha, \beta, m}(t) - s| \end{aligned}$$

Thus

$$|C_\ell(t) - s| \leq CT + \sup_t |U_{\alpha, \beta, m}(t) - s|$$

Applying the second equation of (2.6), we have

$$\tau_\ell(t) = \ell [C_{\ell-1}(t) - C_\ell(t)].$$

Thus

$$C_{\ell-1}(t) - C_\ell(t) = \frac{\tau_\ell(t)}{\ell}$$

or

$$C_{\ell-1}(t) - s = C_{\ell}(t) - s + \frac{\tau_{\ell}(t)}{\ell}$$

Hence

$$\sup_{0 < t < \infty} |C_{\ell-1}(t) - s| = C \sup_{0 < t < \infty} |\tau_{\ell}(t)| + \sup_t |U_{\alpha, \beta, m}(t) - s|$$

Hence the Theorem

Theorem 5.2. Suppose that (2.4) holds. If the function $s(x)$ is summable $(D, \alpha, \beta)(C, \ell)$ to s , and the function $xa(x)$ is summable (C, ℓ) to the sum 0, then the function $s(x)$ is summable $(C, \ell-1)$ to s , $((\alpha > 0, \beta > -1, \ell \geq 1, m \geq 0, \ell > m \geq 0))$. The cases $\beta = 0, \ell = 0$ of Theorems 5.1-5.3 and Theorem 6.1 are due to Mishra and Srivastava⁹ while the case $\beta = 0$ is due to Singh¹⁰.

Proof. The result would follow from Theorems 5.1 but with supremum replaced by $\lim_{t \rightarrow \infty} \sup$ through out provided that limit supremum on the right are 0. For, it is familiar and easily verified that

$$\begin{aligned} \lim_{t \rightarrow \infty} \sup \left| \frac{\Gamma(\ell+1)}{\Gamma(\ell-m)\Gamma(m+1)} t^{-\ell} \int_0^t (t-x)^{\ell-m-1} x^m \{U_{\alpha, \beta, m}(x) - s\} dx \right| \\ \leq \lim_{u \rightarrow \infty} \sup |U_{\alpha, \beta, m}(u) - s|. \end{aligned}$$

In dealing with the second term on the right of (5.3), let us now write $T' = \lim_{u \rightarrow \infty} \sup |\tau_{\ell}(u)|$. Given $\epsilon > 0$, there is u_0 s.t. $|\tau_{\ell}(u)| \leq T' + \epsilon$ for $u \geq u_0$ (or the corresponding equation with x, t interchanged), we can use the estimate $|\tau_{\ell}(u)| \leq T' + \epsilon$ only for that part of the integral $u \geq u_0$. If we take $t > u_0$ then (5.4) (with T replaced by $T' + \epsilon$) holds, as in the proof of Theorem 5.1. For all $x > t$ and for $t > x \geq u_0$, all that we can assert is that

$$\begin{aligned} |C_{\ell}(t) - C_{\ell}(x)| &\leq (T' + \epsilon) \int_{u_0}^t \frac{du}{u} + |C_{\ell}(u_0) - C_{\ell}(x)| \\ &\leq (T' + \epsilon) \log \frac{t}{x} + |C_{\ell}(u_0) - C_{\ell}(x)| \end{aligned}$$

The second term on the right of (5.3) is, in modulus, less than or equal to the sum of the two terms. The first is equal to the expression (5.5) with T replaced by $(T' + \epsilon)$, and can be dealt with Theorem 5.1, the second is

$$(5.7) \quad \frac{\Gamma(\alpha + \beta + 1)}{\Gamma(\alpha)\Gamma(\beta + 1)} t^{\beta+1} \int_0^{u_0} \frac{x^{\alpha-1}}{(x+t)^{\alpha+\beta+1}} |C_\ell(u_0) - C_\ell(x)| dx.$$

It is easy to prove that, for fixed u_0 the expression (5.7) tends to 0 as $t \rightarrow \infty$, thus, once we have chosen u_0 , we can make (5.7) arbitrary small by taking t sufficiently large. This gives the conclusion.

Theorem 5.3. *If the function $s(x)$ is summable $(D, \alpha, \beta) (C, \ell)$ to s , where $\ell > 0, \beta > -1$ and $\alpha > 0$, and*

$$(5.8) \quad t \frac{d}{dt} U_{\alpha, \beta, \ell}(t) = o(1)$$

then the function $s(x)$ is summable $(D, \alpha, \beta) (C, \ell - 1)$ to the same sum.

Proof. Proof of above theorem follows from (2.5).

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