

## Generalized Normalized Semi-Parallel $h$ -Vector Field in a Special Finsler Spaces

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**Abstract:** C.M. Fulton<sup>1</sup> and M. Hashiguchi<sup>2</sup> studied the semi-parallel vector field in a Riemannian space and they found that concurrent vector fields<sup>3</sup>, torse forming vector fields<sup>4</sup> and concircular vector fields<sup>5</sup> are semi-parallel. P.N. Pandey and Fahmi Yaseen Abdo discussed concircular and semi-concircular vector fields in a Finsler space<sup>6,7</sup>. U.P. Singh and B.N. Prasad<sup>8</sup> extended the concept of semi-parallel vector field to Finsler spaces. U.P. Singh and Bindu Kumari<sup>9</sup> studied the existence of a normalized semi-parallel  $h$ -vector field in some special Finsler spaces. The purpose of the current paper is to generalize the concept of a normalized semi-parallel  $h$ -vector field and to study the existence of such vector field in some special Finsler spaces.

### 1. Introduction

Let  $F_n$  be an  $n$ -dimensional Finsler space equipped with a metric function  $F^{(1)}$  satisfying the requisite conditions<sup>10</sup>. The metric tensor and C-tensor of such space are  $g_{ij} = \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^{(2)}$  and  $C_{ijk} = \frac{1}{2} \dot{\partial}_i g_{jk}$ . Corresponding to each contravariant vector  $y^i$ , there is a covariant vector  $y_i$  such that

$$(1.1) \quad y_j = g_{ij} y^i.$$

The metric tensor  $g_{ij}$  and its associated tensor  $g^{ij}$  are related by

$$(1.2) \quad g_{ij} g^{jk} = \delta_i^k = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k. \end{cases}$$

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1) Unless stated otherwise, all geometric objects are assumed to be functions of line-element  $(x^i, y^i)$ .

2)  $\dot{\partial}_i = \frac{\partial}{\partial y^i}$ .

Due to homogeneous property, the C-tensor  $C_{ijk}$  satisfies the following identity

$$C_{ijk}y^i = 0,$$

which is equivalent to

$$(1.3) \quad C_{jk}^i y_i = 0.$$

The unit vector  $l^i$  in the direction of  $y^i$  is given by

$$(1.4a) \quad l^i := \frac{y^i}{F}.$$

The associate vector of  $l^i$  is defined by

$$(1.4b) \quad l_i := g_{ij}l^j = \dot{\partial}_i F = \frac{y_i}{F}.$$

E. Cartan deduced two types of covariant derivatives from his fundamental postulates. These covariant derivatives are defined as follows:

$$(1.5a) \quad X^i|_k := \dot{\partial}_k X^i + X^r C_{rk}^i$$

and

$$(1.5b) \quad X_{|k}^i := \partial_k X^i - (\dot{\partial}_r X^i) G_k^r + X^r \Gamma_{rk}^{*i}, \quad \partial_k \equiv \frac{\partial}{\partial x^k},$$

where the functions  $\Gamma_{rk}^{*i}$  are Cartan's connection parameters and  $G_k^r$  are given by  $G_k^r = \Gamma_{sk}^{*r} y^s$ . These covariant derivatives are called v-covariant derivative and h-covariant derivative respectively. The h-covariant derivatives of the metric tensor  $g_{ij}$ , the unit vector  $l^i$  and the metric function  $F$  vanish identically, i.e.

$$(1.6a) \quad g_{ij|k} = 0,$$

$$(1.6b) \quad l_{|k}^i = 0$$

and

$$(1.6c) \quad F_{|k} = 0.$$

The  $hv$ -curvature tensor which is defined by

$$P_{jkh}^i = \dot{\partial}_h \Gamma_{jk}^{*i} + C_{jm}^i P_{kh}^m - C_{jh}^i P_{mk}^m,$$

satisfies

$$(1.7) \quad P_{jkh}^i y^j = \Gamma_{jkh}^{*i} y^j = P_{kh}^i = C_{kh}^i y^r$$

where  $\dot{\partial}_j \Gamma_{hk}^{*r} = \Gamma_{jhk}^{*r}$  and  $P_{kh}^i$  is called  $\nu$  ( $h\nu$ ) – torsion tensor.

## 2. Normalized Semi-Parallel $h$ - Vector Field

U.P. Singh and B.N. Prasad<sup>8</sup> extended the concept of semi-parallel vector field to a Finsler space and they gave the following:

**Definition 2.1:** A non-zero vector field  $v_i$  in a Finsler space is said to be *semi-parallel* if there exists a scalar  $\lambda(x^i) \neq 0$  and  $\varepsilon(x^i)$  satisfying the following conditions

$$(2.1a) \quad v_i \text{ is a function of position coordinates only,}$$

$$(2.1b) \quad C_{jk}^i v_i = 0$$

and

$$(2.1c) \quad v_{i|j} = \lambda(g_{ij} + \varepsilon v_i v_j)$$

where  $v_{i|j}$  denotes the  $h$ -covariant derivative of  $v_i$  with respect to  $x^j$ .

U.P. Singh and Bindu Kumari<sup>9</sup> denoted the normalized vector field by  $X_i$ . Thus,  $X_i = \frac{v_i}{\nu}$  where  $\nu$  denotes the length of the vector  $v_i$ , i.e.  $\nu^2 = v_i v^i$ . They showed that the condition (2.1c) is equivalent to

$$X_{i|j} = \frac{\lambda}{\nu} (g_{ij} - X_i X_j).$$

In view of it, they defined a normalized semi-parallel vector field  $X_i$  by the following conditions

$$(2.2a) \quad X_i \text{ is a function of position coordinates only,}$$

$$(2.2b) \quad C_{jk}^i X_i = 0$$

and

$$(2.2c) \quad X_{i|j} = \rho(g_{ij} - X_i X_j).$$

The scalar function  $\rho$  in (2.2c) was assumed by U.P. Singh and B.N. Prasad<sup>8</sup> as a scalar point function but U.P. Singh and Bindu Kumari<sup>9</sup> proved that it is actually so.

## 3. On Generalized Normalized Semi-Parallel $h$ -Vector Field in Special Finsler Spaces

Generalizing the concept of normalized semi-parallel vector field to a Finsler space U.P. Singh and Bindu Kumari<sup>9</sup> gave the following:

**Definition 3.1:** A Normalized vector field  $X_i$  in a Finsler space  $F_n$  is said to be a normalized semi-parallel  $h$ -vector field if

$$(3.1a) \quad X_i|_j = 0,$$

$$(3.1b) \quad FC_{jk}^i X_i = \sigma h_{jk}$$

$$(3.1c) \quad X_{i|j} = \rho(g_{ij} - X_i X_j)$$

where

$$(3.1d) \quad h_{jk} = g_{jk} - l_j l_k,$$

and  $\sigma$  and  $\rho$  are non-zero scalar functions of  $x^i$  only. They also assumed that the vector  $X_i$  is positively homogeneous of degree zero in  $y^i$  and dimension  $n$  of Finsler space is greater than two. In terms of contravariant components

$$(3.2) \quad X^i = g^{ij} X_j,$$

the equation (3.1) can be written as

$$(3.3a) \quad X^i|_j = 0$$

$$(3.3b) \quad FC_{ijk} X^i = \sigma h_{jk}$$

$$(3.3c) \quad X_{i|j}^i = \rho(\delta_j^i - X^i X_j).$$

They studied the properties of  $X_i$ ,  $\sigma$  and  $\rho$  in a Finsler space  $F_n (n > 2)$  and the existence of normalized semi-parallel  $h$ -vector field in some special Finsler spaces.

Differentiating (3.1d) covariantly with respect to  $x^s$  in the sense of Cartan and using (1.6a) and (1.6b), we get

$$(3.4) \quad h_{ij|s} = 0.$$

Taking  $h$ -covariant derivative of (3.1b) with respect to  $x^s$  and using (1.6c), (3.1c), (3.4) and (1.5), we get

$$(3.5) \quad FC_{jks}^i X_i + \rho FC_{jk}^i (g_{is} - X_i X_s) = \sigma_s h_{jk}$$

where  $\sigma_{|s} = \partial_s \sigma = \sigma_s$ . Transvecting (3.5) by  $y^s$  and using (1.7), we get

$$(3.6) \quad FP_{jk}^i X_i + \rho FC_{jk}^i y^s (g_{is} - X_i X_s) = \sigma_s h_{jk} y^s.$$

Utilizing (1.1), (1.3) and (3.1b) in (3.6), we get

$$(3.7) \quad P_{jk}^i X_i = \frac{1}{F} y^s h_{jk} (\sigma_s + \rho \sigma X_s).$$

Using (1.4a) in (3.7), we get  $P_{jk}^i X_i = l^s (\sigma_s + \rho \sigma X_s) h_{jk}$  or

$$(3.8) \quad P_{jk}^i X_i = \mu h_{jk}$$

where  $\mu = l^s (\sigma_s + \rho \sigma X_s)$ . Thus, we see that a normalized semi-parallel  $h$ -vector field characterized by (3.1) satisfies (3.8). However, a vector field  $X_i$  satisfying the condition (3.8), (3.1a) and (3.1c) need not be a normalized semi-parallel  $h$ -vector field. In view of these facts, we wish to generalize the concept of a normalized semi-parallel  $h$ -vector field and give the following:

**Definition 3.2:** A normalized vector field  $X_i$  in a Finsler space  $F_n$  is said to be a generalized normalized semi-parallel  $h$ -vector field if

$$(3.9a) \quad X_i|_j = 0,$$

$$(3.9b) \quad P_{jk}^i X_i = \mu h_{jk}$$

and

$$(3.9c) \quad X_{i|j} = \rho (g_{ij} - X_i X_j)$$

where  $\mu$  and  $\rho$  are non-zero scalar functions, the vector field  $X_i$  is positively homogeneous function of degree zero in  $y^i$  and the dimension of the Finsler space is greater than two. The condition (3.8) may be written as  $P_{jhh} X^j = \mu h_{hk}$ , where  $P_{jhh} = P_{jk}^i g_{hi}$ . Therefore, in terms of contravariant components  $X^i$ , the equations (3.9) can be written as

$$(3.10a) \quad X^i|_j = 0,$$

$$(3.10b) \quad P_{ijk} X^i = \mu h_{jk}$$

and

$$(3.10c) \quad X_{i|j}^i = \rho (\delta_j^i - X^i X_j).$$

Since  $P_{ijk}$ ,  $X^i$ ,  $h_{jk}$  and the scalar function  $\mu$  are positively homogeneous functions of degree zero in  $y^i$ .

**Lemma 3.1:** The angular metric tensor  $h_{ij}$  of a Finsler space  $F_n (n > 2)$  can never be written as a product of two covariant vectors.

**Proof:** Suppose

$$(3.11) \quad h_{ij} = \lambda_i \lambda_j.$$

Transvecting (3.11) by  $g^{hi}$  and using (3.1d), we get

$$(3.12) \quad \delta_j^h - l^h l_j = \lambda^h \lambda_j.$$

Transvecting (3.12) by  $l^j$ , we get  $\lambda^h \lambda_j l^j = 0$ , which implies  $\lambda_j l^j = 0$ , for  $\lambda^h \neq 0$ . Contracting the indices  $h$  and  $j$  in (3.12), we have  $\lambda_j \lambda^j = n-1$ . Transvecting (3.11) by  $\lambda^i$  and using  $\lambda_j \lambda^j = n-1$  and  $\lambda_j l^j = 0$ , we have  $(n-2)\lambda_j = 0$ . Since  $n > 2$ ,  $(n-2)\lambda_j = 0$  implies  $\lambda_j = 0$ . Therefore  $h_{ij} = 0$ , a contradiction. Thus, we have the lemma.

**Lemma 3.2:** *The concept of generalized normalized semi-parallel vector field and normalized semi-parallel vector field coincide in  $P^*$ -Finsler space.*

**Proof:** The  $P^*$ -Finsler space is characterized by  $P_{ijk} = \lambda C_{ijk}$ , where  $\lambda$  is a scalar function. For such space, the equation (3.9b) reduces to

$$FC_{jk}^i X_i = \sigma h_{jk}, \quad \text{where } \sigma = \frac{F\mu}{\lambda}.$$

**Theorem 3.1:** *In a Landsberg space there exists no generalized normalized semi-parallel  $h$ -vector field.*

**Proof:** In a Landsberg space, we have the condition  $P_{ijk} = 0$ . In view of this condition, the condition (3.1b) gives  $\mu h_{jk} = 0$ . This implies  $\mu = 0$  for  $h_{jk} \neq 0$ . But  $\mu$  is assumed to be non-zero. Therefore, we have the theorem.

**Corollary 3.1:** *There exists no generalized normalized semi-parallel  $h$ -vector field in a Berwald space.*

**Proof:** A Finsler space  $F_n$  is said to be a Berwald space if its Berwald connection coefficients  $G_{jk}^i$  are independent of  $y^i$  or equivalently its Cartan's connection coefficients  $\Gamma_{jk}^{*i}$  are independent of  $y^i$ . Since every Berwald space is a Landsberg space and there exists no generalized normalized semi-parallel  $h$ -vector field characterized by (3.10) in a Landsberg space, we have the Corollary 3.1.

**Corollary 3.2:** *In a locally Minkowskian space, there exists no generalized normalized semi-parallel  $h$ -vector field.*

**Proof:** The Finsler space  $F_n$  is locally Minkowskian space if and only if  $R_{ijk}^h = 0$  and  $C_{ijk|h} = 0$ . Hence the Finsler space  $F_n$  is a Berwald space. Therefore, we have the Corollary 3.2.

**Theorem 3.2:** *A  $P$ -reducible Finsler space does not admit a generalized normalized semi-parallel  $h$ -vector field characterized by (3.10).*

**Proof:** A  $P$ -reducible Finsler space of dimension  $n > 2$  is characterized by the condition

$$(3.13) \quad (n+1)P_{ijk} = P_i h_{jk} + P_j h_{ki} + P_k h_{ij}$$

where  $P_i = g^{jk} P_{ijk}$ . Transvecting (3.10b) by  $g^{jk}$  and using (1.4b), we get

$$(3.14) \quad X^i P_i = (n-1)\mu.$$

Transvecting (3.13) by  $X^i$  and using (3.10b) and (3.14), we have

$$(3.15) \quad (n+1)\mu h_{jk} = (n-1)\mu h_{jk} + P_j(X_k - X_0 l_k) + P_k(X_j - X_0 l_j)$$

where  $X_0 = X^i l_i = X_i l^i$ . Equation (3.15) can be written as

$$(3.16) \quad 2\mu h_{jk} = P_j X_k + P_k X_j - X_0(P_j l_k + P_k l_j).$$

Transvecting (3.16) by  $P^j P^k$  and using  $l_j P^j = 0$  and (3.14), we get  $(n-2)\mu P^2 = 0$ .

Since  $n > 2$ ,  $\mu P^2 = 0$ , which implies  $\mu = 0$  or  $P^2 = 0$ .  $P^2 = 0$  means  $P_k P^k = 0$ . Since the metric of our space is positive definite, we have  $P_k = 0$ . This gives a contradiction to the fact that  $P_k \neq 0$ . Therefore,  $P^2 \neq 0$ . Hence  $\mu = 0$ , which is contradicting the fact that  $\mu$  is non-zero scalar function. Hence the theorem.

**Corollary 3.3:** *A C-reducible Finsler space does not admit a generalized normalized semi-parallel h-vector field.*

**Proof:** Suppose that a C-reducible Finsler space admits a generalized normalized semi-parallel h-vector field. Since a C-reducible Finsler space is P-reducible, a P-reducible Finsler space admits a generalized normalized semi-parallel h-vector field. This contradicts theorem 3.2.

We know that a Finsler space with  $(\alpha, \beta)$  metric is C-reducible if and only if the space is Randers space or a Kropina space<sup>11</sup>. Therefore, in view of Corollary (3.3), we have the following:

**Theorem 3.3:** *A Randers or Kropina space does not admit a generalized normalized semi-parallel h-vector field.*

**Theorem 3.4:** *If a semi-P-reducible Finsler space  $F_n$  admits a generalized normalized semi-parallel h-vector field, then  $F_n$  is a semi-P-reducible Finsler space with constant coefficients.*

**Proof:** A non-Riemannian Finsler space is said to be a semi-P-reducible Finsler space if its P-tensor  $P_{ijk}$  is written as

$$(3.17) \quad P_{ijk} = \frac{p}{n+1} (h_{ij} P_k + h_{jk} P_i + h_{ki} P_j) + \frac{q}{p^2} P_i P_j P_k.$$

In general,  $p$  and  $q$  are functions of  $x^i$  and  $y^i$ . If they are constants, then  $F_n$  is called semi-P-reducible Finsler space with constant coefficients. Transvecting (3.17) by  $g^{ij}$  and

using  $P_{ijk}g^{ij} = P_k$ ,  $h_{ij}g^{ij} = n-1$  and  $P^i P_i = P^2$ , we find  $(1-p-q)P_k = 0$ . Since  $P_k \neq 0$ ,  $1-p-q=0$ . Thus,  $p$  and  $q$  appearing in equation (3.17) satisfy  $p+q=1$ . Transvecting (3.17) by  $X^i$  and using (3.10b) and (3.14) which is a direct consequence of (3.10b), we get

$$(3.18) \quad \mu h_{jk} = \frac{P}{n+1} \left\{ h_{ij} P_k X^i + (n-1) \mu h_{jk} + h_{ki} P_j X^i \right\} + \frac{q}{P^2} (n-1) \mu P_j P_k.$$

Transvecting (3.18) by  $P^j P^k$  and using (3.14) and  $P_j P^j = P^2$ , we get

$$\{(n+1) - 3(n-1)p - (n-1)(n+1)q\} \mu P^2 = 0.$$

Since  $\mu \neq 0$  and  $P^2 \neq 0$ , we have

$$(n+1) = (n-1)\{3p + (n+1)q\}.$$

But  $p+q=1$ . Therefore  $p = \frac{n+1}{n-1}$ . This gives  $q = -\frac{2}{n-1}$ . Thus,  $p$  and  $q$  are constants.

This completes the proof.

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