

Fuzzy Anti n- Inner Product Space

Parijat Sinha

Department of Mathematics, V.S.S.D. College, Kanpur

(Received December 31, 2020)

Abstract: In the present paper, I have introduced the concept of fuzzy anti n-inner product function on a linear space and discussed its properties. Fuzzy anti n-inner product has been defined relative to fuzzy anti n-norm. Lastly Parallelogram law and polarization identity has been proved.

Keywords: fuzzy anti n-inner product space, α -n-norm, Parallelogram law, Polarization identity.

1. Introduction

Majumdar and Samanta¹ were the first to define fuzzy inner product function on a linear space and discussed properties of fuzzy inner product function. They also induced a fuzzy norm from the fuzzy inner product function and derived several results. Sinha, Mishra and Lal² introduced the definition of fuzzy anti inner product space. and proved parallelogram law and polarization identity on fuzzy anti inner product space. In 2007, Vijayabalaji and Thillaligovindan ³ introduced the concept of fuzzy n-inner product space.

Here, I have introduced the concept of fuzzy anti n-inner product space and established some theorems on fuzzy anti n-inner product space in the present paper. I have also proved parallelogram law and polarization identity on fuzzy anti n-inner product space.

2. Preliminaries

This section contains some definitions and preliminary results which will be used in this paper.

Definition⁴ 2.1: Let X be a linear space over a real field F . A fuzzy subset N^* of $X \times X \times \dots \times X \times R$ such that $x_1, x_2, \dots, x_n, y \in X$ and $c \in F$ is called a fuzzy anti n -norm on X if and only if it satisfies,

(Fa-nN1) For all $t \in R$ with $t \leq 0$, $N^*(x_1, x_2, \dots, x_n, t) = 1$

(Fa-nN2) For all $t \in R$ with $t > 0$, $N^*(x_1, x_2, \dots, x_n, t) = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent.

(Fa-nN3) $N^*(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \dots, x_n$.

(Fa-nN4) For all $t \in R$ with $t > 0$, $N^*(x_1, x_2, \dots, cx_n, t)$

$$= N^*\left(x_1, x_2, \dots, x_n, \frac{t}{|c|}\right) \text{ if } c \neq 0,$$

(Fa-nN5) For $s, t \in R$ with $t > 0$,

$$N^*(x_1, x_2, \dots, x_{n-1}, x_n + y, s + t) \leq \max\{N^*(x_1, x_2, \dots, x_{n-1}, x_n, s), N^*(x_1, x_2, \dots, x_{n-1}, y, t)\}$$

(Fa-nN6) $N^*(x_1, x_2, \dots, x_n, t)$ is non-increasing function of $t \in R$ and

$$\lim_{t \rightarrow \infty} N^*(x_1, x_2, \dots, x_n, t) = 0$$

(X, N^*) is called a fuzzy anti n -normed linear space.

Following condition of fuzzy anti n -norm N^* will be required later,

(Fa-nN7) for all $t \in R$ with $t > 0$, $N^*(x_1, x_2, \dots, x_n, t) < 1$ implies that $x_1, x_2, \dots, x_n$ are linearly dependent.

Example⁴ 2.1: Let $(X, \|\bullet, \bullet, \dots, \bullet\|)$ be n -normed linear space. Define

$$N^*(x_1, x_2, \dots, x_n, t) = \frac{\|x_1, x_2, \dots, x_n\|}{t + \|x_1, x_2, \dots, x_n\|}, \text{ when } t > 0, t \in R,$$

$$x_1, x_2, \dots, x_n \in X$$

$$= 1, \quad \text{when } t \leq 0, x_1, x_2, \dots, x_n \in X$$

Then (X, N^*) will be a fuzzy anti n -normed linear space.

Theorem⁴ 2.1: Let (X, N^*) be a fuzzy anti n-normed linear space, if we assume that for all $t > 0$, $N^*(x_1, x_2, \dots, x_n, t) < 1$ implies that $x_1, x_2, \dots, x_n$ are linearly dependent.

Define, $\|x_1, x_2, \dots, x_n\|_\alpha^* = \inf \{t : N^*(x_1, x_2, \dots, x_n, t) < \alpha, \alpha \in (0, 1)\}$ is called α - n-norm. Then $\{\|\bullet, \bullet, \dots, \bullet\|_\alpha^* : \alpha \in (0, 1)\}$ is a decreasing family of α -n-norm on linear space X .

3. Fuzzy Anti n-Inner Product Space

In this section, I have introduced the definition of fuzzy anti n-inner product space and discussed its properties.

Definition 3.1: Let X be a linear space over the field C of complex numbers. Let $\mu^* : X^{n+1} \times C \rightarrow [0, 1]$ be a mapping, such that the following conditions hold,

(Fan-IP1) For $s, t \in C$,

$$\mu^*(x + x', y | x_2, \dots, x_n, |t| + |s|) \leq \max \{ \mu^*(x, y | x_2, \dots, x_n, |t|), \mu^*(x', y | x_2, \dots, x_n, |s|) \}$$

(Fan-IP2) For $s, t \in C$,

$$\mu^*(x, y | x_2, \dots, x_n, |st|) \leq \max \{ \mu^*(x, x | x_2, \dots, x_n, |s|^2), \mu^*(y, y | x_2, \dots, x_n, |t|^2) \}$$

(Fan-IP3) For $t \in C$, $\mu^*(x, y | x_2, \dots, x_n, t) = \mu^*(y, x | x_2, \dots, x_n, \bar{t})$,

(Fan-IP4) $\mu^*(\alpha x, y | x_2, \dots, x_n, t) = \mu^*\left(x, y | x_2, \dots, x_n, \frac{t}{|\alpha|}\right)$, $\alpha (\neq 0) \in C$, $t \in C$

(Fan-IP5) For all $t \in C$, $\mu^*(x, x | x_2, \dots, x_n, t) = 1, \forall t \in C \setminus R^+$

(Fan-IP6) $\mu^*(x, x | x_2, \dots, x_n, t) = 0$, for every $t > 0$, iff and only if $x, x_2, \dots, x_n$ are linearly dependent.

(Fan-IP7) $\mu^*(x, x | x_2, \dots, x_n, \bullet) : R \rightarrow [0, 1]$ is a monotonic, non-increasing function of R and

$$\lim_{t \rightarrow \infty} \mu^*(x, x | x_2, \dots, x_n, t) = 0 \text{ as } t \rightarrow \infty.$$

Then μ^* is said to be fuzzy anti n-inner product (Fan-IP) function on X and the pair (X, N^*) is called a fuzzy anti n-inner product space.

Example 3.1: Let $(X, (\bullet, \bullet | \bullet, \dots, \bullet))$ be an n -inner product space. Define a function $\mu^* : X^{n+1} \times C \rightarrow [0,1]$

$$\mu^*(x, y | x_2, \dots, x_n, t) = \frac{|\langle x, y | x_2, \dots, x_n \rangle|}{|t| + |\langle x, y | x_2, \dots, x_n \rangle|} \quad \forall t \in C \setminus \{0\}$$

and linearly independent $x, \dots, x_n, y \in X$

$$= 1 \text{ for } t = 0 \text{ and } \forall \text{ linearly independent } x, \dots, x_n, y \in X$$

$$= 1 \quad \forall t \in (C \setminus R^+) \cup \{0\} \text{ and } \forall \text{ linearly dependent } x, \dots, x_n, y \in X$$

$$= \frac{|\langle x, y | x_2, \dots, x_n \rangle|}{|t| + |\langle x, y | x_2, \dots, x_n \rangle|}, \quad \forall t \in R^+ \setminus \{0\}$$

and $\forall$ linearly dependent $x, \dots, x_n, y \in X$

Then μ^* is a fuzzy anti n -inner product function.

Proof: (Fan-IP1) If at least one of $|t|$ and $|s|$ is zero, then the result is trivial. Consider the case when none of $|t|$ and $|s|$ is zero.

Let us assume, without loss of generality, that

$$\mu^*(x, y | x_2, \dots, x_n, |t|) \geq \mu^*(x', y | x_2, \dots, x_n, |s|)$$

$$\Rightarrow \frac{|\langle x, y | x_2, \dots, x_n \rangle|}{|t| + |\langle x, y | x_2, \dots, x_n \rangle|} \geq \frac{|\langle x', y | x_2, \dots, x_n \rangle|}{|s| + |\langle x', y | x_2, \dots, x_n \rangle|}$$

$$\Rightarrow \frac{|t| + |\langle x, y | x_2, \dots, x_n \rangle|}{|\langle x, y | x_2, \dots, x_n \rangle|} \leq \frac{|s| + |\langle x', y | x_2, \dots, x_n \rangle|}{|\langle x', y | x_2, \dots, x_n \rangle|}$$

$$\Rightarrow \frac{|t|}{|\langle x, y | x_2, \dots, x_n \rangle|} + 1 \leq \frac{|s|}{|\langle x', y | x_2, \dots, x_n \rangle|} + 1$$

$$\Rightarrow \frac{|t|}{|\langle x, y | x_2, \dots, x_n \rangle|} \leq \frac{|s|}{|\langle x', y | x_2, \dots, x_n \rangle|}$$

$$\begin{aligned}
&\Rightarrow \left| \langle x', y \mid x_2, \dots, x_n \rangle \right| \leq \frac{|s|}{|t|} \cdot \left| \langle x, y \mid x_2, \dots, x_n \rangle \right| \\
&\Rightarrow \left| \langle x, y \mid x_2, \dots, x_n \rangle \right| + \left| \langle x', y \mid x_2, \dots, x_n \rangle \right| \leq \left| \langle x, y \mid x_2, \dots, x_n \rangle \right| \\
&\quad + \frac{|s|}{|t|} \cdot \left| \langle x, y \mid x_2, \dots, x_n \rangle \right| \\
&\Rightarrow \left| \langle x, y \mid x_2, \dots, x_n \rangle \right| + \left| \langle x', y \mid x_2, \dots, x_n \rangle \right| \leq \left(1 + \frac{|s|}{|t|} \right) \left| \langle x, y \mid x_2, \dots, x_n \rangle \right| \\
&\Rightarrow \left| \langle x + x', y \mid x_2, \dots, x_n \rangle \right| \leq \left| \langle x, y \mid x_2, \dots, x_n \rangle \right| + \left| \langle x', y \mid x_2, \dots, x_n \rangle \right| \\
&\leq \left(1 + \frac{|s|}{|t|} \right) \left| \langle x, y \mid x_2, \dots, x_n \rangle \right| \\
&\Rightarrow \left| \langle x + x', y \mid x_2, \dots, x_n \rangle \right| \leq \left(\frac{|t| + |s|}{|t|} \right) \left| \langle x, y \mid x_2, \dots, x_n \rangle \right| \\
&\Rightarrow \frac{\left| \langle x + x', y \mid x_2, \dots, x_n \rangle \right|}{|t| + |s|} \leq \frac{\left| \langle x, y \mid x_2, \dots, x_n \rangle \right|}{|t|} \\
&\Rightarrow \frac{|t|}{\left| \langle x, y \mid x_2, \dots, x_n \rangle \right|} \leq \frac{|t| + |s|}{\left| \langle x + x', y \mid x_2, \dots, x_n \rangle \right|} \\
&\Rightarrow \frac{|t|}{\left| \langle x, y \mid x_2, \dots, x_n \rangle \right|} + 1 \leq \frac{|t| + |s|}{\left| \langle x + x', y \mid x_2, \dots, x_n \rangle \right|} + 1 \\
&\Rightarrow \frac{|t| + \left| \langle x, y \mid x_2, \dots, x_n \rangle \right|}{\left| \langle x, y \mid x_2, \dots, x_n \rangle \right|} \leq \frac{|t| + |s| + \left| \langle x + x', y \mid x_2, \dots, x_n \rangle \right|}{\left| \langle x + x', y \mid x_2, \dots, x_n \rangle \right|} \\
&\Rightarrow \mu^* (x + x', y \mid x_2, \dots, x_n, |t| + |s|) \leq \max \left\{ \mu^* (x, y \mid x_2, \dots, x_n, |t|), \right. \\
&\quad \left. \mu^* (x', y \mid x_2, \dots, x_n, |s|) \right\}
\end{aligned}$$

(Fan-IP2) If at least one of $|t|$ and $|s|$ is zero, the result is trivial. Now, consider the case when none of $|t|$ and $|s|$ is zero. Let us assume, without loss of generality, that

$$\begin{aligned}
 & \mu^*(x, x \mid x_2, \dots, x_n, |s|^2) \geq \mu^*(y, y \mid x_2, \dots, x_n, |t|^2) \\
 \Rightarrow & \frac{|\langle x, x \mid x_2, \dots, x_n \rangle|}{|s|^2 + |\langle x, x \mid x_2, \dots, x_n \rangle|} \geq \frac{|\langle y, y \mid x_2, \dots, x_n \rangle|}{|t|^2 + |\langle y, y \mid x_2, \dots, x_n \rangle|} \\
 \Rightarrow & \frac{|s|^2 + |\langle x, x \mid x_2, \dots, x_n \rangle|}{|\langle x, x \mid x_2, \dots, x_n \rangle|} \leq \frac{|t|^2 + |\langle y, y \mid x_2, \dots, x_n \rangle|}{|\langle y, y \mid x_2, \dots, x_n \rangle|} \\
 \Rightarrow & \frac{|s|^2}{|\langle x, x \mid x_2, \dots, x_n \rangle|} + 1 \leq \frac{|t|^2}{|\langle y, y \mid x_2, \dots, x_n \rangle|} + 1 \\
 \Rightarrow & \frac{|s|^2}{|\langle x, x \mid x_2, \dots, x_n \rangle|} \leq \frac{|t|^2}{|\langle y, y \mid x_2, \dots, x_n \rangle|} \\
 \Rightarrow & \frac{|s|^2}{|\langle x, x \mid x_2, \dots, x_n \rangle| |\langle x, x \mid x_2, \dots, x_n \rangle|} \leq \frac{|t|^2}{|\langle x, x \mid x_2, \dots, x_n \rangle| |\langle y, y \mid x_2, \dots, x_n \rangle|} \\
 \Rightarrow & \frac{|s|^2}{|\langle x, x \mid x_2, \dots, x_n \rangle|^2} \leq \frac{|t|^2}{\|x, x_2, \dots, x_n\|^2 \|y, x_2, \dots, x_n\|^2} \\
 \Rightarrow & \frac{|s|^4}{|\langle x, x \mid x_2, \dots, x_n \rangle|^2} \leq \frac{|s|^2 |t|^2}{\|x, x_2, \dots, x_n\|^2 \|y, x_2, \dots, x_n\|^2}
 \end{aligned}$$

Taking square root on both side,

$$\Rightarrow \frac{|s|^2}{|\langle x, x \mid x_2, \dots, x_n \rangle|} \leq \frac{|s| |t|}{\|x, x_2, \dots, x_n\| \|y, x_2, \dots, x_n\|} \leq \frac{|st|}{|\langle x, y \mid x_2, \dots, x_n \rangle|}$$

$$\Rightarrow \frac{|s|^2}{|\langle x, x | x_2, \dots, x_n \rangle|} \leq \frac{|st|}{|\langle x, y | x_2, \dots, x_n \rangle|}$$

$$\Rightarrow \frac{|s|^2}{|\langle x, x | x_2, \dots, x_n \rangle|} + 1 \leq \frac{|st|}{|\langle x, y | x_2, \dots, x_n \rangle|} + 1$$

$$\Rightarrow \frac{|s|^2 + |\langle x, x | x_2, \dots, x_n \rangle|}{|\langle x, x | x_2, \dots, x_n \rangle|} \leq \frac{|st| + |\langle x, y | x_2, \dots, x_n \rangle|}{|\langle x, y | x_2, \dots, x_n \rangle|}$$

$$\Rightarrow \frac{|\langle x, x | x_2, \dots, x_n \rangle|}{|s|^2 + |\langle x, x | x_2, \dots, x_n \rangle|} \geq \frac{|\langle x, y | x_2, \dots, x_n \rangle|}{|st| + |\langle x, y | x_2, \dots, x_n \rangle|}$$

$$\Rightarrow \mu^*(x, x | x_2, \dots, x_n, |s|^2) \geq \mu^*(x, y | x_2, \dots, x_n, |st|)$$

$$\Rightarrow \mu^*(x, y | x_2, \dots, x_n, |st|) \leq \max \left\{ \mu^*(x, x | x_2, \dots, x_n, |s|^2), \mu^*(y, y | x_2, \dots, x_n, |t|^2) \right\}$$

(Fan-IP3) for $t \in C$, $|t| = |\bar{t}|$

$$\begin{aligned} \mu^*(y, x | x_2, \dots, x_n, \bar{t}) &= \frac{|\langle y, x | x_2, \dots, x_n \rangle|}{|\bar{t}| + |\langle y, x | x_2, \dots, x_n \rangle|} = \frac{|\overline{\langle y, x | x_2, \dots, x_n \rangle}|}{|t| + |\langle y, x | x_2, \dots, x_n \rangle|} \\ &= \frac{|\langle x, y | x_2, \dots, x_n \rangle|}{|t| + |\langle x, y | x_2, \dots, x_n \rangle|} = \mu^*(x, y | x_2, \dots, x_n, t) \end{aligned}$$

(Fan-IP4) $\alpha (\neq 0) \in C$, Consider the following cases:

Case (i) If $t (\neq 0) \in C$, $x, \dots, x_n, y$ are linearly independent. Then:

(a) If $\langle x, y | x_2, \dots, x_n \rangle = 0$, then

$$\mu^*(\alpha x, y | x_2, \dots, x_n, t) = \frac{|\langle \alpha x, y | x_2, \dots, x_n \rangle|}{|t| + |\langle \alpha x, y | x_2, \dots, x_n \rangle|} = 0$$

$$\begin{aligned}
 &= \mu^* \left(x, y \mid x_2, \dots, x_n, \frac{t}{|\alpha|} \right) \\
 &= \frac{|\langle x, y \mid x_2, \dots, x_n \rangle|}{\frac{|t|}{|\alpha|} + |\langle x, y \mid x_2, \dots, x_n \rangle|}
 \end{aligned}$$

(b) If $\langle x, y \mid x_2, \dots, x_n \rangle \neq 0$, then $\langle \alpha x, y \mid x_2, \dots, x_n \rangle = \alpha \langle x, y \mid x_2, \dots, x_n \rangle \neq 0$ if $\alpha \neq 0$

$$\begin{aligned}
 \mu^*(\alpha x, y \mid x_2, \dots, x_n, t) &= \frac{|\langle \alpha x, y \mid x_2, \dots, x_n \rangle|}{|t| + |\langle \alpha x, y \mid x_2, \dots, x_n \rangle|} = \frac{|\alpha| \cdot |\langle x, y \mid x_2, \dots, x_n \rangle|}{|t| + |\alpha| \cdot |\langle x, y \mid x_2, \dots, x_n \rangle|} \\
 &= \frac{|\langle x, y \mid x_2, \dots, x_n \rangle|}{\frac{|t|}{|\alpha|} + |\langle x, y \mid x_2, \dots, x_n \rangle|} \\
 &= \mu^* \left(x, y \mid x_2, \dots, x_n, \frac{t}{|\alpha|} \right)
 \end{aligned}$$

$$\mu^*(\alpha x, y \mid x_2, \dots, x_n, t) = \mu^* \left(x, y \mid x_2, \dots, x_n, \frac{t}{|\alpha|} \right)$$

Case (ii) If $t=0$ and $x, \dots, x_n, y$ are linearly independent, then

$$\mu^*(\alpha x, y \mid x_2, \dots, x_n, t) = \mu^* \left(x, y \mid x_2, \dots, x_n, \frac{t}{|\alpha|} \right) = 1$$

Case (iii) $t \in (C \setminus R^+) \cup \{0\}$, $\forall$ linearly dependent $x, x_2, \dots, x_n, y \in X$, then

$$\mu^*(\alpha x, y \mid x_2, \dots, x_n, t) = \mu^* \left(x, y \mid x_2, \dots, x_n, \frac{t}{|\alpha|} \right) = 1$$

Case (iv) Let $t \in R^+ \setminus \{0\}$, If $x, x_2, \dots, x_n, y$ are linearly dependent then,

$$\begin{aligned} \mu^*(\alpha x, y \mid x_2, \dots, x_n, t) &= \frac{|\langle \alpha x, y \mid x_2, \dots, x_n \rangle|}{|t| + |\langle \alpha x, y \mid x_2, \dots, x_n \rangle|} = \frac{|\alpha| \cdot |\langle x, y \mid x_2, \dots, x_n \rangle|}{|t| + |\alpha| \cdot |\langle x, y \mid x_2, \dots, x_n \rangle|} \\ &= \frac{|\langle x, y \mid x_2, \dots, x_n \rangle|}{\frac{|t|}{|\alpha|} + |\langle x, y \mid x_2, \dots, x_n \rangle|} \\ &= \mu^*\left(x, y \mid x_2, \dots, x_n, \frac{t}{|\alpha|}\right) \end{aligned}$$

$$(\text{Fan-IP5}) \mu^*(x, x \mid x_2, \dots, x_n, t) = \frac{|\langle x, x \mid x_2, \dots, x_n \rangle|}{|t| + |\langle x, x \mid x_2, \dots, x_n \rangle|} = 1 \quad \forall t \in C \setminus R^+$$

(By definition) (Fan-IP6) $x, \dots, x_n, y \in X$ are linearly dependent

$$\Leftrightarrow \langle x, x \mid x_2, \dots, x_n \rangle = \|x, x_2, \dots, x_n\|^2 = 0$$

$$\Leftrightarrow \forall t > 0, \mu^*(x, x \mid x_2, \dots, x_n, t) = \frac{|\langle x, x \mid x_2, \dots, x_n \rangle|}{|t| + |\langle x, x \mid x_2, \dots, x_n \rangle|} = 0$$

$$\begin{aligned} (\text{Fan-IP7}) \quad \forall t \in R \text{ and } t > 0, \mu^*(x, y \mid x_2, \dots, x_n, t) &= \frac{|\langle x, y \mid x_2, \dots, x_n \rangle|}{|t| + |\langle x, y \mid x_2, \dots, x_n \rangle|} \\ &= \frac{1}{\frac{|t|}{|\langle x, y \mid x_2, \dots, x_n \rangle|} + 1} \rightarrow 0 \quad \text{as } t \rightarrow \infty \end{aligned}$$

Thus it is monotonic non-increasing.

Hence (X, μ^*) is a fuzzy anti n-inner product space.

Theorem 3.1: Let μ^* be a fuzzy anti n -inner product on X . Then the function $N^* : X^n \times C \rightarrow [0,1]$ defined by

$$N^*(x_1, x_2, \dots, x_n, t) = \mu^*(x_1, x_1 | x_2, \dots, x_n, t^n), \quad \forall t \in R \text{ and } t > 0 \\ = 1, \quad \forall t \in R \text{ and } t > 0$$

is a fuzzy anti n -norm on X .

Proof: (Fa-nN1) $\forall t \in R$ with $t \leq 0$, $N^*(x_1, x_2, \dots, x_n, t) = 1$
(From definition) (Fa-nN2)

$$\forall t \in R \text{ with } t > 0, N^*(x_1, x_2, \dots, x_n, t) = \mu^*(x_1, x_1 | x_2, \dots, x_n, t^n)$$

this implies that $x_1, x_2, \dots, x_n$ are linearly dependent [by (Fan-IP6)]

(Fa-nN3) By definition it is clear that $N^*(x_1, x_2, \dots, x_n, t)$ is invariant under any permutation of $x_1, x_2, \dots, x_n$.

(Fa-nN4) For all $\forall t \in R$ with $t > 0$ and $c \neq 0$

$$N^*(cx_1, cx_2, \dots, cx_n, t) = \mu^*(cx_1, cx_1 | cx_2, \dots, cx_n, t^n) \\ = \mu^*\left(cx_1, x_1 | x_2, \dots, cx_n, \frac{t^n}{|c|}\right)$$

[by (Fan-IP4)]

$$= \mu^*\left(cx_1, x_1 | x_2, \dots, cx_n, \frac{t^n}{|c|}\right) \\ = \mu^*\left(x_1, x_1 | x_2, \dots, cx_n, \frac{t^n}{|c|^2}\right) \\ = \mu^*\left(x_1, x_1 | x_2, \dots, x_n, \left(\frac{t}{|c|}\right)^n\right)$$

[by (Fan-IP3), (Fan-IP4)]

$$= N^* \left(x_1, x_2, \dots, x_n, \frac{t}{|c|} \right)$$

(Fa-nN5) It has to be proved that $\forall s, t \in R$ and $x_1, x_2, \dots, x_{n-1}, x_n, y \in X$

$$N^*(x_1, x_2, \dots, x_{n-1}, x_n + y, s + t) \leq \max \left\{ N^*(x_1, x_2, \dots, x_{n-1}, x_n, s), N^*(x_1, x_2, \dots, x_{n-1}, y, t) \right\}$$

Consider the following cases

- (a) $s+t < 0$ (b) $s = t = 0$ (c) $s+t > 0, s > 0, t < 0$,
or $s < 0, t > 0$, (d) $s > 0, t > 0, s+t > 0$

For (a), (b) and (c) the results are obvious. For (d),

$$\begin{aligned} N^*(x + y, x_2, x_3, \dots, x_{n-1}, x_n, s + t) &= \mu^*(x + y, x + y | x_2, \dots, x_n, (s + t)^n) \\ &\leq \max \left\{ \mu^*(x, x | x_2, \dots, x_n, |s|^n), \mu^*(y, y | x_2, \dots, x_n, |t|^n) \right\} \end{aligned}$$

[by (Fan-IP1), (Fan-IP2) and by using induction hypothesis]

$$= \max \{ N^*(x, x_2, \dots, x_n, s), N^*(y, x_2, \dots, x_n, t) \}$$

(Fa-nN6) $N^*(x_1, x_2, \dots, x_n, \bullet) = \mu^*(x_1, x_1 | x_2, \dots, x_n, \bullet)$ from (Fan-IP7) is a monotonic non-increasing function of R and tends to 0 as $t \rightarrow \infty$.

Theorem 3.2: If μ_1^* and μ_2^* are two fuzzy anti n-inner product on X , then $\mu^* = \max \{ \mu_1^*, \mu_2^* \}$ is a fuzzy anti n-inner product on X .

Proof: (Fan-IP1)

$$\mu^*(x + x', y | x_2, \dots, x_n, |t| + |s|) = \max \left\{ \mu_1^*(x + x', y | x_2, \dots, x_n, |t| + |s|), \mu_2^*(x + x', y | x_2, \dots, x_n, |t| + |s|) \right\}$$

$$\begin{aligned}
&\leq \max \left\{ \begin{array}{l} \max \left[\mu_1^*(x, y \mid x_2, \dots, x_n, |t|), \mu_1^*(x', y \mid x_2, \dots, x_n, |s|) \right], \\ \max \left[\mu_2^*(x, y \mid x_2, \dots, x_n, |t|), \mu_2^*(x', y \mid x_2, \dots, x_n, |t|) \right] \end{array} \right\} \\
&= \max \left\{ \begin{array}{l} \max \left[\mu_1^*(x, y \mid x_2, \dots, x_n, |t|), \mu_2^*(x, y \mid x_2, \dots, x_n, |t|) \right], \\ \max \left[\mu_1^*(x', y \mid x_2, \dots, x_n, |s|), \mu_2^*(x', y \mid x_2, \dots, x_n, |s|) \right] \end{array} \right\} \\
&= \max \left[\mu^*(x, y \mid x_2, \dots, x_n, |t|), \mu^*(x', y \mid x_2, \dots, x_n, |s|) \right] \\
&\Rightarrow \mu^*(x + x', y \mid x_2, \dots, x_n, |t| + |s|) \leq \max \left[\begin{array}{l} \mu^*(x, y \mid x_2, \dots, x_n, |t|), \\ \mu^*(x', y \mid x_2, \dots, x_n, |s|) \end{array} \right]
\end{aligned}$$

(Fan-IP2) For $s, t \in C$,

$$\begin{aligned}
\mu^*(x, y \mid x_2, \dots, x_n, |st|) &= \max \left[\mu_1^*(x, y \mid x_2, \dots, x_n, |st|), \mu_2^*(x, y \mid x_2, \dots, x_n, |st|) \right] \\
&\leq \max \left\{ \begin{array}{l} \max \left[\mu_1^*(x, x \mid x_2, \dots, x_n, |s|^2), \mu_1^*(y, y \mid x_2, \dots, x_n, |t|^2) \right], \\ \max \left[\mu_2^*(x, x \mid x_2, \dots, x_n, |s|^2), \mu_2^*(y, y \mid x_2, \dots, x_n, |t|^2) \right] \end{array} \right\} \\
&= \max \left[\mu^*(x, x \mid x_2, \dots, x_n, |s|^2), \mu^*(y, y \mid x_2, \dots, x_n, |t|^2) \right]
\end{aligned}$$

(Fan-IP3)

$$\begin{aligned}
\mu^*(x, y \mid x_2, \dots, x_n, |t|) &= \max \left[\mu_1^*(x, y \mid x_2, \dots, x_n, |t|), \mu_2^*(x, y \mid x_2, \dots, x_n, |t|) \right] \\
&= \max \left[\mu_1^*(y, x \mid x_2, \dots, x_n, \bar{t}), \mu_2^*(y, x \mid x_2, \dots, x_n, t) \right] \\
&= \mu^*(y, x \mid x_2, \dots, x_n, \bar{t})
\end{aligned}$$

(Fan-nIP4) For $\alpha \neq 0$,

$$\begin{aligned}
\mu^*(\alpha x, y \mid x_2, \dots, x_n, t) &= \max \{ \mu_1^*(\alpha x, y \mid x_2, \dots, x_n, t), \mu_2^*(\alpha x, y \mid x_2, \dots, x_n, t) \} \\
&= \max \left\{ \mu_1^* \left(\alpha x, y \mid x_2, \dots, x_n, \frac{t}{|\alpha|} \right), \mu_2^* \left(\alpha x, y \mid x_2, \dots, x_n, \frac{t}{|\alpha|} \right) \right\} \\
&= \mu^* \left(\alpha x, y \mid x_2, \dots, x_n, \frac{t}{|\alpha|} \right)
\end{aligned}$$

(Fan-IP5)

$$\begin{aligned}
\mu^*(x_1, x_1 \mid x_2, \dots, x_n, t) &= \max \{ \mu_1^*(x_1, x_1 \mid x_2, \dots, x_n, t), \mu_2^*(x_1, x_1 \mid x_2, \dots, x_n, t) \} \\
&= 1, \quad \forall t \in C/R^+
\end{aligned}$$

(Fan-IP6) Follows from the definition.

(Fan-IP7) We know that if $\mu_1^*(x, y \mid x_2, \dots, x_n, \bullet)$ and $\mu_2^*(x, y \mid x_2, \dots, x_n, \bullet)$ are monotonic non-increasing in R and tends to 0 as $t \rightarrow \infty$, so also is $\max \{ \mu_1^*, \mu_2^* \}$. Hence the result follows.

Theorem 3.3: (i) For $x, y, z \in X$ and $t, s \in C$,

$$\mu^*(x + x', y \mid x_2, \dots, x_n, |t| + |s|) \leq \max \left\{ \mu^*(x, y \mid x_2, \dots, x_n, |t|), \mu^*(x', y \mid x_2, \dots, x_n, |s|) \right\}$$

(ii) For $\lambda \in C$ and $\lambda \neq 0$, $\mu^*(x, \lambda y \mid x_2, \dots, x_n, t) = \mu^*(\lambda x, y \mid x_2, \dots, x_n, t)$

(iii) For all $t \in R$ and $t > 0$,

$$\mu^*(0, 0 \mid x_2, \dots, x_n, t) = 0 \leq \mu^*(x, y \mid x_2, \dots, x_n, t), \quad \forall x_1, \dots, x_n \in X$$

Proof: (i) From the definition,

$$\mu^*(x + x', y \mid x_2, \dots, x_n, |t| + |s|) \leq \max \left\{ \mu^*(x, y \mid x_2, \dots, x_n, |t|), \mu^*(x', y \mid x_2, \dots, x_n, |s|) \right\}$$

(ii) For $\lambda \in C$ and $\lambda \neq 0$,

$$\begin{aligned}\mu^*(x, \lambda y \mid x_2, \dots, x_n, t) &= \mu^*(\lambda y, x \mid x_2, \dots, x_n, \bar{t}) \\ &= \mu^*\left(y, x \mid x_2, \dots, x_n, \frac{\bar{t}}{|\lambda|}\right) \\ &= \mu^*\left(x, y \mid x_2, \dots, x_n, \frac{t}{|\lambda|}\right) \\ &= \mu^*(\lambda x, y \mid x_2, \dots, x_n, t)\end{aligned}$$

(iii) For all $t \in R$ and $t > 0$,

$$(3.1) \quad \mu^*(0, 0 \mid x_2, \dots, x_n, t) = 0$$

(by Fa-nIP6)

and

$$(3.2) \quad \mu^*(x, y \mid x_2, \dots, x_n, t) \leq 1$$

So from (3.1) and (3.2)

$$\mu^*(0, 0 \mid x_2, \dots, x_n, t) = 0 \leq \mu^*(x, y \mid x_2, \dots, x_n, t), \quad \forall x_1, \dots, x_n \in X$$

Hence the results follows.

Further, assume that μ^* satisfies the condition :

(Fan-IP8) $\mu^*(x_1, x_1 \mid x_2, \dots, x_n, t^2) < 1 \Rightarrow x$ and y are linearly independent

Definition 3.2: Let (X, μ^*) be a fuzzy anti n -inner product space satisfying (Fan-IP8), $\forall \alpha \in (0, 1)$,

$\|x_1, \dots, x_n\|_\alpha^* = \bigwedge \{t > 0 : \mu^*(x_1, x_1 \mid x_2, \dots, x_n, t^2) < \alpha, \alpha \in (0, 1)\}$ is called the α - n -norm on X generated from μ^*

(Fan-IP9) for all $x, y \in X$ and $p, q \in R$,

$$\inf \{ \mu^*(x + y, x + y \mid x_2, \dots, x_n, 2q^2), \mu^*(x - y, x - y \mid x_2, \dots, x_n, 2p^2) \}$$

$$\leq \inf \left\{ \mu^* (x, x | x_2, \dots, x_n, p^2), \mu^* (y, y | x_2, \dots, x_n, q^2) \right\}$$

Theorem 3.4 : (Parallelogram Law) : Let μ^* be a fuzzy anti n -inner product on X , satisfying (Fan-IP8) and (Fan-IP9). Let $\alpha \in (0,1)$ and $\|x_1, \dots, x_n\|_\alpha^*$ be the α - n -norm, generated from the fuzzy anti n -inner product μ^* on X . Then

$$\|x - y, x_2, \dots, x_n\|_\alpha^{*2} + \|x + y, x_2, \dots, x_n\|_\alpha^{*2} = 2(\|x, x_2, \dots, x_n\|_\alpha^{*2} + \|y, x_2, \dots, x_n\|_\alpha^{*2}).$$

Proof: We have, $\|x - y, x_2, \dots, x_n\|_\alpha^{*2} + \|x + y, x_2, \dots, x_n\|_\alpha^{*2}$

$$= \inf \left\{ t^2 : t \in R^+ \text{ and } N^*(x - y, x_2, \dots, x_n, t) < \alpha \right\} \\ + \inf \left\{ s^2 : s \in R^+ \text{ and } N^*(x + y, x_2, \dots, x_n, t) < \alpha \right\}$$

Where N^* is the fuzzy anti n -norm induced from μ^* .

$$(3.3) \quad = \inf \left\{ \begin{array}{l} t^2 + s^2 : t, s \in R^+ \text{ and } N^*(x - y, x_2, \dots, x_n, t) < \alpha, \\ N^*(x + y, x_2, \dots, x_n, t) < \alpha \end{array} \right\} \\ = \inf \left\{ \begin{array}{l} t^2 + s^2 : t, s \in R^+ \text{ and } N^*(x - y, x_2, \dots, x_n, t) \\ \wedge N^*(x + y, x_2, \dots, x_n, t) < \alpha \end{array} \right\}$$

Also,

$$(3.4) \quad 2(\|x, x_2, \dots, x_n\|_\alpha^{*2} + \|y, x_2, \dots, x_n\|_\alpha^{*2}) \\ = 2 \inf \left\{ p^2 : p \in R^+ \text{ and } N^*(x, x_2, \dots, x_n, p) < \alpha \right\} \\ + 2 \inf \left\{ q^2 : q \in R^+ \text{ and } N^*(y, x_2, \dots, x_n, q) < \alpha \right\} \\ = 2 \inf \left\{ \begin{array}{l} p^2 + q^2 : p, q \in R^+ \text{ and } N^*(x, x_2, \dots, x_n, p) \\ \wedge N^*(y, x_2, \dots, x_n, q) < \alpha \end{array} \right\}$$

Now by (Fan-IP9)

$$N^*(x - y, x_2, \dots, x_n, \sqrt{2}p) \wedge N^*(x + y, x_2, \dots, x_n, \sqrt{2}q) \\ < N^*(x, x_2, \dots, x_n, p) \wedge N^*(y, x_2, \dots, x_n, q)$$

From (3.3) and (3.4) we get,

$$(3.5) \quad \begin{aligned} & \|x - y, x_2, \dots, x_n\|_\alpha^{*2} + \|x + y, x_2, \dots, x_n\|_\alpha^{*2} \\ & \leq 2 \left(\|x, x_2, \dots, x_n\|_\alpha^{*2} + \|y, x_2, \dots, x_n\|_\alpha^{*2} \right) \end{aligned}$$

Again,

$$\begin{aligned} \|x, x_2, \dots, x_n\|_\alpha^{*2} + \|y, x_2, \dots, x_n\|_\alpha^{*2} &= 2 \left(\left\| \frac{x+y}{2} + \frac{x-y}{2}, x_2, \dots, x_n \right\|_\alpha^{*2} \right. \\ & \quad \left. + \left\| \frac{x+y}{2} - \frac{x-y}{2}, x_2, \dots, x_n \right\|_\alpha^{*2} \right) \\ &= \frac{1}{2} \left(\|(x+y) + (x-y), x_2, \dots, x_n\|_\alpha^{*2} + \|(x+y) - (x-y), x_2, \dots, x_n\|_\alpha^{*2} \right) \\ &\geq \frac{1}{2} \left[\|x+y, x_2, \dots, x_n\|_\alpha^{*2} + \|x-y, x_2, \dots, x_n\|_\alpha^{*2} + 2\|x+y, x_2, \dots, x_n\|_\alpha^* \right. \\ & \quad \left. \|x-y, x_2, \dots, x_n\|_\alpha^* \right. \\ & \quad \left. + \left[\|y+x, x_2, \dots, x_n\|_\alpha^{*2} \|x-y, x_2, \dots, x_n\|_\alpha^{*2} - 2\|y+x, x_2, \dots, x_n\|_\alpha^* \right. \right. \\ & \quad \left. \left. \|x-y, x_2, \dots, x_n\|_\alpha^* \right] \right] \\ &\geq \frac{1}{2} \times 2 \left[\|x+y, x_2, \dots, x_n\|_\alpha^{*2} + \|x-y, x_2, \dots, x_n\|_\alpha^{*2} \right] \\ &= \|x+y, x_2, \dots, x_n\|_\alpha^{*2} + \|x-y, x_2, \dots, x_n\|_\alpha^{*2} \\ (3.6) \quad & \|x-y, x_2, \dots, x_n\|_\alpha^{*2} + \|x+y, x_2, \dots, x_n\|_\alpha^{*2} \geq 2 \left(\|x, x_2, \dots, x_n\|_\alpha^{*2} + \|y, x_2, \dots, x_n\|_\alpha^{*2} \right) \end{aligned}$$

From (3.5) and (3.6) we get,

Therefore

$$\|x - y, x_2, \dots, x_n\|_{\alpha}^{*2} + \|x + y, x_2, \dots, x_n\|_{\alpha}^{*2} = 2[\|x, x_2, \dots, x_n\|_{\alpha}^{*2} + \|y, x_2, \dots, x_n\|_{\alpha}^{*2}]$$

Theorem 3.5 : Let (X, μ^*) be a fuzzy anti n-inner product space. If $\mu^*(x, y|x_2, \dots, x_n, t) = \mu^*(x', y|x_2, \dots, x_n, t)$ for all $y \in X$ then $x, \dots, x_n, y$ are dependent.

Proof: By definition,

$$\begin{aligned} \mu^*(x - x', y|x_2, \dots, x_n, t) &\leq \max \{ \mu^*(x, y|x_2, \dots, x_n, t), \mu^*(-x', y|x_2, \dots, x_n, t) \} \\ &= \max \left\{ \mu^*(x, y|x_2, \dots, x_n, t), \mu^* \left(x', y|x_2, \dots, x_n, \frac{t}{|-1|} \right) \right\} \\ &= \max \{ \mu^*(x, y|x_2, \dots, x_n, t), \mu^*(x', y|x_2, \dots, x_n, t) \} \\ &= \max \mu^*(x, y|x_2, \dots, x_n, t) \end{aligned}$$

So $x, \dots, x_n, y$ are dependent.

Theorem 3.6: (Polarization identity): If $x, y, x_2, \dots, x_n$ are the elements of the fuzzy anti n- inner product space (X, μ^*) , then

$$\begin{aligned} 4\mu^*(x, y|x_2, \dots, x_n, st) &= \|x + y, x_2, \dots, x_n\|_{\alpha}^{*2} - \|x - y, x_2, \dots, x_n\|_{\alpha}^{*2} \\ &\quad + i\|x + iy, x_2, \dots, x_n\|_{\alpha}^{*2} - i\|x - iy, x_2, \dots, x_n\|_{\alpha}^{*2}. \end{aligned}$$

Proof : We know that,

$$\begin{aligned} &\|x + y, x_2, \dots, x_n\|_{\alpha}^{*2} - \|x - y, x_2, \dots, x_n\|_{\alpha}^{*2} + i\|x + iy, x_2, \dots, x_n\|_{\alpha}^{*2} - i\|x - iy, x_2, \dots, x_n\|_{\alpha}^{*2} \\ &= \mu^*((x + y), (x + y)|x_2, \dots, x_n, (s + t)^2) - \mu^*((x - y), (x - y)|x_2, \dots, x_n, (s - t)^2) \\ &\quad + i\mu^*((x + iy), (x + iy)|x_2, \dots, x_n, (s + t)^2) - i\mu^*((x - iy), (x - iy)|x_2, \dots, x_n, (s - t)^2) \\ &= \mu^*((x + y), (x + y)|x_2, \dots, x_n, s^2 + st + st + t^2) \\ &\quad - \mu^*((x - y), (x - y)|x_2, \dots, x_n, s^2 - st - st + t^2) \\ &\quad + i\mu^*((x + iy), (x + iy)|x_2, \dots, x_n, s^2 + st + st + t^2) \end{aligned}$$

$$\begin{aligned}
& -i\mu^* \left((x-iy), (x-iy) \middle| x_2, \dots, x_n, s^2 - st - st + t^2 \right) \\
& = \mu^* \left(x, x \middle| x_2, \dots, x_n, s^2 \right) + \mu^* \left(x, y \middle| x_2, \dots, x_n, st \right) + \mu^* \left(y, x \middle| x_2, \dots, x_n, st \right) \\
& + \mu^* \left(y, y \middle| x_2, \dots, x_n, t^2 \right) - \mu^* \left(x, x \middle| x_2, \dots, x_n, s^2 \right) \\
& - \mu^* \left(-y, x \middle| x_2, \dots, x_n, -st \right) - \mu^* \left(-y, x \middle| x_2, \dots, x_n, -st \right) \\
& - \mu^* \left(-y, -y \middle| x_2, \dots, x_n, t^2 \right) \left(x, iy \middle| x_2, \dots, x_n, st \right) \\
& + i\mu^* \left(x, x \middle| x_2, \dots, x_n, s^2 \right) + i\mu^* \left(x, iy \middle| x_2, \dots, x_n, st \right) \\
& + i\mu^* \left(iy, x \middle| x_2, \dots, x_n, st \right) + i\mu^* \left(iy, iy \middle| x_2, \dots, x_n, t^2 \right) - i\mu^* \left(x, x \middle| x_2, \dots, x_n, s^2 \right) \\
& - i\mu^* \left(x, -iy \middle| x_2, \dots, x_n, -st \right) - i\mu^* \left(-iy, x \middle| x_2, \dots, x_n, -st \right) \\
& - i\mu^* \left(-iy, -iy \middle| x_2, \dots, x_n, t^2 \right) \\
& = \mu^* \left(x, x \middle| x_2, \dots, x_n, s^2 \right) + \mu^* \left(x, y \middle| x_2, \dots, x_n, st \right) + \mu^* \left(y, x \middle| x_2, \dots, x_n, st \right) \\
& + \mu^* \left(y, y \middle| x_2, \dots, x_n, t^2 \right) - \mu^* \left(x, x \middle| x_2, \dots, x_n, s^2 \right) \\
& - \mu^* \left(y, x \middle| x_2, \dots, x_n, \frac{st}{|-1|} \right) - \mu^* \left(y, x \middle| x_2, \dots, x_n, \frac{st}{|-1|} \right) \\
& - \mu^* \left(y, y \middle| x_2, \dots, x_n, t^2 \right) + i\mu^* \left(x, x \middle| x_2, \dots, x_n, s^2 \right) \\
& + i\mu^* \left(x, iy \middle| x_2, \dots, x_n, st \right) + i\mu^* \left(iy, x \middle| x_2, \dots, x_n, st \right) \\
& + i\mu^* \left(iy, iy \middle| x_2, \dots, x_n, t^2 \right) - i\mu^* \left(x, x \middle| x_2, \dots, x_n, s^2 \right) \\
& - i\mu^* \left(x, iy \middle| x_2, \dots, x_n, \frac{-st}{|-1|} \right) - i\mu^* \left(iy, x \middle| x_2, \dots, x_n, \frac{-st}{|-1|} \right) - i\mu^* \left(iy, iy \middle| x_2, \dots, x_n, t^2 \right) \\
& = \mu^* \left(x, y \middle| x_2, \dots, x_n, st \right) + \mu^* \left(y, x \middle| x_2, \dots, x_n, st \right) - \mu^* \left(x, y \middle| x_2, \dots, x_n, -st \right) \\
& - \mu^* \left(y, x \middle| x_2, \dots, x_n, -st \right) + i\mu^* \left(x, iy \middle| x_2, \dots, x_n, st \right) \\
& + i\mu^* \left(iy, x \middle| x_2, \dots, x_n, st \right) + i\mu^* \left(x, iy \middle| x_2, \dots, x_n, st \right) + i\mu^* \left(iy, x \middle| x_2, \dots, x_n, st \right) \\
& = \mu^* \left(x, y \middle| x_2, \dots, x_n, st \right) + \mu^* \left(y, x \middle| x_2, \dots, x_n, st \right) + \mu^* \left(x, y \middle| x_2, \dots, x_n, st \right) \\
& + \mu^* \left(y, x \middle| x_2, \dots, x_n, st \right) + i\mu^* \left(x, iy \middle| x_2, \dots, x_n, st \right) \\
& + i\mu^* \left(iy, x \middle| x_2, \dots, x_n, st \right) + i\mu^* \left(x, iy \middle| x_2, \dots, x_n, st \right) + i\mu^* \left(iy, x \middle| x_2, \dots, x_n, st \right)
\end{aligned}$$

$$\begin{aligned}
&= \mu^*(x, y | x_2, \dots, x_n, st) + \mu^*(y, x | x_2, \dots, x_n, st) + \mu^*(x, y | x_2, \dots, x_n, st) \\
&+ \mu^*(y, y | x_2, \dots, x_n, st) + i(-i)\mu^*(x, y | x_2, \dots, x_n, st) \\
&+ i(i)\mu^*(y, x | x_2, \dots, x_n, st) + i(-i)\mu^*(x, y | x_2, \dots, x_n, st) + i(i)\mu^*(y, x | x_2, \dots, x_n, st) \\
&= \mu^*(x, y | x_2, \dots, x_n, st) + \mu^*(y, x | x_2, \dots, x_n, st) + \mu^*(x, y | x_2, \dots, x_n, st) \\
&+ \mu^*(y, x | x_2, \dots, x_n, st) + \mu^*(x, y | x_2, \dots, x_n, st) \\
&- \mu^*(y, x | x_2, \dots, x_n, st) + \mu^*(x, y | x_2, \dots, x_n, st) - \mu^*(y, x | x_2, \dots, x_n, st) \\
&= 4 \mu^*(x, y | x_2, \dots, x_n, st) .
\end{aligned}$$

Hence the result.

References

1. Pinaki and S. K. Samanta, On fuzzy inner product spaces, *The journal of fuzzy mathematics*, **16 (2)** (2008), 377-392.
2. P. Sinha, Divya Mishra and Ghanshyam Lal, Fuzzy Anti Inner Product Space and Its Properties, *The Journal of Fuzzy Mathematics*, **22(4)** (2014).
3. S. Vijaybalaji and N. Thillalgovindan, Fuzzy n- inner product space, *Bull Korean Math, Soc.*, **43(3)** (2007), 447-459.
4. B.Surender Reddy, Fuzzy Anti-n-normed linear space, *Journal of Mathematics and Teachnogy*, **2(1)** (2011).
5. T. Bag and S. K. Samanta, Product fuzzy normed linear spaces, *The journal of fuzzy mathematics*, **13(3)** (2005).
6. T. Bag and S. K. Samanta, Finite Dimensional fuzzy normed linear spaces, *J. Fuzzy Math.*, **11(3)** (2003), 687-705.
7. Erwin Kreyszig, *Introductory Function Analysis with application*, John Wiley & Sons.
8. C. Felbin, Finite dimensional fuzzy Normed linear space, *Fuzzy sets and systems*, (1992).
9. Iqbal, H. Jebril and T. K. Samanta, Fuzzy Anti-normed linear space, *Journal of Mathematics and Technology*, (2010).
10. S. Vijayabalaji and N. Thillaigovindan, Fuzzy n-inner product space, *Bull. Korean Math.Soc.*, **43(3)** (2007), 447-459.
11. P. Sinha, Divya Mishra and Ghanshyam Lal, Some Results On Fuzzy Anti 2- Normed Linear Space, *International Journal of Applied Engineering Research*, **7(1)** (2012).
12. P. Sinha, Ghanshyam Lal and Divya Mishra, On Fuzzy 2- Inner Product Space, *International Journal of Fuzzy Mathematics and System*, **3(3)** (2013), 243-250.

13. Some Results On 2- Inner Product Space, *Novi Sad J. Math*, **37(2)** (2007), 35-49.
14. R. M. Soma sundaram and Thangaraj Beaula, Some Aspects of 2-fuzzy 2-normed linear spaces, *Bull. Malays. Math. Sci. Soc.*, **32(2)** (2009), 211-221.
15. L. A. Zadeh, fuzzy Sets, *Information and Control*, 8 (1965), 338-353.