

Certain Linear Generating Relations for the Hermite Polynomials using Fractional Calculus

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Abstract: The main aim of the present paper is to apply the extended Riemann Liouville fractional derivative operator for finding some linear generating relations for the Hermite polynomials. Four main results are obtained, which are presented in the form of four theorems.

Keywords: Gamma function, Beta function, Riemann-Liouville fractional derivative operator, extended Gauss hypergeometric function, Fox H-function, generating functions, Hermite polynomials.

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1. Introduction

The subject of fractional calculus, nowadays, is one of the most rapidly growing subjects of mathematical analysis. The fractional integral operators, involving various Special functions have found significant importance and applications in various Subfields of applicable mathematical analysis. The applications of fractional calculus are also seen in various fields, including turbulence and fluid dynamics, stochastic dynamical system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological system, astrophysics etc.¹⁻⁶.

In the last three decades, a number of workers like Love⁷, Mc Bride⁸, Kalla⁹⁻¹⁰, Kalla and Saxena¹¹, Saigo¹²⁻¹³, Kilbas¹⁴ have studied the properties, applications and different extensions of various operators of fractional calculus on a number of classical & non classical Special functions & polynomials. A sufficient account of fractional calculus operators along with their properties and applications can be found in the research monographs by Kiryakova¹⁵ and Miller and Rose¹⁶. The first application of fractional calculus was due to Abel¹⁷ in the solution to the fractional problem. In fractional calculus, the fractional derivatives are defined via fractional integrals.

In recent years, certain extended fractional derivative operators, associated with Special functions have been actively investigated and applied on various Special functions. Authors Agarwal *et al.*¹⁸ have introduced certain extended fractional derivative operators, and applied them on various Special functions. Motivated by these recent developments in the field of applications of extended fractional derivatives to various Special functions, in the present paper an attempt has been made to obtain some linear generating relations including Hermite polynomial, using extended Riemann Liouville fractional derivative operator, defined by Choi & Pairs in their very recent paper¹⁸, published in the year 2015.

(1) The extended Gauss hypergeometric function $F_p^{(\alpha, \beta, \kappa, \mu)}(a, b; c; z)$ is defined by Agrawal and Choi¹⁸ as follows:

$$(1.1) \quad F_p^{(\alpha, \beta, \kappa, \mu)}(a, b; c; z) = \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\alpha, \beta, \kappa, \mu)}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!},$$

($|z| < 1$; $\min \{ \operatorname{Re}(\alpha), \operatorname{Re}(\beta), \operatorname{Re}(\kappa), \operatorname{Re}(\mu) \} > 0$; $\operatorname{Re}(c) > \operatorname{Re}(b) > 0$; $\operatorname{Re}(p) \geq 0$), where, $B(u, v)$ is the familiar Beta function defined as:

$$(1.2) \quad B(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} dt, \quad (\operatorname{Re}(u) > 0; \operatorname{Re}(v) > 0) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} (u, v \in \mathbb{C}),$$

where Γ denotes the Euler's Gamma function¹⁹. It is to note here that, for $p = 0$, (1.1) reduces to the ordinary Gauss hypergeometric function ${}_2F_1(a, b; c; z)$, defined as follows:

$$(1.3) \quad {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z| < 0$$

(2) The Hermite Polynomial $H_n(x)$ is defined by Rainville¹⁹ s:

$$(1.4) \quad H_n(x) = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m n!}{m! (n-2m)!} (2x)^{n-2m},$$

(3) The extended beta function $B_p^{(\alpha, \beta, \kappa, \mu)}(x, y)$ is defined by Srivastava *et al*²⁰ as:

$$(1.5) \quad B_p^{(\alpha, \beta, \kappa, \mu)}(x, y) = \left\{ \int_0^1 t^{x-1} (1-t)^{y-1} {}_1F_1(\alpha; \beta; \frac{-p}{t^k(1-t)^k}) dt \right\},$$

where $k \geq 0$, $\mu \geq 0$, $\min \{ \operatorname{Re}(\alpha), \operatorname{Re}(\beta) \} > 0$; $\operatorname{Re}(x) > -\operatorname{Re}(\kappa a) > 0$; $\operatorname{Re}(y) \geq -\operatorname{Re}(\mu a) \& \operatorname{Re}(\rho)$.

(4) A further extension of the extended Gauss hypergeometric function $F_{p; \kappa, \mu}(a, b; c; z; m)$ is defined by Srivastava *et al.*²⁰:

$$(1.6) \quad F_{p; \kappa, \mu}(a, b; c; z; m) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{B_p^{(\alpha, \beta, \kappa, \mu)}(b+n, c-b+m)}{B(b+n, c-b+m)} \frac{z^n}{n!},$$

where ($p \geq 0$; $\operatorname{Re}(\kappa) > 0$, $\operatorname{Re}(\mu) > 0$; $\operatorname{Re}(c) > \operatorname{Re}(b) > m$; $\operatorname{Re}(p) \geq 0$).

(5) The extended Riemann -Liouville fractional derivative of $F(z)$ of order v is defined by Agarwal *et al.*¹⁸ by the following relations:

$$(1.7) \quad D_z^{v,p,k,\mu} f(z) = \frac{1}{\Gamma_{-v}} \int_0^z (z-t)^{-v-1} f(t) dt \times {}_1F_1(\alpha; \beta; \frac{pz^{k+\mu}}{t^k(z-t)^\mu}) dt,$$

and

$$(1.8) \quad \begin{aligned} D_z^{v,p,k,\mu} f(z) &= \frac{d^m}{dz^m} D_z^{v-m;p;k,\mu} f(z) \\ &= \frac{d^m}{dz^m} \left\{ \frac{1}{\Gamma_{m-v}} \int_0^z (z-t)^{m-v-1} f(t) dt \right. \\ &\quad \left. \times {}_1F_1(\alpha; \beta; \frac{pz^{k+\mu}}{t^k(z-t)^\mu}) dt \right\}, \end{aligned}$$

where $m-1 \leq \operatorname{Re}(v) < m$; $\operatorname{Re}(v) < 0$; $\operatorname{Re}(p) > 0$; $\operatorname{Re}(k) > 0$; $\operatorname{Re}(\mu) > 0$. From (1.7) and (1.8), it may easily be seen that for $p = 0$, we obtain the classical-Riemann Liouville fractional derivative.

In the present paper, we have obtained certain linear generating relations, involving Hermite polynomials (1.4), using operators (1.7) and (1.8).

2. Preliminaries

While proving the main results, the following well-known identities & results will be used:

2.1. The elementary identity (Kiryakova¹⁵):

$$(2.1) \quad [(1-x) - t]^{-\alpha} = (1-t)^{-\alpha} \left(1 - \frac{x}{1-t}\right)^{-\alpha}$$

2.2. The identity (K. S. Millar²¹):

$$(2.2) \quad [(1-x) - t]^{-\alpha} = (1-t)^{-\alpha} \left(1 + \frac{xt}{1-t}\right)^{-\alpha}$$

2.3. The result [Agarwal *et al.*¹⁸, p. 458]:

$$(2.3) \quad \begin{aligned} \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} F_{p,k,\mu} \alpha + n, \lambda, v; z; m) t^n \\ = (1-t)^{-\alpha} F_{p,k,\mu} \left(\alpha, \lambda; v; \frac{x}{(1-t)}; m \right) \end{aligned}$$

2.4. The generalized binomial theorem, [Agarwal *et al.*¹⁸, p.456]:

$$(2.4) \quad (1-z)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!}, (\alpha \in \mathbb{C})$$

2.5.

$$(2.5) \quad (z)_n = \frac{\Gamma_{z+n}}{\Gamma_z},$$

2.6 The result, [Agarwal *et al.*¹⁸, p. 456]:

$$(2.6) \quad D_z^{\lambda-v,p,k,\mu} \{ z^{\lambda-1} (1-z)^{-\alpha} \}$$

$$\begin{aligned}
&= \frac{\Gamma(\lambda)z^{(v-1)}}{\Gamma v} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\lambda)_n}{(v)_n} \frac{B_p^{\alpha,\beta,k,\mu}(\lambda+n, v-\lambda+m)}{B(\lambda+n, v-\lambda+m)} \frac{z^n}{n!}, \\
&= \frac{\Gamma(\lambda)z^{(v-1)}}{\Gamma v} F_{p,\kappa,\mu}(\alpha, \lambda; v; z; m),
\end{aligned}$$

where, $m-1 \leq \operatorname{Re}(v) < m$, for some $m \in \mathbb{N}$ & $\operatorname{Re}(v) < \operatorname{Re}(\lambda)$.

2.7. The result, [Agarwal *et al.*¹⁸, p. 457]:

$$\begin{aligned}
(2.7) \quad D_z^{\lambda-v,p,k,\mu} \{(1-az)^{-\alpha}(1-az)^{-\beta}\} \\
= \frac{\Gamma(\lambda)z^{(v-1)}}{\Gamma v} F_{1,p,k,\mu}(\alpha, \beta, \lambda; v; az; bz; m),
\end{aligned}$$

where, $m-1 \leq \operatorname{Re}(v) < m$, for some $m \in \mathbb{N}$ & $\operatorname{Re}(v) < \operatorname{Re}(\lambda)$.

2.8. The result, [Agarwal *et al.*¹⁸, p. 457]:

$$\begin{aligned}
(2.8) \quad D_z^{\lambda-v,p,k,\mu} \{(1-z)^{-\alpha} z^{\lambda-1} F_{p,k,\mu} \left(\alpha, \lambda; v; \frac{x}{(1-t)}; m \right)\}, \\
= \frac{\Gamma(\lambda)z^{(v-1)}}{\Gamma v} F_{2,p,k,\mu}(\alpha, \beta, \lambda; y; v; x; z; m),
\end{aligned}$$

where, $m-1 \leq \operatorname{Re}(v) < m$, for some $m \in \mathbb{N}$, & $\operatorname{Re}(v) < \operatorname{Re}(\lambda)$.

3. Main Results

3.1. Linear generating relations for the Hermite polynomial $H_n(x)$:

We use extended fractional derivatives, defined in (1.8), for establishing some linear generating relations for the Hermite polynomial:

Theorem. I The following linear generating relation holds:

$$\begin{aligned}
(3.1) \quad D_x^{v,\rho;\kappa,\mu} \{H_n(x)\} \\
= n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{m!(n-2m)!} D_x^{v,\rho;\kappa,\mu} \frac{\Gamma(n-2m+1) D_x^{v,\rho;\kappa,\mu}(n-2m+1, m-v)}{\Gamma(n-2m-v+1), \beta(n-2m+1, m-v)} (2x)^{n-2m-v}
\end{aligned}$$

where, $\{|x| < \min(1, |1-t|)\}$, $(\alpha \in \mathbb{C}, |z| < 1; |\frac{t}{1-y}| < 1;)$, & $m-1 \leq \operatorname{Re}(\beta-x) < m < \operatorname{Re}(\beta)$, for some $m \in \mathbb{N}$, & $\operatorname{Re}(x) < \operatorname{Re}(v)$.

Proof: Applying (1.8) in (1.4) to the function, x^{n-2m} , we obtain:

$$\begin{aligned}
D_x^{v,\rho;\kappa,\mu} \{H_n(x)\} &= n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{m!(n-2m)!} 2^{n-2m} \frac{d^m}{dx^m} \left\{ \frac{1}{\Gamma m-v} \int_0^x (x-t)^{m-v-1} \right. \\
&\quad \left. \times {}_1F_1(\alpha; \beta; -\frac{p x^{k+\mu}}{t^{k(x-t)^\mu}}) dt \right\}
\end{aligned}$$

Setting $t = xu$, on the right hand of the above equation, we obtain:

$$(3.2) \quad D_x^{v,\rho;\kappa,\mu} \{H_n(x)\} = \left(\frac{d^m}{dx^m} (2x)^{n-2m-v-1} \right) \times n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{m!(n-2m)!}$$

$${}_1F_1 \frac{1}{\Gamma(m-v)} \int_0^1 (1-u)^{m-v-1} u^{n-2m} \left(\alpha; \beta; -\frac{p x^{k+\mu}}{u^k(1-u)^\mu} \right) du,$$

$$\text{where } \frac{d^m}{dx^m} (x)^{n-m-v-1} = \frac{\Gamma(n-m-v)}{\Gamma(n-v)} x^{n-2m-v}.$$

Using (1.4) & (1.5) on the right hand side of the last equation, we obtain:

$$D_x^{v,\rho;\kappa,\mu} \{H_n(x)\} = n! \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^m}{m! (n-2m)!} D_x^{v,\rho;\kappa,\mu} \frac{\Gamma(n-2m+1) D_x^{v,\rho;\kappa,\mu} (n-2m+1, m-v)}{\Gamma(n-2m-v+1), \beta(n-2m+1, m-v)} (2x)^{n-2m-v},$$

which is the desired result (3.1), and thus theorem I is established.

Theorem. II. The following linear generating relation holds:

$$(3.3) \quad D_x^{v,\rho;\kappa,\mu} \{H_n(x)\} = n! \sum_{m=0}^{\frac{n}{2}} (\alpha)_n \frac{(-1)^m}{m! (n-2m)!} D_x^{v,\rho,\kappa,\mu} (2x)^{n-2m}$$

where, $\{|x| < \min(1, |1-t|)\}$, $(\alpha \in \mathbb{C}, |x| < 1; |\frac{t}{1-y}| < 1;)$, & $m-1 \leq \operatorname{Re}(\beta-x) < m < \operatorname{Re}(\beta)$, for some $m \in \mathbb{N}$, & $\operatorname{Re}(x) < \operatorname{Re}(v)$.

Proof: Applying (1.8) in (1.4) to the Hermite function $H_n(x)$ with its series expansion, we obtain:

$$\begin{aligned} D_x^{v,\rho;\kappa,\mu} \{H_n(x)\} &= n! \frac{d^m}{dx^m} \left\{ \frac{1}{\Gamma(m-v)} \int_0^1 (x-t)^{m-v-1} t^{n-2m} \times \right. \\ &\quad \left. {}_1F_1(\alpha; \beta; -\frac{p x^{k+\mu}}{t^k(x-t)^\mu}) \sum_{m=0}^{\frac{n}{2}} (\alpha)_n \frac{(-1)^m}{m! (n-2m)!} dt \right\} \\ (3.4) \quad D_x^{v,\rho;\kappa,\mu} \{H_n(x)\} \\ &= n! \sum_{m=0}^{\frac{n}{2}} (\alpha)_n \frac{(-1)^m}{m! (n-2m)!} \frac{d^m}{dx^m} \left\{ \frac{1}{\Gamma(m-v)} \int_0^x (x-t)^{n-2m-v-1} t^{n-2m} dt \right. \\ &\quad \left. {}_1F_1(\alpha; \beta; -\frac{p x^{k+\mu}}{t^k(x-t)^\mu}) \right\} \end{aligned}$$

$$(3.5) \quad D_x^{v,\rho;\kappa,\mu} \{H_n(x)\} = n! \sum_{m=0}^{\frac{n}{2}} (\alpha)_n \frac{(-1)^m}{m! (n-2m)!} D_x^{v,\rho,\kappa,\mu} (2x)^{n-2m}$$

which is the desired result (3.3), and thus theorem II is established.

Theorem. III. The following linear generating relation holds:

$$\begin{aligned} (3.6) \quad D_x^{v,\rho;\kappa,\mu} \{x^{\lambda-1} H_n(x)\} \\ &= n! \sum_{m=0}^{\frac{n}{2}} \frac{(-1)^m}{m! (n-2m)!} (\alpha)_n D_x^{v,\rho,\kappa,\mu} (2x)^{\lambda+n-2m-1} \\ &= \frac{\Gamma(\lambda)x^{\lambda-v-1}}{\Gamma(\lambda-v)} n! \sum_{m=0}^{\frac{n}{2}} \frac{(-1)^m}{m! (n-2m)!} \frac{(\lambda)_n}{(\lambda-v)_n} \frac{\beta_{\rho}^{\alpha,\beta,\kappa,\mu} \{(\lambda+n-2m), (m-v)\}}{\beta(\lambda+n-2m, m-v)} (2x)^{n-2m}, \end{aligned}$$

where $\{|x| < \min(1, |1-t|)\}$, $(\alpha \in \mathbb{C}, |x| < 1; |\frac{t}{1-y}| < 1)$ and $m-1 \leq \operatorname{Re}(\beta-x) < m < \operatorname{Re}(\beta)$, for some $m \in \mathbb{N}$ & $\operatorname{Re}(x) < \operatorname{Re}(v)$.

Proof: Using Maclaurin expansion,

$$f(x) = \sum_{n=0}^{\infty} (\alpha)_n x^n,$$

we can write:

$$D_x^{v,\rho;\kappa,\mu} \{f(x)\} = n! \sum_{m=0}^{\frac{n}{2}} (\alpha)_n \frac{(-1)^m}{m! (n-2m)!} (2x)^{n-2m}$$

Applying (3.1), on the right-hand side of the last equation, we obtain:

$$\{H_n(x)\} = n! \sum_{m=0}^{\frac{n}{2}} (\alpha)_n \frac{(-1)^m}{m! (n-2m)!} \frac{\Gamma(n-2m+1) D_x^{v,\rho;\kappa,\mu} (n-2m+1, m-v)}{\Gamma(n-2m-v+1), \beta(n-2m+1, m-v)} (2x)^{n-2m-v}$$

Applying (3.3), on the right-hand side of the last equation, we obtain:

$$(3.7) \quad x^{\lambda-1} D_x^{v,\rho;\kappa,\mu} \{H_n(x)\} = \frac{\Gamma\lambda(x)^{\lambda-v-1}}{\Gamma(\lambda-v)} n! \sum_{m=0}^{\frac{n}{2}} (\alpha)_n \frac{(-1)^m}{m! (n-2m)!} \frac{(\lambda)_n}{(\lambda-v)_n} \frac{\beta_{\rho}^{\alpha,\beta,\kappa,\mu} \{(\lambda+n-2m), (m-v)\}}{\beta(\lambda+n-2m, m-v)} (2x)^{n-2m},$$

Applying (2.6), on the right-hand side of the last equation, we obtain:

$$D_x^{v,\rho;\kappa,\mu} x^{\lambda-1} \{H_n(x)\} = \frac{\Gamma\lambda(x)^{\lambda-v-1}}{\Gamma(\lambda-v)} F_{\rho;\kappa,\mu}(\alpha; \lambda; v; x; m).$$

which is the desired result (3.6), and thus theorem III is established.

Theorem. IV. The following linear generating relation holds:

$$(3.8) \quad D_x^{v,\rho;\kappa,\mu} \{x^{\lambda-1} (1-x)^{-\alpha} H_n(x)\} = \frac{\Gamma\lambda(x)^{v-1}}{\Gamma(v)} \sum_{m=0}^{\frac{n}{2}} (\alpha)_n \frac{(-1)^m}{m! (n-2m)!} (2x)^{n-2m} \times \frac{(\lambda)_n}{(v)_n} \frac{\beta_{\rho}^{\alpha,\beta,\kappa,\mu} \{(\lambda+n-2m), (m+v-\lambda)\}}{\beta(\lambda+n-2m, m+v-\lambda)} = \frac{\Gamma\lambda(x)^{n-2m+v-1}}{\Gamma(v)} F_{p,k,\mu}(\alpha, \lambda; v; x; m),$$

where $\{|x| < \min(1, |1-t|)\}$, $(\alpha \in \mathbb{C}, |x| < 1; |\frac{t}{1-y}| < 1)$, $m-1 \leq \operatorname{Re}(\beta-x) < m < \operatorname{Re}(\beta)$, for some $m \in \mathbb{N}$ & $\operatorname{Re}(x) < \operatorname{Re}(v)$.

Proof: Using the generalized binomial theorem,

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{n!} x^n,$$

we can write:

$$D_x^{\lambda-v,\rho;\kappa,\mu} \{(2x)^{n-2m+\lambda-1} (1-x)^{-\alpha}\} n! \sum_{m=0}^{\frac{n}{2}} (\alpha)_n \frac{(-1)^m}{m! (n-2m)!}$$

$$= D_x^{\lambda-v, \rho; \kappa, \mu} \{x^{n-2m+\lambda-1} n! \sum_{m=0}^{\frac{n}{2}} (\alpha)_n \frac{(-1)^m}{m! (n-2m)!} (2x)^{n-2m}\},$$

Applying (3.1), on the right-hand side of the last equation, we obtain:

$$\begin{aligned} &= n! \sum_{m=0}^{\frac{n}{2}} (\alpha)_n \frac{(-1)^m}{m! (n-2m)!} \{D_x^{\lambda-v, \rho; \kappa, \mu} (2x)^{n-2m+\lambda-1} \\ &= \frac{\Gamma\lambda + n - 2m}{\Gamma(v + n - 2m)} (x)^{n-2m+v-1} \times \\ &n! \sum_{m=0}^{\frac{n}{2}} (\alpha)_n \frac{(-1)^m}{m! (n-2m)!} \frac{\beta_{\rho}^{\alpha, \beta, \kappa, \mu}(\lambda + n - 2m), (m + v - \lambda)}{\beta(\lambda + n - 2m, m + v - \lambda)} (2x)^{v+n-2m-1} \end{aligned}$$

Applying (3.3), on the right-hand side of the last equation, we obtain:

$$= \frac{\Gamma\lambda (x)^{n-2m+v-1}}{\Gamma(v)} n! \sum_{m=0}^{\frac{n}{2}} (\alpha)_n \frac{(-1)^m}{m! (n-2m)!} \frac{\beta_{\rho}^{\alpha, \beta, \kappa, \mu}\{(\lambda+n-2m), (m+v-\lambda)\}}{\beta(\lambda+n-2m, m+v-\lambda)} (2x)^{n-2m},$$

Applying (2.6) on both of the last equation, we obtain:

$$D_z^{v, \rho; \kappa, \mu} \{x^{\lambda-1} (1-x)^{-\alpha} H_n(x)\} = \frac{\Gamma\lambda (x)^{n-2m+v-1}}{\Gamma(v)} F_{p, k, \mu}(\alpha, \lambda; v; x; m).$$

which is the desired result (3.8), and thus theorem IV is established.

4. Conclusion

Linear and bilateral generating relations have been of much interest to various researchers in the recent past. Various mathematicians are investigating and introducing certain extended fractional derivative and integral operators and applying them on various Special functions and obtaining linear and bilateral generating relations involving some Special functions. In the present paper, an attempt has been made to obtain some linear generating relations for Hermite Polynomials, applying the extended Riemann Liouville fractional derivative operator. The results, which we have obtained in this paper, may be of much use to the Researchers, working in this field.

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