

Construction of Cubic Spline Interpolants

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Abstract : In earlier results Chatterjee and Dikshit¹, Meir and Sharma² existence of cubic splines were established under some restrictions on the interpolating points of uniform partition. A general case of this problem was studied by Morsche³. The cubic splines do not exist if the points of interpolation are taken mid points Morsche³. Here, we construct cubic splines which can be interpolated at any point of the mesh which has the same order of convergence.

1. Introduction

Let $\Delta_n = \{0 = x_0 < x_1 < \dots < x_n = 1\}$ be a uniform partition of the interval $[0, 1]$, i.e., $x_i - x_{i-1} = h$, $i = 1, 2, \dots, n$. Suppose that the class of all algebraic polynomial of degree m (> 1), or less is represented by π_m . Let $S(m, \Delta_n) = \{S : S(x) \in \pi_m, x \in [x_{i-1}, x_i], i = 1, 2, \dots, n, S \in C^{m-1}[0, 1]\}$ denote the class of all polynomial splines, where m is a positive integer. Concerning the existence of cubic polynomial interpolating at certain points of the mesh, the following problem was studied in Chatterjee and Dikshit¹.

Problem. Let $y_i = x_{i-1} + \lambda_i h$, $0 \leq \lambda_i \leq 1$. Under what restrictions on λ_i , there exists a unique spline in $S(3, \Delta)$ satisfying the interpolatory condition.

$$(1) \quad f(y_i) = s(y_i), \quad i = 1, 2, \dots, n,$$

where $f(y_i)$'s are given functional values. In the paper¹, existence of the cubic splines was obtained under the condition that $\lambda_i \in [0, 1/3] \cup [2/3, 1]$. A more general case, i.e., splines of degree m was studied in Meir and Sharma².

In the above studies splines were taken periodic. Further, it was shown that the cubic splines do not exist, if points of interpolation are taken mid points. From the practical point of view, a common choice is cubic splines and a desired point of interpolation is mid points besides other points. In view of this, a natural question is to construct cubic splines, so that the splines satisfying (1) for all λ exist. In this direction we establish the following.

2. Main Results

Theorem. Let f be a given function defined on $[0, 1]$. Then there exist cubic splines $s(x) \in s(3, \Delta_n)$, $n \geq 4$, which interpolate at any point in the mesh i.e.,

$$f(y_i) = s(y_i), \quad 0 \leq \lambda \leq 1, \quad 1 \leq i \leq n$$

under the boundary conditions $s''(0) = f''(0)$, $s''(1) = f''(1)$, and

$$6(s'_0 - f'_0) = \lambda^3 h(f''_0 - s''_1),$$

The moment periodicity is replaced by another restriction the matrix involved is establishing the existence of splines is non-circulants. To establish its non-singularity, we make use of the following result.

Lemma. Let $D_n(0, 0)$, $n \geq 4$, be the following n -square quadruple diagonal matrix

$$D_n(0, 0) = \begin{bmatrix} q & p & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ r & q & p & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ s & r & q & p & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & s & r & q & p & 0 & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & s & r & q & p \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & s & r & q \end{bmatrix}$$

where s, r, q, p are positive real numbers and $s \neq 0$. Then the matrix $D_n(0, 0)$, is non-singular if and only if $q \neq 0$ when $p = 0$.

When $s = 0$, then this becomes the case of tridigonal, and this has been dealt with the result of Kumar and Govil⁴.

3. Proof of the Results

Proof of the Lemma. Let $D_n = |D_n(0, 0)|$ be the determinant of the matrix $D_n(0, 0)$. Obviously, when $p = 0$, then $D_n = q^n$ and therefore, in this case the matrix is non-singular if and only if $q \neq 0$.

Now, we assume here onward that $p \neq 0$. It can be seen that

$$(2) \quad D_n = q D_{n-1} - p r D_{n-2} + p^2 s D_{n-3}$$

Set $\alpha + \beta + \gamma = q$, $\alpha \beta + \beta \gamma + \gamma \alpha = r p$, $\alpha \beta \gamma = p^2 s$. Then the determinant D_n of the matrix can be written as

$$D_n = \frac{1}{\alpha - \beta} \sum_{j=0}^n \mu^j (\alpha^{n+1-j} - \beta^{n+1-j}).$$

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We first of all observe that, since p and s are not zero, none of α , β and γ is zero.

In case $\alpha = \beta = \gamma$, by induction we can see (2) that

$$(3) \quad D_n = (n+1)(n+2)\alpha^n/2$$

which is not zero, since $\alpha^3 = p^2s$.

Now we consider the case $\alpha = \beta \neq \gamma$, that is, $2\alpha + \gamma = q$, $\alpha^2 + 2\alpha\gamma = rp$, $\alpha^2\gamma = p^2s$. Again, by induction, we obtain

$$(4) \quad D_n = -\frac{(n+1)\alpha^{n+1}}{\gamma - \alpha} + \frac{\gamma(\gamma^{n+1} - \alpha^{n+1})}{(\gamma - \alpha)^2}$$

Thus, writing $\alpha/\gamma = a \neq 1$, we find that the matrix is non-singular if and only if

$$(5) \quad (n+1)a^{n+1} \neq \frac{(1 - a^{n+1})}{(1 - a)}$$

i.e., $(n+1)a^{n+1} \neq 1 + a + a^2 + \dots + a^n$.

For $0 < a < 1$, each term on the right hand side i.e. a^i , $0 \leq i \leq n$, is bigger than a^{n+1} . Similarly for $a > 1$, a^{n+1} is bigger than each term a^i , $0 \leq i \leq n$. Thus, for these cases the inequality holds good. By a similar arguments it follows that the inequality is true for the other values of a .

Now, consider the case $|a| < 1$, when n is finite, the inequality follows. For $n \rightarrow \infty$, when $|a| < 1$, we have $(n+1)a^{n+1} \rightarrow 0$, or in other words $(n+1)a^{n+1}$ can be made arbitrary small for sufficiently large n , while $(1 - a^{n+1}) \rightarrow 1$, since $a^{n+1} \rightarrow 0$, as $n \rightarrow \infty$. Thus (5) is always true.

The cases $\alpha = \gamma \neq \beta$ and $\beta = \gamma \neq \alpha$ is similar to that of $\alpha = \beta \neq \gamma$.

Now we consider the case $\alpha \neq \beta \neq \gamma$. We claim that

$$(6) \quad D_n = \frac{1}{\alpha - \beta} \left[\frac{\alpha(\gamma^{n+1} - \alpha^{n+1})}{\gamma - \alpha} - \frac{\beta(\gamma^{n+1} - \beta^{n+1})}{\gamma - \beta} \right]$$

We prove this by induction. The statement given in (6) holds for $n = 3, 4, 5$ is obvious. Assuming that it holds for $n-2, n-1$ and n , we have from (2)

$$\begin{aligned} & qD_n - prD_{n-1} + p^2sD_{n-2} \\ &= \frac{1}{\alpha - \beta} \left[(\alpha + \beta + \gamma) \left\{ \frac{\alpha(\gamma^{n+1} - \alpha^{n+1})}{\gamma - \alpha} - \frac{\beta(\gamma^{n+1} - \beta^{n+1})}{\gamma - \beta} \right\} \right. \\ & \quad - (\alpha\beta + \beta\gamma + \gamma\alpha) \left\{ \frac{\alpha(\gamma^n - \alpha^n)}{\gamma - \alpha} - \frac{\beta(\gamma^n - \beta^n)}{\gamma - \beta} \right\} \\ & \quad \left. + \alpha\beta\gamma \left\{ \frac{\alpha(\gamma^{n-1} - \alpha^{n-1})}{\gamma - \alpha} - \frac{\beta(\gamma^{n-1} - \beta^{n-1})}{\gamma - \beta} \right\} \right] \end{aligned}$$

By solving the above expression, we see that

$$\begin{aligned} & q D_n - p r D_{n-1} + p^2 s D_{n-2} \\ &= \frac{1}{\alpha - \beta} \left[\frac{\alpha(\gamma^{n+2} - \alpha^{n+2})}{\gamma - \alpha} - \frac{\beta(\gamma^{n+2} - \beta^{n+2})}{\gamma - \beta} \right] \\ &= D_{n+1} \end{aligned}$$

This establishes (6). $\therefore$

The determinant given in (6) is non-zero, if and only if

$$(7) \quad \frac{(1 - a_1)}{(1 - b_1)} \neq \frac{(1 - a_1^{n+1})a_1}{(1 - b_1^{n+1})b_1}$$

where $\frac{\alpha}{\gamma} = a_1$, $\frac{\beta}{\gamma} = b_1$. Obviously $a_1 \neq b_1$, a_1 and $b_1 \neq 1$

The inequality (7) means

$$(1 + b_1 + b_1^2 + \dots + b_1^n) b_1 \neq (1 + a_1 + a_1^2 + \dots + a_1^n) a_1$$

which is always true for $a_1 \neq b_1$, for any finite n .

Now consider the case when $n \rightarrow \infty$. If $a_1 < 1$, $b_1 < 1$ then the factor $(1 - a_1^{n+1})/(1 - b_1^{n+1})$ can be made very close to 1 by taking n sufficiently large. From this we can see that the above inequality is true. If $a_1 < 1$, and $b_1 < 1$ the right hand side of (7) becomes ultimately zero. Similarly we can find for the case $a_1 > 1$, $b_1 < 1$ the inequality (7) is true if n is taken suitably large.

Hence the result.

Proof of the Theorem. For any function g , we write $g(x_i) = g_i$. We set, for $i = 1, 2, \dots, n$,

$$f(y_i) = s(y_i), \quad s''(x_i) = M_i \quad \text{and} \quad s''(x_0) = M_0$$

For the subinterval $[x_{i-1}, x_i]$, $i = 1, 2, \dots, n$, we see that

$$(9) \quad s'(x) = \frac{M_{i-1}(x_i - x)^2}{2h} + \frac{M_i(x - x_{i-1})^2}{2h} + \frac{A_i}{2h}$$

where A_i 's are constants. Continuity of $s'(x)$ yields

$$(10) \quad A_{i+1} - A_i = 2M_i h^2, \quad i = 1, 2, \dots, n$$

From equation (9) we get,

$$(11) \quad s(x) = \frac{M_{i-1}(x_i - x)^3}{6h} + \frac{M_i(x - x_{i-1})^3}{6h} + \frac{3A_i(x - y_i)}{6h} + \frac{B_i}{6h}$$

where B_i 's are constants. Again continuity of $s(x)$ gives, for $i = 1, 2, \dots, n-1$

$$(12) \quad B_{i+1} - B_i = 3A_{i+1}\lambda h + 3A_i(1 - \lambda)h$$

Applying the interpolatory condition (1), we obtain from (11)

$$6h^{-2}f(y_i) = M_{i-1}(1 - \lambda)^3 + M_i\lambda^3 + B_i h^{-3} \quad i = 1, 2, \dots, n$$

By equations (10), (12) and (13), we obtain, for $i = 1, 2, \dots, n - 1$

$$(14) \quad \begin{aligned} 6h^{-2}[f(y_{i+1}) - f(y_i)] \\ = -M_{i-1}(1 - \lambda)^3 + M_i(1 + 3\lambda + 3\lambda^2 - 2\lambda^3) + M_{i+1}\lambda^3 + 3A_i h^{-2} \end{aligned}$$

From (10) and (14) we find the following system of linear equations, for $i = 1, 2, \dots, n - 2$,

$$(15) \quad \begin{aligned} 6h^{-2}[f(y_{i+2}) - 2f(y_{i+1}) + f(y_i)] \\ = M_{i-1}(1 - \lambda)^3 + M_i[4 + 3(\lambda^3 - 2\lambda^2)] + M_{i+1}[1 + 3\lambda(1 + \lambda - \lambda^2)] + M_{i+2}\lambda^3 \end{aligned}$$

From (9), (14) and the boundary conditions of the theorem, we obtain

$$(16) \quad \begin{aligned} 6h^{-2}[f(y_2) - f(y_1)] - 6h^{-1}f'_0 - (2 + 3\lambda - 3\lambda^2 + 2\lambda^3)f''_0 \\ = M_1[1 + 3\lambda(1 + \lambda - \lambda^2)] + M_2\lambda^3 \end{aligned}$$

Thus the system of equations on M_i 's can be written as

$$(17) \quad AX = U$$

where $A = D_{n-1}(0, 0)$ with $q = [1 + 3\lambda(1 + \lambda - \lambda^2)]$, $p = \lambda^3$, $r = [4 + 3(\lambda^3 - 2\lambda^2)]$

$s = (1 - \lambda)^3$, $X = [M_1, M_2, \dots, M_{n-1}]'$ and $U = [u_1, u_2, \dots, u_{n-1}]'$ with

$$\begin{aligned} u_1 &= 6h^{-2}[f(y_2) - f(y_1)] - 6h^{-1}f'_0 - (2 + 3\lambda - 3\lambda^2 + 2\lambda^3)f''_0 \\ u_i &= 6h^{-2}[f(y_{i+2}) - 2f(y_{i+1}) + f(y_i)], \quad i = 2, 3, \dots, n - 1 \end{aligned}$$

Since, $q \neq 0$, from the lemma we get that the matrix is non-singular. Hence the spline exists.

4. Error bounds

We estimate error bounds for the above splines, we write $e_i'' = M_i'' - f_i''$. For $i = 1, 2, \dots, n - 2$, and $y_i < \eta_{i+1} < y_{i+1}$, from (15) we have

$$\begin{aligned}
(18) \quad & e_{i-1}''(1-\lambda)^3 + e_i''[4+3(\lambda^3-2\lambda^2)] + e_{i+1}''[1+3\lambda(1+\lambda-\lambda^2)] + e_{i+2}''\lambda^3 \\
& = 6[f''(\eta_{i+1}) - f_{i+1}''] + (1-\lambda)^3 [f_{i+1}'' - f_{i-1}''] \\
& \quad + \{4+3(\lambda^3-2\lambda^2)\} [f_{i+1}'' - f_i''] - \lambda^3 [f_{i+1}'' - f_{i+2}''] \\
& \leq 6\omega(f'', h) + (1-\lambda)^3 \omega(f'', h) + \{4+3(\lambda^3-2\lambda^2)\} \omega(f'', h) + \lambda^3 \omega(f'', h) \\
& = (11 - 3\lambda - 3\lambda^2 + 3\lambda^3) \omega(f'', h)
\end{aligned}$$

We can write

$$f(x_1 + \lambda h) - f(x_0 + \lambda h) = h[f'(x_0) + \lambda h f''(\eta) + h(1/2)f''(\zeta)]$$

where $\zeta \in (x_0 + \lambda h, x_1 + \lambda h)$, $\eta \in (x_0, x_0 + \lambda h)$,

In view of this equation (16) gives

$$\begin{aligned}
(19) \quad & e_1''[1+3\lambda(1+\lambda-\lambda^2)] + e_2''\lambda^3 \\
& = 6h^{-2}[f(x_1 + \lambda h) - f(x_0 + \lambda h)] - 6h^{-1}f_0' + (-2-3\lambda+3\lambda^2-2\lambda^3)f_0'' \\
& \quad - (1+3\lambda+3\lambda^2-3\lambda^3)f_1'' - \lambda^3 f_2'' \\
& = 6\lambda[f''(\eta) - f_1''] + 3[f''(\zeta) - f_1''] + \lambda^3[f_1'' - f_2''] \\
& \quad + (-2-3\lambda+3\lambda^2-2\lambda^3)[f_0'' - f_1''] \\
& \leq (1+3\lambda+3\lambda^2-3\lambda^3)\omega(f'', h)
\end{aligned}$$

From equation (18) and (19) we get the following $n-1$ system of the equations of e_i'' 's as

$$AY = V$$

where $A = D_{n-1}(0, 0)$, $y = [e_1'', e_2'', \dots, e_{n-1}'']'$, $e_0'' = 0 = e_n''$,

since $s''(0) = f''(0)$, $s''(1) = f''(1)$, $V = [\theta_1, \theta_2, \dots, \theta_{n-1}]$

$$|\theta_1| \leq (1+3\lambda+3\lambda^2-3\lambda^3)\omega(f'', h),$$

$$|\theta_i| \leq (11-3\lambda-3\lambda^2-3\lambda^3)\omega(f'', h), \quad i = 2, 3, \dots, n-1.$$

By using the result of the lemma, we get that the matrix is non-singular, since

$$1+3\lambda(1+\lambda-\lambda^2) \neq 0 \quad \text{for } 0 \leq \lambda \leq 1.$$

Thus,

$$\max |e_i''| \leq \text{constant } \omega(f'', h)$$

Since $s''(x)$ is linear between knots, we get

$$|e''(x)| \leq \text{constant } \omega(f'', h)$$

uniformly in $[0, 1]$

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