

A Concurrent Vector Field in Special Finsler Spaces

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(Received October 15, 2003)

Abstract : S. Sasaki¹ and K. Yano² introduced the concept of a concurrent vector field in a Riemannian space. This concept was extended to a Finsler space by S. Tachibana³, Makoto Matsumoto and K. Eguchi⁴. P. N. Pandey⁵ has also discussed a concurrent vector field in a Finsler space equipped with Berwald connection. In the present paper, concurrent vector fields in certain special Finsler spaces have been discussed and it has been shown that an affinely connected space, a C^h -recurrent space, a C^h -birecurrent space, a Finsler space having vanishing T -tensor and a Finsler space of scalar curvature do not admit a concurrent vector field.

1. Preliminaries

Let us consider an n -dimensional Finsler space F_n equipped with the metric function $F(x^i, \dot{x}^i)$, the metric tensor g_{ij} and Cartan's Euclidean connection $\bar{\Gamma}_{jk}^i$. If T_j^i is an arbitrary tensor field of type $(1, 1)$, its h -covariant derivative and v -covariant derivative are defined by^{6,7}

$$(1.1) \quad T_j^i|_k = \partial_k T_j^i - \left(\partial_r T_j^i \right) \bar{\Gamma}_{sk}^r \dot{x}^s + T_j^r \bar{\Gamma}_{rk}^i - T_r^i \bar{\Gamma}_{jk}^r$$

and

$$(1.2) \quad T_j^i|_k = F \left(\partial_k T_j^i + T_j^r C_{rk}^i - T_r^i C_{jk}^r \right)$$

respectively. The symbols ∂_k and $\dot{\partial}_k$ stand for partial differentiations with respect to x^k and $\dot{x}^k$ respectively.

The tensor C_{jk}^i is defined by

$$(1.3) \quad C_{jk}^i = g^{il} C_{jlk},$$

where g^{il} is the associate metric tensor and

$$(1.4) \quad C_{jlk} = \frac{1}{2} \dot{\partial}_j g_{lk}.$$

The tensor C_{jlk} is positively homogeneous of degree -1 in $\dot{x}^j$ and satisfies

$$(1.5) \quad C_{jlk} = C_{lkj} = C_{kjl}$$

and

$$(1.6) \quad C_{jlk} \dot{x}^j = 0.$$

The commutation formulae for Cartan's covariant differentiation and directional differentiation are given by

$$(1.7) \quad \partial_h (X^i|_k) - (\partial_h X^i)|_k = X^r \partial_h \bar{\Gamma}_{rk}^i - (\partial_r X^i) P_{hk}^r$$

and

$$(1.8) \quad X^i|_{k|h} - X^i|_{h|k} = X^r K_{rkh}^i - (\partial_r X^i) K_{skh}^r \dot{x}^s$$

where

$$(1.9) \quad P_{hk}^r = C_{hk|s}^r \dot{x}^s$$

and

$$(1.10) \quad K_{rkh}^i = \partial_h \bar{\Gamma}_{kr}^i + \left(\partial_l \bar{\Gamma}_{rh}^i \right) \bar{\Gamma}_{sk}^l \dot{x}^s + \bar{\Gamma}_{mh}^i \bar{\Gamma}_{kr}^m - k/h.$$

The tensor, defined above, is called Cartan's curvature tensor and the symbol $-k/h$ means the subtraction from the former terms obtained by interchanging the indices k and h . The Cartan's curvature tensor K_{rkh}^i is skew-symmetric in its last two lower indices and its associate tensor K_{rjkh} , defined by $K_{rjkh} = g_{ij} K_{rkh}^i$, is also skew-symmetric in its last two indices.

2. A Concurrent Vector Field

Makoto Matsumoto and K. Eguchi⁴ called a vector field $X^i(x^k)$ as concurrent vector field if it satisfies

$$(2.1) \quad C_{ijk} X^k = 0$$

and

$$(2.2) \quad X^i|_j = -\delta_j^i$$

or, equivalently,

$$(2.3) \quad X_{i|j} = -g_{ij}.$$

Since $C_{ijk} = \frac{1}{2} \partial_i g_{jk}$ and the vector field X^i is independent of $\dot{x}^i$, (2.1) may be written as $\partial_i (g_{jk} X^k) = 0$, i.e. $\partial_i X_j = 0$. This shows that the vector field X_j is also independent of $\dot{x}^i$. In view of this fact and the independence of X^i from $\dot{x}^i$, we may prove that the square of the magnitude ($X^2 = g_{ij} X^i X^j$) of X^i is also independent of the directional arguments.

The covariant derivative of a vector field X^i in the sense of Berwald is defined as

follows :

$$(2.4) \quad X_{(k)}^i = \partial_k X^i - (\dot{\partial}_r X^i) G_k^r + X^r G_{rk}^i,$$

where G_{rk}^i are Berwald's connection coefficients and $G_k^r = G_{kh}^r \dot{x}^h$. Berwald's connection coefficients G_{jk}^i and Cartan's connection coefficients Γ_{jk}^i are connected by

$$(2.5) \quad G_{jk}^i = \Gamma_{jk}^{*i} + C_{j|k|h}^i \dot{x}^h$$

and

$$G_j^i = G_{jk}^i \dot{x}^k = \Gamma_{jk}^{*i} \dot{x}^k.$$

Therefore, (2.4) may be written as

$$X_{(k)}^i = \partial_k X^i - (\dot{\partial}_r X^i) \Gamma_{kh}^{*r} \dot{x}^h + X^r \Gamma_{rk}^{*i} + X^r C_{rk|h}^i \dot{x}^h$$

or

$$(2.6) \quad X_{(k)}^i = X_{|k}^i + X^r C_{rk|h}^i \dot{x}^h.$$

If X^i is a concurrent vector field, it satisfies (2.1) and (2.2); in view of which, we have

$$(2.7) \quad X_{(k)}^i = -\delta_k^i.$$

Similarly, by reversing the arguments, we may show that the equation (2.1) and (2.7) imply (2.1) and (2.2). Therefore, a concurrent vector field $X^i(x^j)$ may also be characterized by the equations (2.1) and (2.7).

Let us consider a 2-dimensional Finsler space. In such space the tensor C_{ijk} has the form [Rund⁶]

$$(2.8) \quad F C_{ijk} = J m_i m_j m_k,$$

where m_i is a unit vector field orthogonal to $\dot{x}^i$ and J is a scalar introduced by Berwald. If the space admits a vector field X^i which is independent of $\dot{x}^i$ and satisfies (2.1), then we have at least one of the conditions

$$(2.9) \text{ (a)} \quad m_i X^i = 0$$

$$(b) \quad J = 0.$$

The condition (2.9a) implies that the vector field X^i is orthogonal to m^i and hence parallel to $\dot{x}^i$. Thus, there exists a scalar λ such that $X^i = \lambda \dot{x}^i$, which implies

$$(\dot{\partial}_j \lambda) \dot{x}^j + \lambda \delta_j^i = 0.$$

Contracting the indices i and j , we have $\lambda = 0$. Therefore $X^i = 0$. Hence (2.9a) can not

hold. Therefore we must have (2.9b). In view of (2.9b), (2.8) implies $C_{ijk} = 0$, i.e. the space considered is Riemannian.

Hence a 2-dimensional Finsler space admitting a vector field $X^i(x')$ satisfying (2.1) is necessarily Riemannian. Since a concurrent vector field $X^i(x')$ satisfies (2.1), we conclude that a 2-dimensional Finsler space admitting a concurrent vector field is necessarily Riemannian.

3. Non-Existence of Concurrent Vector Fields in Certain Finsler Spaces

Let us consider an affinely connected Finsler space which admits a concurrent vector field X^i . An affinely connected Finsler space is characterized by the condition

$$(3.1) \quad C_{ijk| h} = 0.$$

Since X^i is a concurrent vector field, it satisfies (2.1) and (2.2). Differentiating (2.1) covariantly with respect to x^h in the sense of Cartan and using (2.2), we have

$$(3.2) \quad -C_{ijh} + C_{ijk| h} X^k = 0.$$

From (3.1) and (3.2), we find $C_{ijk} = 0$, i.e. the space is Riemannian.

This leads to :

Theorem 3.1 : *An affinely connected Finsler space admitting a concurrent vector field is necessarily Riemannian.*

Theorem 3.1 may be restated as :

A non-Riemannian affinely connected Finsler space does not admit a concurrent vector field.

Makoto Matsumoto⁸ introduced a C^h -recurrent Finsler space characterized by the condition

$$(3.3) \quad C_{ijk| h} = \lambda_h C_{ijk}, \quad C_{ijk} \neq 0.$$

If such space admits a concurrent vector field X^i characterized by (2.1) and (2.2), we immediately find (3.2), which, in view of (3.3), gives $C_{ijh} = 0$.

Thus, we get a contradiction to the fact that the tensor C_{ijk} is non-vanishing.

Thus, we have :

Theorem 3.2 : *A C^h -recurrent Finsler space does not admit a concurrent vector field.*

Now, we propose :

Theorem 3.3 : *A non-Riemannian C^h -birecurrent Finsler space does not admit a concurrent vector field.*

Proof : Let F_n be a non-Riemannian C^h -birecurrent Finsler space characterized by the condition⁹ :

$$(3.4) \quad C_{ijk|h|m} = a_{hm} C_{ijk},$$

where $C_{ijk} \neq 0$ and a_{hm} is a non-null tensor field. If this space admits a concurrent vector field X^i characterized by (2.1) and (2.2), we have (3.2). Differentiating (3.2) covariantly in the sense of Cartan, we get

$$(3.5) \quad -C_{ijh|m} - C_{ijm|h} + C_{ijk|h|m} X^k = 0.$$

Using (3.4) and (2.1), we have

$$(3.6) \quad C_{ijh|m} + C_{ijm|h} = 0.$$

Differentiating (3.6) covariantly in the sense of Cartan and using (3.4), we get

$$(3.7) \quad a_{ml} C_{ijh} + a_{hl} C_{ijm} = 0.$$

Multiplying (3.7) by g^{jk} , taking summation over j and using (1.3), we have

$$(3.8) \quad a_{ml} C_{ih}^k + a_{hl} C_{im}^k = 0.$$

Contracting the indices k and i in (3.8) and putting $C_{rh}^r = C_h$, we have

$$(3.9) \quad a_{ml} C_h + a_{hl} C_m = 0.$$

Since the space is non-Riemannian, the vector C_h is non-vanishing and therefore, there exists at least one h for which $C_h \neq 0$. Without any loss of generality, suppose $C_1 \neq 0$.

Putting $m = l = h = 1$ in (3.9), we find $a_{11} C_1 = 0$; which implies $a_{11} = 0$. Putting $m = h = 1$ in (3.9), we find $a_{1l} C_1 = 0$; which implies $a_{1l} = 0$ for $C_1 \neq 0$. Again putting $m = 1$ in (3.9), we find $a_{1l} C_h + a_{hl} C_1 = 0$. Since $a_{1l} = 0$, we have $a_{hl} C_1 = 0$. This implies $a_{hl} = 0$ for $C_1 \neq 0$. Thus, we have a contradiction. Therefore, we have the theorem.

Theorem 3.4 : A non-Riemannian Finsler space satisfying the condition

$$(3.10) \quad T \equiv F C_{hij|k} + C_{kij} y_h + C_{h kj} Y_i + C_{ihk} y_j = 0$$

can not admit a concurrent vector field.

Proof : Suppose a non-Riemannian Finsler space satisfying the condition $T = 0$ admits a concurrent vector field X^i characterized by (2.1) and (2.2). Cartan's covariant derivative of X^i with respect to $\dot{x}^j$ is given by

$$(3.11) \quad X^i|_k = F \left(\partial_k X^i + C_{jk}^i X^j \right).$$

In view of (2.2) and the fact that the vector X^i is independent of $\dot{x}^i$, (3.11) becomes

$$(3.12) \quad X^i|_k = 0.$$

Transvecting (3.10) by X^j and using (2.1) and (3.12), we have $(X^j y_j) C_{ihk} = 0$.

Since the space is non-Riemannian, $C_{ihk} \neq 0$. Therefore we have $X^j y_j = 0$.

Differentiating $X^j y_j = 0$ with respect to $\dot{x}^k$ and using $\dot{\partial}_k y_j = g_{jk}$, we get $g_{jk} X^j = X_k = 0$, which implies $X^i = 0$. This gives a contradiction.

Theorem 3.5 : *A Finsler space of scalar curvature does not admit any concurrent vector field.*

Proof : Let $F_n(n > 2)$ be a Finsler space of scalar curvature. Then it satisfies^{6,10}

$$(3.13) \quad H^i_h = F^2 R \left(\delta^i_h - l^i l_h \right), \quad R \neq 0,$$

where H^i_h and R denotes Berwald's deviation tensor and the Riemannian curvature of the Finsler space, and $l^i = \frac{\dot{x}^i}{F}$. If it admits a concurrent vector field X^i characterized by (2.1) and (2.2), we have $X^i|_k|_h = 0$. Utilizing the commutation formula exhibited by (1.8) for X^i , we get $X^r K^i_{rkh} = 0$ which implies $X^r K_{rikh} = 0$. In view of

$$K_{jikh} = -K_{ijkh} - 2C_{ijl} K^l_{skh} \dot{x}^s,$$

the equation $X^r K_{rikh} = 0$ implies

$$(3.14) \quad K_{ijkh} X^j + 2C_{ijl} X^j K^l_{skh} \dot{x}^s = 0$$

using (2.1) in (3.14), we get $K_{ijkh} X^j = 0$ or $X_j K^j_{ikh} = 0$.

Transvecting this equation by $\dot{x}^i \dot{x}^k$ and using

$$K^j_{ikh} \dot{x}^i \dot{x}^k = H^j_h,$$

we get

$$(3.15) \quad X_j H^j_k = 0..$$

Form (3.13) and (3.15), we have

$$(3.16) \quad X_k = \left(X_j l^j \right) l_k.$$

Transvecting (3.16) by g^{ik} , we get

$$(3.17) \quad X^i = \left(X_j l^j / F \right) \dot{x}^i.$$

Since the vector field X^i is independent of $\dot{x}^i$, the equation (3.17) is true only when $X^i = 0$. This gives a contradiction.

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