

Hypersurface of a Recurrent Finsler Space

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Abstract : Recurrent Finsler spaces were introduced and studied by A. Moór¹ initially, and thereafter several mathematicians including P. N. Pandey² and Shalini Dikshit³, contributed much to such space. U. P. Singh and G. C. Chaubey⁴ studied a hypersurface of a Finsler space whose associate Berwald curvature tensor H_{ijkh} is recurrent and called such space as recurrent space. This definition of a recurrent space differs from the definition of such space given by Moor, Pandey and Shalini Dikshit. In general, they are not equivalent. However, they are equivalent if the space is affinely connected. The current paper is devoted to the study of a hypersurface of a recurrent space being characterized by the recurrence of Berwald curvature tensor. Special attention has been given to the study of an umbilical hypersurface of such space.

1. Preliminaries

Let F_n be an n -dimensional Finsler space equipped with a metric function $F(x^i, y^j)$ satisfying the requisite conditions⁵. The metric tensor and $(h) - hv$ torsion tensor are defined by $g_{ij}(x, y) = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2(x, y)$ and $C_{ijk} = \frac{1}{2} \dot{\partial}_k g_{ij}$ respectively, where $\dot{\partial}_k$ denotes the partial differentiation with respect to y^k . Corresponding to contravariant vector y^i there is a covariant vector y_i such that

$$(1.1) \quad y_i = g_{ij} y^j.$$

The $(h) hv$ - torsion tensor C_{ijk} satisfies the following identities

$$(1.2) \quad (a) \quad C_{ijk} y^i = C_{kij} y^i = C_{jki} y^i = 0,$$

$$(b) \quad C_{jk}^i y_i = C_{kj}^i y_i = 0$$

where

$$C_{jk}^i = g^{ir} C_{jrk}.$$

The v -covariant differentiation and h -covariant differentiation of an arbitrary vector field X^i are defined as follows :

$$(1.3) \quad (a) \quad X^i|_k = \dot{\partial}_k X^i + X^r C_{rk}^i$$

$$(b) \quad X^i|_k = \dot{\partial}_k X^i - \left(\dot{\partial}_r X^i \right) G_k^r + X^r \Gamma_{hk}^{*i}$$

where Γ_{jk}^{*i} are Cartan's connection coefficients, $G_k^r = \Gamma_{sk}^{*r} y^s$ and $\dot{\partial}_k \equiv \partial/x^k$.

The vector y^j vanishes under h -covariant differentiation i.e.,

$$(1.4) \quad y^i|_j = 0.$$

The Berwald covariant derivative of an arbitrary tensor field T_j^i with respect to x^k is given by

$$(1.5) \quad \mathcal{B}_k T_j^i = \dot{\partial}_k T_j^i - \left(\dot{\partial}_s T_j^i \right) G_k^s + T_j^s G_{sk}^i - T_s^i G_{jk}^s.$$

The Berwald covariant derivative of the metric tensor is given by

$$(1.6) \quad \mathcal{B}_k g_{ij} = -2 C_{ijk|h} y^h = -2 y^h \mathcal{B}_h C_{ijk}.$$

The Berwald covariant differential operator commutes with the partial differential operator with respect to y^k according to

$$(1.7) \quad \left(\dot{\partial}_k \mathcal{B}_h - \mathcal{B}_h \dot{\partial}_k \right) T_j^i = T_j^s G_{khs}^i - T_s^i G_{khj}^s$$

where $G_{khs}^i = \dot{\partial}_k G_{hs}^i.$

The commutation formula for Berwald's covariant differentiation is given as

$$(1.8) \quad \mathcal{B}_k \mathcal{B}_h X^i - \mathcal{B}_h \mathcal{B}_k X^i = X^r H_{rhk}^i - \left(\dot{\partial}_r X^i \right) H_{hk}^r$$

where Berwald curvature tensor H_{rhk}^i and the tensor H_{hk}^r are defined by

$$H_{rhk}^i := \dot{\partial}_k G_{rh}^i + G_{rh}^s G_{sk}^i + G_{srk}^i G_h^s - k/h$$

and $H_{hk}^r := H_{shk}^r y^s.$

The notation $-k/h$ means the subtraction from the former term after interchanging the indices k and h .

2. Hypersurface of a Finsler Space

Let F_{n-1} be a hypersurface of F_n given by the equation $x^i := x^i(u^\alpha)$,

($\alpha = 1, 2, \dots, n-1$). The metric tensor $g_{\alpha\beta}(u, \dot{u})$ and the torsion tensor $C_{\alpha\beta\gamma}$ of F_{n-1} are given by

$$(2.1) \quad (a) \quad g_{\alpha\beta}(u, v) = g_{ij}(x, y) B_{\alpha}^i B_{\beta}^j$$

$$(b) \quad C_{\alpha\beta\gamma} = C_{ijk} B_{\alpha}^i B_{\beta}^j B_{\gamma}^k$$

where

$$B_{\alpha\beta\gamma\delta}^{ijk} = B_{\alpha}^i B_{\beta}^j B_{\gamma}^k B_{\delta}^l$$

$$B_{\alpha}^i = \frac{\partial x^i}{\partial u^{\alpha}} \text{ and } v = \dot{u}.$$

At each point u^{α} of F_{n-1} , a unit normal vector $N^{*i}(u, v)$ is defined by

$$(2.2) \quad (a) \quad g_{ij}(x(u), y(u, v)) B_{\alpha}^i N^{*j} = 0,$$

$$(b) \quad g_{ij} N^{*i} N^{*j} = 1$$

and is called as the secondary normal to the hypersurface at the point.

The induced and intrinsic cartan connection coefficients $\Gamma_{\beta\gamma}^{*\alpha}$ and $\Gamma_{\beta\gamma}^{*\alpha 5}$ are related by

$$(2.3) \quad \Gamma_{\beta\gamma}^{*\alpha} = \Lambda_{\beta\gamma}^{\alpha} + \Gamma_{\beta\gamma}^{*\alpha}$$

where

$$(2.4) \quad g_{\varepsilon\gamma} \Lambda_{\alpha\beta}^{\varepsilon} = \Lambda_{\alpha\gamma\beta} = \left(M_{\beta\gamma} \Omega_{\alpha\sigma}^{*} + M_{\alpha\gamma} \Omega_{\beta\sigma}^{*} - M_{\alpha\beta} \Omega_{\gamma\sigma}^{*} \right) \dot{u}^{\sigma} \\ - \left(M_{\lambda\alpha} C_{\beta\gamma}^{\lambda} + M_{\lambda\beta} C_{\alpha\gamma}^{\lambda} - M_{\lambda\gamma} C_{\beta\alpha}^{\lambda} \right) \Omega_{\sigma\mu}^{*} \dot{u}^{\sigma} \dot{u}^{\mu}$$

and

$$M_{\beta\gamma} = N_i^{*} C_{jk}^i B_{\beta}^j B_{\gamma}^k.$$

The normal curvature of the hypersurface F_{n-1} of a Finsler space F_n in the direction of $\dot{u}^{\sigma}$ is given by

$$(2.5) \quad k_n(u, \dot{u}) = \left(\Omega_{\sigma\lambda}^{*} \dot{u}^{\sigma} \dot{u}^{\lambda} \right) F^{-2}(u, \dot{u}).$$

If the vector $\dot{u}^{\sigma}$ is of unit length i.e. $F(u, \dot{u}) = 1$ then the expression for the normal curvature k_n may be written as

$$(2.6) \quad k_n(u, \dot{u}) = \Omega_{\sigma\lambda}^* \dot{u}^\sigma \dot{u}^\lambda$$

In view of (2.6) the quantities $\Lambda_{\alpha\beta}^\varepsilon$ satisfy the following condition

$$(2.7) \quad (a) \quad \Lambda_{\alpha\beta}^\varepsilon \dot{u}^\alpha = k_n M_\beta^\varepsilon,$$

$$(b) \quad g_{\varepsilon\gamma} \dot{u}^\gamma \Lambda_{\alpha\beta}^\varepsilon = -k_n M_{\alpha\beta}$$

where

$$(c) \quad M_\beta^\varepsilon = g^{\varepsilon\gamma} M_{\gamma\beta}.$$

The mixed covariant derivative of an arbitrary tensor field T_α^i is given by

$$(2.8) \quad B_\gamma T_\alpha^i = \partial_\gamma T_\alpha^i - \left(\partial_\varepsilon T_\alpha^i \right) \dot{\partial}_\gamma G^\varepsilon - T_\alpha^i G_{\alpha\gamma}^\varepsilon + T_\alpha^r G_{rh}^i B_\gamma^h$$

where $G_{\alpha\gamma}^\varepsilon$ are the induced Berwald connection parameters. In particular,

$$(2.9) \quad \mathcal{B}_\beta B_\alpha^i = V_{\alpha\beta}^i = B_{\alpha\beta}^i - B_\varepsilon^i G_{\alpha\beta}^\varepsilon + G_{hk}^i B_{\alpha\beta}^{hk}.$$

This equation can be written as⁷

$$(2.10) \quad V_{\alpha\beta}^i = N^{*i} \Omega_{\alpha\beta}^* - B_\varepsilon^i \left(\Lambda_{\alpha\beta}^\varepsilon + C_{\alpha\beta|\sigma}^\varepsilon \dot{u}^\sigma \right) + C_{hk|\gamma}^i \gamma^\gamma B_{\alpha\beta}^{hk}$$

where $\Omega_{\alpha\beta}^*$ are the components of the second fundamental tensor. Gauss equations for Berwald curvature tensor, obtained by Sinha and Singh⁶ are given by

$$(2.11) \quad \begin{aligned} H_{\varepsilon\delta\beta\gamma} &= H_{hlkj} B_{\varepsilon\delta\beta\gamma}^{hlkj} + \left(\Omega_{\varepsilon\beta}^* \Omega_{\delta\gamma}^* - \Omega_{\varepsilon\gamma}^* \Omega_{\delta\beta}^* \right) \\ &+ 2 M_{lh} B_\delta^h \left(\Omega_{\varepsilon\beta}^* V_{\gamma\sigma}^l - \Omega_{\varepsilon\gamma}^* V_{\beta\sigma}^l \right) \dot{u}^\sigma - \left(2 N^{*h} C_{hlk|r} \gamma^r \Omega_{\varepsilon\beta}^* B_{\gamma\delta}^{kl} \right. \\ &- 2 B_\delta^l g_{il} \Lambda_{\varepsilon\beta}^\sigma V_{\gamma\sigma}^l - 2 \mathcal{B}_\gamma \Lambda_{\varepsilon\beta}^\alpha g_{\alpha\delta} - 2 \mathcal{B}_j C_{hlk|r} \gamma^r B_{\varepsilon\delta\beta\gamma}^{hlkj} \\ &- 2 \dot{\partial}_j C_{hlk|r} \gamma^r B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\sigma}^j \dot{u}^\sigma - 2 C_{hlk|r} B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\sigma}^r \dot{u}^\sigma \\ &\left. - 2 C_{hlk|r} \gamma^r B_{\delta\beta}^{lk} V_{\gamma\sigma}^h - \beta/\gamma \right). \end{aligned}$$

Associate Berwald curvature tensors of the Finsler space F_n and the hypersurface F_{n-1} being denoted by H_{hlkj} and $H_{\varepsilon\delta\beta\gamma}$ respectively.

3. Hypersurface of a Recurrent Finsler Space

A recurrent Finsler space F_n equipped with Berwald connection G_{jk}^i is characterized by the condition

$$(3.1) \quad \mathcal{B}_m H_{hkj}^i = \lambda_m H_{hkj}^i, \quad H_{hkj}^i \neq 0.$$

The non-zero covariant vector λ_m is the recurrence vector. This vector is independent of directional arguments y^i .

Let the hypersurface F_{n-1} of the recurrent Finsler space F_n be umbilical. Then the Gauss equations for umbilical hypersurface F_{n-1} , obtained by Singh and Chaubey⁴ are as follows :

$$(3.2) \quad H_{\varepsilon\delta\beta\gamma} = H_{hlkj} B_{\varepsilon\delta\beta\gamma}^{hlkj} + M^{*i} M_i^* \left(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta} \right) \\ + 2 M_{\delta} k_n^2 \left(g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta} \right) \dot{u}^\sigma + P_{\varepsilon\delta\beta\gamma}$$

where

$$(3.3) \quad (a) M^{*i} M_i^* = k_n^2, \quad (b) \Omega_{\alpha\beta}^*(u, \dot{u}) = k_n g_{\alpha\beta}(u, \dot{u})$$

$$(c) \quad P_{\varepsilon\delta\beta\gamma} = 2 M_{lh} B_{\delta\lambda}^{hl} \left(\Lambda_{\beta\sigma}^\lambda \Omega_{\varepsilon\gamma}^* - \Lambda_{\gamma\sigma}^\lambda \Omega_{\varepsilon\beta}^* \right) \dot{u}^\sigma \\ - \left\{ 2 N^{*h} C_{h|k|l} y^r \Omega_{\varepsilon\beta}^* B_{\gamma\delta}^{kl} - 2 B_{\delta}^l g_{il} \Lambda_{\varepsilon\beta}^\sigma V_{\gamma\sigma}^i - 2 \mathcal{B}_\gamma \Lambda_{\varepsilon\beta}^\alpha g_{\alpha\delta} \right. \\ \left. + 2 \mathcal{B}_j C_{h|k|l} y^r B_{\varepsilon\delta\beta\gamma}^{h|k|j} + 2 \partial_j C_{h|k|l} y^r B_{\varepsilon\delta\beta}^{h|k|l} V_{\gamma\sigma}^i \dot{u}^\sigma \right. \\ \left. + 2 C_{h|k|l} B_{\varepsilon\delta\beta}^{h|k|l} V_{\gamma\sigma}^r \dot{u}^\sigma + 2 C_{h|k|l} y^r B_{\delta\beta}^{lk} V_{\gamma\varepsilon}^h - \beta/\gamma \right\}$$

The covariant derivative of $H_{\varepsilon\delta\beta\gamma}$ is given by

$$(3.4) \quad \mathcal{B}_\theta H_{\varepsilon\delta\beta\gamma} = \left(\mathcal{B}_m H_{h|k|j} B_\theta^m + \frac{\partial H_{h|k|j}}{\partial y^r} V_{\sigma\theta}^r \dot{u}^\sigma \right) B_{\varepsilon\delta\beta\gamma}^{h|k|j} \\ + \mathcal{B}_\theta \left[M^{*i} M_i^* \left(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta} \right) + 2 M_{\delta} k_n^2 \left(g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta} \right) \dot{u}^\sigma \right. \\ \left. + P_{\varepsilon\delta\beta\gamma} \right] + H_{h|k|j} \left(B_{\delta\beta\gamma}^{lkj} V_{\varepsilon\theta}^h + B_{\varepsilon\beta\gamma}^{hkj} V_{\delta\theta}^l + B_{\varepsilon\delta\gamma}^{h|j} V_{\beta\theta}^k + B_{\varepsilon\delta\beta}^{h|k} V_{\gamma\theta}^j \right)$$

Transvecting (3.1) by g_{il} and using equation (3.2) and (3.4), we get

$$\begin{aligned}
 (3.5) \quad \mathcal{B}_\theta H_{\varepsilon\delta\beta\gamma} &= \lambda_\theta \left[H_{\varepsilon\delta\beta\gamma} - M^{*i} M_i^* (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \right. \\
 &\quad \left. - 2M_\delta k_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma - P_{\varepsilon\delta\beta\gamma} \right] \\
 &\quad + \left[\left(\mathcal{B}_m g_{il} \right) H_{h\delta\beta\gamma}^i B_\theta^m + \frac{\partial H_{hlkj}}{\partial y^r} V_{\sigma\theta}^r \dot{u}^\sigma \right] B_{\varepsilon\delta\beta\gamma}^{hlkj} \\
 &\quad + \mathcal{B}_\theta \left[M^{*i} M_i^* (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \right. \\
 &\quad \left. + 2M_\delta k_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma - P_{\varepsilon\delta\beta\gamma} \right] \\
 &\quad + H_{hlkj} \left(B_{\delta\beta\gamma}^{lkj} V_{\varepsilon\theta}^h + B_{\varepsilon\beta\gamma}^{hkj} V_{\delta\theta}^l + B_{\varepsilon\delta\gamma}^{hlj} V_{\beta\theta}^k + B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\theta}^j \right)
 \end{aligned}$$

where $\lambda_\theta = \lambda_m B_\theta^m$.

Let us assume

$$\begin{aligned}
 (3.6) \quad T_{\varepsilon\delta\beta\gamma} &= H_{\varepsilon\delta\beta\gamma} - M^{*i} M_i^* (g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta}) \\
 &\quad - 2M_\delta k_n^2 (g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta}) \dot{u}^\sigma - P_{\varepsilon\delta\beta\gamma}.
 \end{aligned}$$

In view of the equation (3.6), the equation (3.5) may be written as

$$\begin{aligned}
 (3.7) \quad \mathcal{B}_\theta T_{\varepsilon\delta\beta\gamma} &= \lambda_\theta T_{\varepsilon\delta\beta\gamma} + \left[\left(\mathcal{B}_m g_{il} \right) H_{h\delta\beta\gamma}^i B_\theta^m + \frac{\partial H_{hlkj}}{\partial y^r} V_{\sigma\theta}^r \dot{u}^\sigma \right] B_{\varepsilon\delta\beta\gamma}^{hlkj} \\
 &\quad + H_{hlkj} \left[B_{\delta\beta\gamma}^{lkj} V_{\varepsilon\theta}^h + B_{\varepsilon\beta\gamma}^{hkj} V_{\delta\theta}^l + B_{\varepsilon\delta\gamma}^{hlj} V_{\beta\theta}^k + B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\theta}^j \right].
 \end{aligned}$$

Therefore $\mathcal{B}_\theta T_{\varepsilon\delta\beta\gamma} = \lambda_\theta T_{\varepsilon\delta\beta\gamma}$ if and only if

$$\begin{aligned}
 (3.8) \quad &\left[\left(\mathcal{B}_m g_{il} \right) H_{h\delta\beta\gamma}^i B_\theta^m + \frac{\partial H_{hlkj}}{\partial y^r} V_{\sigma\theta}^r \dot{u}^\sigma \right] B_{\varepsilon\delta\beta\gamma}^{hlkj} \\
 &+ H_{hlkj} \left[B_{\delta\beta\gamma}^{lkj} V_{\varepsilon\theta}^h + B_{\varepsilon\beta\gamma}^{hkj} V_{\delta\theta}^l + B_{\varepsilon\delta\gamma}^{hlj} V_{\beta\theta}^k + B_{\varepsilon\delta\beta}^{hlk} V_{\gamma\theta}^j \right] = 0.
 \end{aligned}$$

Hence, we conclude :

Theorem 3.1 : If an umbilical hypersurface F_{n-1} is immersed in a recurrent Finsler space F_n whose recurrence vector field λ_m is not normal to the hypersurface F_{n-1} , then $T_{\varepsilon\delta\beta\gamma}$ is recurrent with the recurrence vector $\lambda_\theta = \lambda_m B_\theta^m$ if and only if (3.8) holds good.

Suppose

$$(3.9) \quad J_{\varepsilon\delta\beta\gamma} = \left(M^{*i} M_i^* \right) \left(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta} \right) \\ + 2M_\delta k_n^2 \left(g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta} \right) \dot{u}^\sigma + P_{\varepsilon\delta\beta\gamma}.$$

In view of the equation (3.7), the equation (3.9) may be written as

$$(3.10) \quad H_{\varepsilon\delta\beta\gamma} = T_{\varepsilon\delta\beta\gamma} + J_{\varepsilon\delta\beta\gamma}.$$

This equation shows that $H_{\varepsilon\delta\beta\gamma}$ is recurrent with the recurrence vector field λ_θ if the tensors $T_{\varepsilon\delta\beta\gamma}$ and $J_{\varepsilon\delta\beta\gamma}$ are recurrent with the same recurrence vector field λ_θ . This fact and the theorem 3.1 prove the following :

Theorem 3.2 : The sufficient conditions that the curvature tensor $H_{\varepsilon\delta\beta\gamma}$ of an umbilical hypersurface immersed in a recurrent space, whose recurrence vector field λ_m is not normal to the hypersurface F_{n-1} , be recurrent with the recurrence vector field λ_θ are that the relation (3.8) holds and $J_{\varepsilon\delta\beta\gamma}$ is recurrent with the recurrence vector field λ_θ .

Now, suppose that the recurrence vector field λ_m is normal to the hypersurface F_{n-1} i.e. $\lambda_m B_\theta^m = \lambda_\theta = 0$. The Bianchi identity for a recurrent Finsler space is given by²

$$(3.11) \quad \lambda_r H_{h k j}^i + \lambda_j H_{h r k}^i + \lambda_k H_{h j r}^i = 0.$$

Transvecting (3.11) by g_{il} , we get

$$(3.12) \quad \lambda_r H_{h l k j} + \lambda_j H_{h l r k} + \lambda_k H_{h l j r} = 0.$$

Multiplying the above equation by $B_{\varepsilon\delta\beta\gamma}^{h l k j}$ and using the conditions $\lambda_\theta = \lambda_m B_\theta^m = 0$ and $\lambda_r \neq 0$, we get

$$(3.13) \quad H_{h l k j} B_{\varepsilon\delta\beta\gamma}^{h l k j} = 0.$$

In view of equation (3.13) the equation (3.2) may be written as

$$(3.14) \quad H_{\varepsilon\delta\beta\gamma} = M^{*i} M_i^* \left(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta} \right) \\ + 2M_\delta k_n^2 \left(g_{\varepsilon\beta} g_{\sigma\gamma} - g_{\varepsilon\gamma} g_{\sigma\beta} \right) \dot{u}^\sigma + P_{\varepsilon\delta\beta\gamma}.$$

This leads to :

Theorem 3.3. If an umbilical hypersurface F_{n-1} is immersed in a recurrent Finsler space F_n whose recurrence vector λ_m is normal to the hypersurface F_{n-1} , then the tensor $T_{\varepsilon\delta\beta\gamma}$ defined by (3.6) vanishes.

Transvecting (3.14) by $\dot{u}^\delta$ and using $M_\beta = N^*_i C^i_{jk} B^j_\beta N^{*k}$, we get

$$(3.15) \quad H_{\varepsilon\delta\beta\gamma} \dot{u}^\delta = M^{*i} M^*_i \left(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta} \right) \dot{u}^\delta + P_{\varepsilon\delta\beta\gamma} \dot{u}^\delta.$$

Transvecting the equation (3.3c) by $\dot{u}^\delta$ and using the equations $M_{lh} y^h = 0$, (1.2a), (1.4) and (2.10), we get

$$(3.16) \quad P_{\varepsilon\delta\beta\gamma} \dot{u}^\delta = \left[-2B^l_\delta g_{il} \dot{u}^\delta \Lambda^\sigma_{\varepsilon\beta} \left\{ N^{*i} \Omega^*_{\gamma\sigma} - B^i_\lambda \left(\Lambda^\lambda_{\gamma\sigma} + C^\lambda_{\gamma\sigma|\eta} \dot{u}^\eta \right) \right. \right. \\ \left. \left. + C^i_{hk|\eta} y^\eta B^{hk}_{\gamma\sigma} \right\} - 2\mathcal{B}_\gamma \Lambda^\alpha_{\varepsilon\beta} g_{\alpha\delta} \dot{u}^\delta - \beta/\gamma \right].$$

Using equation (1.1), (1.2b), (1.4) and (2.1) in the equation (3.16), we have

$$(3.17) \quad P_{\varepsilon\delta\beta\gamma} \dot{u}^\sigma = 2g_{\alpha\delta} \dot{u}^\sigma \Lambda^\sigma_{\varepsilon\beta} \left[\Lambda^\alpha_{\gamma\sigma} + C^\alpha_{\gamma\sigma|\eta} \dot{u}^\eta \right] \\ - 2\mathcal{B}_\gamma \Lambda^\alpha_{\varepsilon\beta} g_{\alpha\delta} \dot{u}^\delta - \beta/\gamma.$$

Using the equation (3.17) in the equation (3.15), we get

$$(3.18) \quad H_{\varepsilon\delta\beta\gamma} \dot{u}^\delta = M^{*i} M^*_i \left(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta} \right) \dot{u}^\delta \\ + 2g_{\alpha\delta} \dot{u}^\delta \left[\Lambda^\sigma_{\varepsilon\beta} \left(\Lambda^\alpha_{\gamma\sigma} + C^\alpha_{\gamma\sigma|\eta} \dot{u}^\eta \right) - \mathcal{B}_\gamma \Lambda^\alpha_{\varepsilon\beta} - \beta/\gamma \right]$$

The hypersurface F_{n-1} of constant curvature is characterized by the equation

$$(3.19) \quad H_{\varepsilon\delta\beta\gamma} \dot{u}^\delta = K \left(g_{\varepsilon\beta} g_{\delta\gamma} - g_{\varepsilon\gamma} g_{\delta\beta} \right) \dot{u}^\delta.$$

From equations (3.18) and (3.19), we conclude :

Theorem 3.4 : If an umbilical hypersurface F_{n-1} is immersed in a recurrent Finsler space F_n whose recurrence vector field λ_m is normal to F_{n-1} , then the necessary and sufficient condition that F_{n-1} be a space of constant curvature is that

$$g_{\alpha\delta} \dot{u}^\delta \left[\mathcal{B}_\gamma \Lambda^\sigma_{\varepsilon\beta} - \Lambda^\sigma_{\varepsilon\beta} \left(\Lambda^\alpha_{\gamma\sigma} + C^\alpha_{\gamma\sigma|\eta} \dot{u}^\eta \right) - \beta/\gamma \right] = 0.$$

If the hypersurface F_{n-1} is of minimal variety then curvature vector will vanish and equations (3.3a) and (3.3b) give

$$(3.20) \quad k_n = 0, \quad \Omega_{\alpha\beta}^* = 0.$$

In view of (3.20) the equation (2.4) implies

$$(3.21) \quad \Lambda_{\beta\gamma}^{\epsilon} = 0.$$

Using the equation (3.21) in the equation (3.3c), we get

$$(3.22) \quad P_{\epsilon\delta\beta\gamma} = 2\mathcal{B}_j C_{hlk|r} y^r B_{\epsilon\delta\beta\gamma}^{hlkj} + 2\dot{\partial}_j C_{hlk|r} y^r B_{\epsilon\delta\beta}^{hlk} V_{\gamma\sigma}^i \dot{u}^\sigma \\ + 2C_{hlk|r} B_{\epsilon\delta\beta}^{hlk} V_{\gamma\sigma}^r \dot{u}^\sigma + 2C_{hlk|r} y^r B_{\delta\beta}^{lk} V_{\gamma\epsilon}^h - \beta/\gamma.$$

Using equation (1.6), (2.10), (3.20) and (3.21) in the equation (3.22), we find.

$$(3.23) \quad P_{\epsilon\delta\beta\gamma} = 2\left(\mathcal{B}_j \mathcal{B}_k g_{hl}\right) B_{\epsilon\delta\beta\gamma}^{hlkj} + 2\dot{\partial}_j C_{hlk|r} y^r B_{\epsilon\delta\beta}^{hlk} \\ \left(C_{ms|\eta}^i B_{\gamma\sigma}^{ms} \dot{u}^\sigma - B_{\eta}^i C_{\gamma\sigma|\lambda}^{\eta} \dot{u}^\lambda \dot{u}^\sigma\right) \\ + 2C_{hlk|r} B_{\epsilon\delta\beta}^{hlk} \left[C_{ms|\eta}^r y^{\eta} B_{\gamma\sigma}^{ms} \dot{u}^\sigma - B_{\eta}^r C_{\gamma\sigma|\lambda}^{\eta} \dot{u}^\lambda \dot{u}^\sigma\right] \\ + 2\left(\mathcal{B}_k g_{hl}\right) B_{\delta\beta}^{lk} V_{\gamma\epsilon}^h - \beta/\gamma.$$

Using equations (1.2a), (1.4), (1.6), (3.13), (1.7) and $\dot{u}_{|\lambda}^\sigma = 0$ in the equation (3.23), we get

$$(3.24) \quad P_{\epsilon\delta\beta\gamma} = C_{rhl} H_{kj}^r B_{\epsilon\delta\beta\gamma}^{hlkj} + \left(y_r G_{hkl}^r B_{\delta\beta}^{lk} V_{\gamma\epsilon}^h - \beta/\gamma\right).$$

In view of (3.24) and (3.20), the equation (3.14) gives

$$H_{\epsilon\delta\beta\gamma} = C_{rhl} H_{kj}^r B_{\epsilon\delta\beta\gamma}^{hlkj} + \left(y_r G_{hkl}^r B_{\delta\beta}^{lk} V_{\gamma\epsilon}^h - \beta/\gamma\right).$$

This leads to :

Theorem 3.5 : If an umbilical hypersurface F_{n-1} is of minimal variety and is immersed in a recurrent space F_n whose recurrence vector field is normal to F_{n-1} , then the space F_{n-1} is Minkowskian (i.e. $H_{\epsilon\delta\beta\gamma} = 0$) if and only if

$$C_{rhl} H_{kj}^r B_{\epsilon\delta\beta\gamma}^{hlkj} = -y_r G_{hkl}^r B_{\delta\beta}^{lk} V_{\gamma\epsilon}^h - \beta/\gamma.$$

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