

On the Degree of Approximation by Baskakov-Type Operators

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(Received February 22, 2003)

Abstract. This paper contains the discussion about the following Operator

$$\left(L_{\frac{n}{c}+p} f\right)(x) = c \binom{\frac{n}{c}+p-1}{k} \sum_{k=0}^{\infty} \binom{\frac{n}{c}+p+k-1}{k} \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+p+k}} \\ \int_0^{\infty} \binom{\frac{n}{c}+p+k-1}{k} \frac{(ct)^k}{(1+ct)^{\frac{n}{c}+p+k}} f(t) dt$$

and some approximation results including Voronskya type results.

1. Introduction

For a function f defined on the closed interval $[0, 1]$ of the real x -axis the expression.

$$(1.1) \quad B_n(f; x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) p_{nk}(x)$$

where

$$p_{nk}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

$n = 1, 2, \dots$; $k = 0, 1, 2, \dots$, is called the Bernstein polynomial of order n of the function f . It is well known that if $f \in C[0, 1]$ the space of all real functions of f defined and continuous on $[0, 1]$, then

$$\lim_{n \rightarrow \infty} B_n(f; x) = f(x)$$

uniformly on $[0, 1]$

In 1965, Schurer¹, generalized the Baskakov operators² by introducing a parameter in the definition of these operators. These generalized Baskakov operators are define in

the following way.

Let $M[0, \infty)$ be the class of functions f defined on $[0, \infty)$ for which $|f(t)| \leq m_1(1+t)^{m_2}$, $t \in [0, \infty)$ where $m_1, m_2 \geq 0$ are constants depending on f then Lnp ($n \in N$, $p \in R$, $n+p \geq 1$) is defined on $M[0, \infty)$ by

$$(1.3) \quad (L_{np}f)(x) = (1+x)^{-n-p} \sum_{k=0}^{\infty} \binom{n+p+k-1}{k} x^k (1+x)^{-k} f\left(\frac{k}{n}\right)$$

where $x \in [0, \infty)$ for $p = 0$, the operators reduced to Baskakov operators.

By slight modifications by Lenhnoff³, May⁴, the following operators

$$(1.4) \quad (L_n, c)(f, x) = \sum_{k=0}^{\infty} p_{nk}(c, x) f\left(\frac{k}{n}\right)$$

where c is a real number and

$$(1.5) \quad p_{n,k}(c, x) = \begin{cases} e^{-nx} \frac{(nx)^k}{k!} & c = 0 \\ \binom{k + \frac{n}{c} - 1}{k} \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+k}} & c \neq 0 \end{cases}$$

We note that the cases $c = -1$, $c = 0$ and $c = 1$ give the well known operator Bernstein, Szász, Mirakian and Lupas respectively motivated by Derriennic⁵, Sahai A. *et al.*⁶, studied the following modified Lupas operators defined for functions integrable on $[0, \infty)$ as

$$(1.6) \quad (Mnf)(x) = (n-1) \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} p_{n,k}(y) f(y) dy$$

where

$$(1.7) \quad p_{n,k}(t) = \binom{n+k-1}{k} \frac{t^k}{(1+t)^{n+k}}$$

In the present paper we study the following operator

$$(1.8) \quad \begin{aligned} & \left(L \frac{n}{c} + p f \right)(x) \\ & \left(L \frac{n}{c} + p f \right)(x) = c \binom{\frac{n}{c} + p - 1}{k} \sum_{k=0}^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+p+k}} \\ & \int_0^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \frac{(ct)^k}{(1+ct)^{\frac{n}{c}+p+k}} f(t) dt \end{aligned}$$

by putting $p = 0$ in (1.8) we get

$$(1.9) \quad \left(L_{\frac{n}{c}+p} f \right)(x) = c \left(\frac{n}{c} - 1 \right) \sum_{k=0}^{\infty} \binom{\frac{n}{c} + k - 1}{k} \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+k}} \\ \int_0^{\infty} \binom{\frac{n}{c} + k - 1}{k} \frac{(ct)^k}{(1+ct)^{\frac{n}{c}+k}} f(t) dt$$

By putting $c = 1$ in (1.9) we get the operator due to Sahai, A. *et al.*⁶. Again, by putting $c = 1$ in (1.8) we get the following operator.

$$(1.10) \quad \left(L_{\frac{n}{c}+p} f \right)(x) = (n+p-1) \sum_{k=0}^{\infty} \binom{n+p+k-1}{k} \frac{(x)^k}{(1+x)^{n+p+k}} \\ \int_0^{\infty} \binom{n+p+k-1}{k} \frac{(t)^k}{(1+t)^{n+p+k}} f(t) dt$$

which is extension of the operator due to Sahai A. *et al.*⁶ and an integral analogue of the operator (1.3) due to F. Schurer¹.

2. Basic Results

In this section we write some basic results to prove our theorems.

Result 1

For $n \geq 1$ one obtains,

$$(2.1) \quad \left(L_{\frac{n}{c}+p} 1 \right)(x) = 1$$

$$(2.2) \quad \left(L_{\frac{n}{c}+p} t \right)(x) = \frac{1}{c(\frac{n}{c}+p-2)} \{ c(\frac{n}{c}+p)x + 1 \}$$

Proof. Clearly

$$\left(L_{\frac{n}{c}+p} 1 \right)(x) = c \left(\frac{n}{c} + p - 1 \right) \sum_{k=0}^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+p+k}} \\ \int_0^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \frac{(ct)^k}{(1+ct)^{\frac{n}{c}+p+k}} f(t) dt$$

$$\begin{aligned}
 \left(L_{\frac{n}{c}+p} t \right) (x) &= c \left(\frac{n}{c} + p - 1 \right) \sum_{k=0}^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \frac{(cx)^k}{(1 + cx)^{\frac{n}{c} + p + k}} \\
 &\quad \int_0^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \frac{(ct)^k}{(1 + ct)^{\frac{n}{c} + p + k}} f(t) dt \\
 &= c \left(\frac{n}{c} + p - 1 \right) \sum_{k=0}^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \frac{(cx)^k}{(1 + cx)^{\frac{n}{c} + p + k}} \\
 &\quad \left[\frac{\left(\frac{n}{c} + p + k - 1 \right)!}{k! \left(\frac{n}{c} + p - 1 \right)!} \frac{1}{c} \int_0^{\infty} \frac{(ct)^k}{(1 + ct)^{\frac{n}{c} + p + k}} dt \right] \\
 &= c \left(\frac{n}{c} + p - 1 \right) \sum_{k=0}^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \frac{(cx)^k}{(1 + cx)^{\frac{n}{c} + p + k}} I_1
 \end{aligned}$$

where

$$I_1 = \frac{\left(\frac{n}{c} + p + k - 1 \right)!}{k! \left(\frac{n}{c} + p - 1 \right)!} \frac{1}{c} \int_0^{\infty} \frac{(ct)^k}{(1 + ct)^{\frac{n}{c} + p + k}} dt$$

$$I_1 = \frac{(k + 1)}{c^2 \left(\frac{n}{c} + p - 1 \right) \left(\frac{n}{c} + p - 2 \right)}$$

$$\begin{aligned}
 \left(L_{\frac{n}{c}+p} t \right) (x) &= \frac{c}{c^2} \left(\frac{n}{c} + p - 1 \right) \sum_{k=0}^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \\
 &\quad \frac{(cx)^k}{(1 + cx)^{\frac{n}{c} + p + k}} \frac{(k + 1)}{\left(\frac{n}{c} + p - 1 \right) \left(\frac{n}{c} + p - 2 \right)} \\
 &= \frac{1}{c} \frac{1}{\left(\frac{n}{c} + p - 2 \right)} \sum_{k=0}^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \frac{(cx)^k}{(1 + cx)^{\frac{n}{c} + p + k}} (k + 1)
 \end{aligned}$$

$$(2.3) \quad \left(L_{\frac{n}{c}+p}^t \right) (x) = \frac{1}{c} \frac{1}{\left(\frac{n}{c} + p - 2 \right)} \left[\sum_{k=0}^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \right.$$

$$\left. \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+p+k}} k + \sum_{k=0}^{\infty} \binom{\frac{n}{c} + p + k + 1}{k} \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+p+k}} \right]$$

we know that

$$(2.4) \quad (1+cx)^{\frac{n}{c}+p} = \left(1 - \frac{cx}{1+cx} \right)^{-\frac{n}{c}-p}$$

$$= \sum_{k=0}^{\infty} \frac{\left(\frac{n}{c} + p + k - 1 \right)!}{k! \left(\frac{n}{c} + p - 1 \right)!} \frac{(cx)^k}{(1+cx)^k}$$

Differentiating (2.4) w.r. to x , we get

$$\left(\frac{n}{c} + p \right) (1+cx)^{\frac{n}{c}+p-1} c = \sum_{k=0}^{\infty} \frac{\left(\frac{n}{c} + p + k - 1 \right)!}{k! \left(\frac{n}{c} + p - 1 \right)!} c^k$$

$$\left[\frac{k x^{k-1}}{(1+cx)^k} - \frac{c k x^k}{(1+cx)^{k+1}} \right]$$

$$= \sum_{k=0}^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \frac{(cx)^k}{(1+cx)^k} \frac{k}{x(1+cx)}$$

$$(2.5) \quad c \left(\frac{n}{c} + p \right) x = \sum_{k=0}^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \frac{(cx)^k}{(1+cx)^{k+\frac{n}{c}+p}} k$$

Using (2.4) and (2.5) in (2.3) we get,

$$(2.6) \quad \left(L_{\frac{n}{c}+p}^t \right) (x) = \frac{1}{c} \frac{1}{\left(\frac{n}{c} + p - 2 \right)} \left[c \left(\frac{n}{c} + p \right) x + 1 \right]$$

In particular

$$\begin{aligned}
 \left(L_{\frac{n}{c}+p}(t-x)\right)(x) &= c\left(\frac{n}{c}+p-1\right) \sum_{k=0}^{\infty} \binom{\frac{n}{c}+p+k-1}{k} \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+p+k}} \\
 &\quad \int_0^{\infty} \binom{\frac{n}{c}+p+k-1}{k} \frac{(ct)^k}{(1+ct)^{\frac{n}{c}+p+k}} (t-x) dt \\
 (2.7) \quad \left(L_{\frac{n}{c}+p}(t-x)\right)(x) &= c\left(\frac{n}{c}+p-1\right) \sum_{k=0}^{\infty} \binom{\frac{n}{c}+p+k-1}{k} \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+p+k}} \\
 &\quad \int_0^{\infty} \binom{\frac{n}{c}+p+k-1}{k} \frac{(ct)^k}{(1+ct)^{\frac{n}{c}+p+k}} t dt \\
 &\quad - cx\left(\frac{n}{c}+p-1\right) \sum_{k=0}^{\infty} \binom{\frac{n}{c}+p+k-1}{k} \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+p+k}} \\
 &\quad \int_0^{\infty} \binom{\frac{n}{c}+p+k-1}{k} \frac{(ct)^k}{(1+ct)^{\frac{n}{c}+p+k}} dt
 \end{aligned}$$

Using (2.6) and (2.1) in (2.7) we get,

$$\begin{aligned}
 \left(L_{\frac{n}{c}+p}(t-x)\right)(x) &= \left(L_{\frac{n}{c}+p}t\right)(x) - x\left(L_{\frac{n}{c}+p}1\right)(x) \\
 &= \frac{1}{c} \frac{1}{\left(\frac{n}{c}+p-2\right)} \left[c\left(\frac{n}{c}+p\right)x + 1 \right] - x \\
 &= \frac{1}{c} \frac{1}{\left(\frac{n}{c}+p-2\right)} [1 + 2cx] \\
 &= \frac{1 + 2cx}{n + c(p-2)} \\
 (2.8) \quad \left(L_{\frac{n}{c}+p}(t-x)\right)(x) &= \frac{1 + 2cx}{n + c(p-2)}
 \end{aligned}$$

Result 2Let $\frac{n}{c} + p > m + 2$

$$(2.9) \quad T_{\frac{n}{c}+p,m} = c \left(\frac{n}{c} + p - 1 \right) \sum_{k=0}^{\infty} \binom{\frac{n}{c}+p+k-1}{k} \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+p+k}} \\ \int_0^{\infty} \binom{\frac{n}{c}+p+k-1}{k} \frac{(ct)^k}{(1+ct)^{\frac{n}{c}+p+k}} (t-x)^m dt$$

$$(2.10) \quad c \left(\frac{n}{c} + p - m - 2 \right) T_{\frac{n}{c}+p,m+1} = x(1+cx) \left\{ T'_{\frac{n}{c}+p,m} + 2m T_{\frac{n}{c}+p,m-1} \right\} \\ + (m+1)(1+2cx) T_{\frac{n}{c}+p,m}$$

with

$$(2.11) \quad T_{\frac{n}{c}+p,0} = 1$$

$$(2.12) \quad T_{\frac{n}{c}+p,1} = \frac{1+2xc}{n+c(p-2)}$$

Proof.Differentiating (2.9) w.r.t. x

$$T'_{\frac{n}{c}+p,m} = c \left(\frac{n}{c} + p - 1 \right) \sum_{k=0}^{\infty} \binom{\frac{n}{c}+p+k-1}{k} c^k \left[\frac{(kx)^{k-1}}{(1+cx)^{\frac{n}{c}+p+k}} - \frac{cx^k \left(\frac{n}{c} + p + k \right)}{(1+cx)^{\frac{n}{c}+p+k+1}} \right] \\ \int_0^{\infty} \binom{\frac{n}{c}+p+k-1}{k} \frac{(ct)^k}{(1+ct)^{\frac{n}{c}+p+k}} (t-x)^m dt \\ - c \left(\frac{n}{c} + p - 1 \right) \sum_{k=0}^{\infty} \binom{\frac{n}{c}+p+k-1}{k} \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+p+k}} \\ \int_0^{\infty} \binom{\frac{n}{c}+p+k-1}{k} \frac{(ct)^k}{(1+ct)^{\frac{n}{c}+p+k}} m(t-x)^{m-1} dt \\ T'_{\frac{n}{c}+p,m} = c \left(\frac{n}{c} + p - 1 \right) \sum_{k=0}^{\infty} \binom{\frac{n}{c}+p+k-1}{k} \frac{c^k x^k}{(1+cx)^{\frac{n}{c}+p+k}} \left[\frac{k}{x} - \frac{c \left(\frac{n}{c} + p + k \right)}{(1+cx)} \right]$$

$$\begin{aligned}
& \int_0^\infty \binom{\frac{n}{c}+p+k-1}{k} \frac{(ct)^k}{(1+ct)^{\frac{n}{c}+p+k}} (t-x)^m dt - mT_{\frac{n}{c}+p, m-1} \\
& \sum_{k=0}^\infty \binom{\frac{n}{c}+p+k-1}{k} \frac{c^k x^k}{(1+cx)^{\frac{n}{c}+p+k}} [k+kcx-nx-pcx-kcx] \\
& \int_0^\infty \binom{\frac{n}{c}+p+k-1}{k} \frac{(ct)^k}{(1+ct)^{\frac{n}{c}+p+k}} (t-x)^m dt - mx(1+cx) T_{\frac{n}{c}+p, m-1}
\end{aligned}$$

$$\begin{aligned}
x(1+cx) \left\{ T'_{\frac{n}{c}+p, m} + mT_{\frac{n}{c}+p, m-1} \right\} &= c \left(\frac{n}{c} + p - 1 \right) \\
& \sum_{k=0}^\infty \binom{\frac{n}{c}+p+k-1}{k} \frac{c^k x^k}{(1+cx)^{\frac{n}{c}+p+k}} \left[k - xc \left(\frac{n}{c} + p \right) \right] \\
& \int_0^\infty \binom{\frac{n}{c}+p+k-1}{k} \frac{(ct)^k}{(1+ct)^{\frac{n}{c}+p+k}} (t-x)^m dt \\
&= c \left(\frac{n}{c} + p - 1 \right) \sum_{k=0}^\infty \binom{\frac{n}{c}+p+k-1}{k} \frac{c^k x^k}{(1+cx)^{\frac{n}{c}+p+k}} \\
& \int_0^\infty \left[k - c \left(\frac{n}{c} + p \right) t + c \left(\frac{n}{c} + p \right) t - xc \left(\frac{n}{c} + p \right) \right] \\
& \binom{\frac{n}{c}+p+k-1}{k} \frac{(ct)^k}{(1+ct)^{\frac{n}{c}+p+k}} (t-x)^m dt \\
&= c \left(\frac{n}{c} + p - 1 \right) \sum_{k=0}^\infty \binom{\frac{n}{c}+p+k-1}{k} \frac{c^k x^k}{(1+cx)^{\frac{n}{c}+p+k}} \\
& \int_0^\infty \left[k - c \left(\frac{n}{c} + p \right) t \right] \binom{\frac{n}{c}+p+k-1}{k} \frac{(ct)^k}{(1+ct)^{\frac{n}{c}+p+k}} (t-x)^m dt
\end{aligned}$$

$$\begin{aligned}
& + c \left(\frac{n}{c} + p - 1 \right) \sum_{k=0}^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \frac{c^k x^k}{(1 + cx)^{\frac{n}{c} + p + k}} \\
& \int_0^{\infty} c \left(\frac{n}{c} + p \right) \binom{\frac{n}{c} + p + k - 1}{k} \frac{(ct)^k}{(1 + ct)^{\frac{n}{c} + p + k}} (t - x)^{m+1} dt \\
& = c \left(\frac{n}{c} + p - 1 \right) \sum_{k=0}^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \frac{c^k x^k}{(1 + cx)^{\frac{n}{c} + p + k}} \\
& \int_0^{\infty} \left[k - c \left(\frac{n}{c} + p \right) t \right] \binom{\frac{n}{c} + p + k - 1}{k} \frac{(ct)^k}{(1 + ct)^{\frac{n}{c} + p + k}} (t - x)^m dt \\
& + c \left(\frac{n}{c} + p \right) T_{\frac{n}{c} + p, m+1} \\
& x(1 + cx) \left\{ T'_{\frac{n}{c} + p, m} + m T_{\frac{n}{c} + p, m-1} \right\} - c \left(\frac{n}{c} + p \right) T_{\frac{n}{c} + p, m+1} \\
& = c \left(\frac{n}{c} + p - 1 \right) \sum_{k=0}^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \frac{c^k x^k}{(1 + cx)^{\frac{n}{c} + p + k}} \int_0^{\infty} \left[k - c \left(\frac{n}{c} + p \right) t \right] \\
& \binom{\frac{n}{c} + p + k - 1}{k} \frac{(ct)^k}{(1 + ct)^{\frac{n}{c} + p + k}} (t - x)^m dt \\
& \lim_{\frac{n}{c} + p, k} (t) = \binom{\frac{n}{c} + p + k - 1}{k} \frac{(ct)^k}{(1 + ct)^{\frac{n}{c} + p + k}}
\end{aligned}$$

then we can write on

$$\begin{aligned}
t(1 + ct) p'_{\frac{n}{c} + p, k}(t) & = (k - ct(\frac{n}{c} + p)) p_{\frac{n}{c} + p, k}(t) \\
& x(1 + cx) \left\{ T'_{\frac{n}{c} + p, m} + m T_{\frac{n}{c} + p, m} \right\} - c \left(\frac{n}{c} + p \right) T_{\frac{n}{c} + p, m+1} \\
& = c \left(\frac{n}{c} + p - 1 \right) \sum_{k=0}^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \frac{(ct)^k}{(1 + ct)^{\frac{n}{c} + p + k}}
\end{aligned}$$

$$\begin{aligned}
& \int_0^{\infty} t(1+ct) p'_{\frac{n}{c}+p,k}(t) (t-m)^m dt \\
&= c \left(\frac{n}{c} + p - 1 \right) \sum_{k=0}^{\infty} \binom{\frac{n}{c} + p + k - 1}{k} \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+p+k}} \\
& \quad \int_0^{\infty} \left[(1+2cx)(t-x) + c(t-x)^2 + x(1+cx) \right] p'_{\frac{n}{c}+p,k}(t) (t-m)^m dt \\
&= -(m+1)(1+2cx) T_{\frac{n}{c}+p,m} - c(m+2) T_{\frac{n}{c}+p,m+1} - mx(1+cx) T_{\frac{n}{c}+p,m-1} \\
& \quad x(1+cx) \left\{ T'_{\frac{n}{c}+p,m} + m T_{\frac{n}{c}+p,m-1} \right\} + (m+1)(1+2cx) T'_{\frac{n}{c}+p,m-1} \\
& \quad + mx(1+cx) T_{\frac{n}{c}+p,m-1} = \left[c \left(\frac{n}{c} + p - m - 2 \right) \right] T_{\frac{n}{c}+p,m+1} \\
&\therefore T_{\frac{n}{c}+p,1} = \frac{1+2cx}{c \left(\frac{n}{c} + p - 2 \right)} \\
&\text{and} \\
& T'_{\frac{n}{c}+p,1} = \frac{2}{c \left(\frac{n}{c} + p - 2 \right)}
\end{aligned}$$

In particular by putting $m = 1$ in (2.10)

$$\begin{aligned}
& c \left(\frac{n}{c} + p - 3 \right) T'_{\frac{n}{c}+p,2} = x(1+cx) \left\{ T'_{\frac{n}{c}+p,1} + 2 T_{\frac{n}{c}+p,0} \right\} \\
& \quad + (2)(1+2cx) T'_{\frac{n}{c}+p,1} \\
&= x(1+cx) \left\{ \frac{2c}{c \left(\frac{n}{c} + p - 2 \right)} + 2.1 \right\} + 2 \frac{(1+2cx)(1+2x)}{c \left(\frac{n}{c} + p - 2 \right)} \left\{ \frac{1+2cx}{c \left(\frac{n}{c} + p - 2 \right)} \right\} \\
&= 2x(1+cx) \left\{ \frac{\left(\frac{n}{c} + p - 1 \right)}{c \left(\frac{n}{c} + p - 2 \right)} \right\} + \frac{2(1+2cx)^2}{c \left(\frac{n}{c} + p - 2 \right)}
\end{aligned}$$

$$(2.13) \quad T_{\frac{n}{c}+p,2} = \frac{2x(1+cx)\left(\frac{n}{c}+p-1\right)}{c^2\left(\frac{n}{c}+p-2\right)\left(\frac{n}{c}+p-3\right)} + \frac{2(1+2cx)^2}{c^2\left(\frac{n}{c}+p-2\right)\left(\frac{n}{c}+p-3\right)}$$

and

$$(2.14) \quad T_{\frac{n}{c}+p,4} = 0\left(\frac{1}{n^2}\right)$$

$$\begin{aligned} T_{\frac{n}{c}+p,2} &= \frac{2}{c^2\left(\frac{n}{c}+p-2\right)\left(\frac{n}{c}+p-3\right)} \left[cx(1+cx)\left(\frac{n}{c}+p-1\right)(1+2cx)^2 \right] \\ &= \frac{2\left(\frac{n}{c}+p-3\right)}{c^2\left(\frac{n}{c}+p-2\right)\left(\frac{n}{c}+p-3\right)} \left[cx + cx^2 + c^2x^2 \right] \\ &= \frac{2\left(\frac{n}{c}+p-3\right)}{c^2\left(\frac{n}{c}+p-2\right)\left(\frac{n}{c}+p-3\right)} cx \left[1 + cx \right] \\ &= \frac{2\left(\frac{n}{c}+p-3\right)x(1+cx)}{c^2\left(\frac{n}{c}+p-2\right)\left(\frac{n}{c}+p-3\right)} \\ (2.15) \quad &= \mu_{n,p,c}(x) \quad (\text{say}) \end{aligned}$$

3. Main Results

Theorem 1. If a function 'f' is such that its first and second order derivatives are bounded in $[0, \infty)$; then

$$(3.1) \quad \lim_{n \rightarrow \infty} \left\{ n + c(p-2) \right\} \left\{ \left(L_{\frac{n}{c}+p} f \right)(x) - f(x) \right\} \\ = f'(x)(1+2cx) + x(1+cx)f''(x)$$

Proof. Using Taylor's theorem we write that

$$(3.2) \quad f(t) - f(x) = (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + \frac{(t-x)^2}{2!}\eta(t,x)$$

when $\eta(t,x)$ is a bounded function $\forall t, x$ and

$$\lim_{t \rightarrow x} \eta(t, x) = 0$$

Now, applying (1.8) and (3.2), we get,

$$\begin{aligned} & \left(L_{\frac{n}{c}+p} f \right) (x) - f(x) \left(L_{\frac{n}{c}+p-1} \right) (x) \\ &= f'(x) \left(L_{\frac{n}{c}+p} (t-x) \right) (x) + \frac{f''(x)}{2} \left(L_{\frac{n}{c}+p} (t-x)^2 \right) (x) + I_3 \end{aligned}$$

where

$$I_3 = \frac{1}{2} \left(L_{\frac{n}{c}+p} (t-x)^2 \eta(t, x) \right) (x)$$

Using (2.1), (2.12) and (2.13), we get,

$$\begin{aligned} & \left(L_{\frac{n}{c}+p} f \right) (x) - f(x) = f'(x) T_{\frac{n}{c}+p, 1} + \frac{f''(x)}{2} T_{\frac{n}{c}+p, 2} + I_3 \\ & \left(L_{\frac{n}{c}+p} f \right) (x) - f(x) = f'(x) \left[\frac{1+2xc}{n+c(p-2)} \right] + \frac{f''(x)}{2} 2 \\ & \left[\frac{x(1+cx) \left(\frac{n}{c}+p-1 \right)}{c^2 \left(\frac{n}{c}+p-2 \right) \left(\frac{n}{c}+p-3 \right)} + \frac{(1+2cx)^2}{c^2 \left(\frac{n}{c}+p-2 \right) \left(\frac{n}{c}+p-3 \right)} + I_3 \right] \end{aligned}$$

$$\begin{aligned} & [n+c(p-2)] \left\{ \left(L_{\frac{n}{c}+p} f \right) (x) - f(x) \right\} = f'(x) (1+2x) + f''(x) \\ & \left[x(1+cx) \frac{\left(\frac{n}{c}+p-1 \right)}{c \left(\frac{n}{c}+p-3 \right)} + \frac{(1+2cx)^2}{c \left(\frac{n}{c}+p-3 \right)} \right] + \{n+c(p-2)\} I_3 \end{aligned}$$

Now, we have to show that as $n \rightarrow \infty$, the value of $\{n+c(p-2)\}I_3 \rightarrow 0$. Let $\varepsilon > 0$ be given since $\eta(t, x) \rightarrow 0$ as $t \rightarrow 0$, then there exists $\delta > 0$ such that when $|t-x| < \delta$ we have $|\eta(t, x)| < \varepsilon$ and when $|t-x| \geq \delta$, we write

$$|\eta(t, x)| \leq M. 1 \leq M \frac{(t-x)^2}{\delta^2}$$

Thus, for all $t, x \in [0, \infty)$ we can write

$$|\eta(t, x)| \leq \varepsilon + M \frac{(t-x)^2}{\delta^2}$$

$$\begin{aligned} \{n+c(p-2)\} |I_3| &\leq [n+c(p-2)] \left\{ L_{\frac{n}{c}+p} (t-x)^2 \left(\varepsilon + \frac{M(t-x)^2}{\delta^2} \right) \right\} (x) \\ \therefore \{n+c(p-2)\} |I_3| &\leq \varepsilon [n+c(p-2)] \left(L_{\frac{n}{c}+p} (t-x)^2 \right) (x) \\ &\quad + \frac{M}{\delta^2} (n+c(p-2)) \left(L_{\frac{n}{c}+p} (t-x)^4 \right) (x) \end{aligned}$$

Now, using (2.13) and (2.14) we see that, as $n \rightarrow \infty$

$$\therefore \{n+c(p-2)\} |I_3| \rightarrow 0$$

Thus, we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \{n+c(p-2)\} \left\{ \left(L_{\frac{n}{c}+p} f \right) (x) - f(x) \right\} \\ = f'(x)(1+2x) + x(1+cx)f''(x) \end{aligned}$$

This completes the proof.

Remark. By putting $c = 1$, $p = 0$ in (3.1), we get this following result.

$$\lim_{n \rightarrow \infty} n \left\{ (L_n f)(x) - f(x) \right\} = f'(x)(1+2x) + x(1+x)f''(x)$$

which is due to Sahai A.⁶, et al.

Theorem 2. If $g \in C_B^2[0, \infty)$ then we have

$$(3.3) \quad \left| \left(L_{\frac{n}{c}+p} g \right) (x) - g(x) \right| \leq \lambda_{n,p,c}(x) \|g\| c_B^2$$

$$\text{where} \quad \lambda_{n,p,c}(x) = \max \left\{ \frac{1+2x}{n+c+p-2}, \frac{1}{2} \mu_{n,p,c}(x) \right\}$$

Proof. We write that

$$(3.4) \quad g(t)g(x) = (t-x)g'(x) + \frac{1}{2}(t-x)^2 g''(\xi)$$

where ξ lies between t and x

Now applying (1.8) on (3.4) we write

$$\begin{aligned} \left| \left(L_{\frac{n}{c}+p} g \right) (x) - g(x) \right| &\leq \|g'\| \left| \left(L_{\frac{n}{c}+p} (t-x) \right) (x) \right| \\ &\quad + \frac{1}{2} \|g''\| \left| \left(L_{\frac{n}{c}+p} (t-x)^2 \right) (x) \right| \end{aligned}$$

Now using (2.12) and (2.13) we see that

$$\begin{aligned} \left| \left(L_{\frac{n}{c}+p} g \right) (x) - g(x) \right| &\leq \frac{1+2x}{n+c(p-2)} \|g'\| + \frac{\lambda_{n,p,c}(x)}{2} \|g''\| \\ &\leq \lambda_{n,p,c}(x) \{ \|g'\| + \|g''\| \} \leq \lambda_{n,p,c}(x) \|g\|_{C_B^2} \end{aligned}$$

Theorem 3. For $f \in C_B [0, \infty)$ one obtains

$$\left| \left(L_{\frac{n}{c}+p} f \right) (x) - f(x) \right| \leq 2A \left\{ \omega_2 \left(f, \frac{\sqrt{\lambda_{n,p,c}(x)}}{2} \right) + \min \left(1, \frac{\lambda_{n,p,c}}{2} \right) \|f\|_{C_B} \right\}$$

where constant A depends on f and $\lambda_{n,p,c}(x)$.

Proof. For $f \in C_B [0, \infty)$ and $g \in C_B^2 [0, \infty)$ we write

$$\begin{aligned} \left(L_{\frac{n}{c}+p} f \right) (x) - f(x) &= \left(L_{\frac{n}{c}+p} f \right) (x) - \left(L_{\frac{n}{c}+p} g \right) (x) \\ &\quad + \left(L_{\frac{n}{c}+p} g \right) (x) - g(x) + g(x) - f(x) \end{aligned}$$

$$\begin{aligned} \left| \left(L_{\frac{n}{c}+p} f \right) (x) - f(x) \right| &\leq \left| \left(L_{\frac{n}{c}+p} f \right) (x) - \left(L_{\frac{n}{c}+p} g \right) (x) \right| \\ &\quad + \left| \left(L_{\frac{n}{c}+p} g \right) (x) - g(x) \right| + |g(x) - f(x)| \quad \text{Using (3.3)} \end{aligned}$$

$$\leq \|L_{\frac{n}{c}+p} f\| \|f - g\| + \lambda_{n,p,c} \|g\|_{C_B^2} + \|f - g\|$$

$$\leq 2 \|f - g\| + \lambda_{n,p,c}(x) \|g\|_{C_B^2}$$

$$\leq 2 \left\{ \|f - g\| + \frac{1}{2} \lambda_{n,p,c}(x) \|g\|_{C_B^2} \right\}$$

Using Peetre K functions

$$\leq 2K \left\{ f; \frac{1}{2} \lambda_{n,p,c}(x) \right\}$$

$$\leq 2A \left\{ \omega_2 \left(f, \frac{\sqrt{\lambda_{n,p,c}(x)}}{2} \right) + \min \left(1, \frac{\lambda_{n,p,c}}{2} \right) \|f\|_{C_B} \right\}$$

This completes the proof.

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