

Generalized Poisson Distribution Series and its Application

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Abstract: The tenacity of the current paper is to attain sufficient conditions for generalized Poisson distribution series in the various subclasses of univalent functions of conic domains. Further, we obtain some inclusion relations of these subclasses with the class $\mathcal{P}_\gamma^r(\beta)$.

Finally, we obtain sufficient conditions for an integral operator which is defined in terms of generalized Poisson distribution series.

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1. Introduction

Let $\mathcal{A}$ represent the class of all analytic functions f defined in $\mathbb{U} = \{z : |z| < 1\}$, with normalized by the condition $f(0) = f'(0) - 1 = 0$. Further, $\mathcal{S}$ represent the subclass of $\mathcal{A}$ consisting of the functions that are

univalent in $\mathbb{U}$. We can write the functions of the class $\mathcal{S}$ by the power series expansion about the origin

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

A function f of the form (1.1) belongs to the class of k -uniformly convex functions of order α represented by $k-\mathcal{UCV}(\alpha)$, iff, the following condition holds

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq k \left| \frac{zf''(z)}{f'(z)} \right| + \alpha,$$

where $0 \leq k < \infty, 0 \leq \alpha < 1$.

By applying the Alexander transform, we define the corresponding class of k -starlike functions of order α represented by $k-\mathcal{S}_p(\alpha)$ as

$$f(z) \in k-\mathcal{UCV}(\alpha) \Leftrightarrow zf'(z) \in k-\mathcal{S}_p(\alpha).$$

The classes $k-\mathcal{UCV}(\alpha)$ and $k-\mathcal{S}_p(\alpha)$ were initially investigated and studied in¹.

The study of class $k-\mathcal{UCV}(\alpha)$ is of special interest because it generalizes several classes of univalent functions.

1. $k-\mathcal{UCV}(0) \equiv k-\mathcal{UCV}$ studied by Kanas and Wisniowska², (see also^{3,4}).
2. $k-\mathcal{S}_p(0) \equiv k-\mathcal{S}_p$ studied by Kanas and Wisniowska⁵, (see also^{3,4}).
3. $1-\mathcal{UCV}(0) \equiv \mathcal{UCV}$ studied by Goodman⁶.
4. $1-\mathcal{S}_p(0) \equiv \mathcal{S}_p$ studied by Goodman⁷.
5. $0-\mathcal{UCV}(\alpha) \equiv \mathcal{C}(\alpha)$ and $0-\mathcal{S}_p(\alpha) \equiv \mathcal{S}^*(\alpha)$ studied by Robertson⁸ and Silverman⁹.
6. $0-\mathcal{UCV}(0) \equiv \mathcal{C}(0)$ and $0-\mathcal{S}_p(0) \equiv \mathcal{S}^*$ studied by Robertson⁸ and Silverman⁹.

In¹⁰, the classes $\mathcal{S}_\lambda^*$ and $\mathcal{C}_\lambda$ are introduced as follows

$$\mathcal{S}_\lambda^* = \left\{ f \in A : \left| \frac{zf'(z)}{f(z)} \right| < \lambda \right\},$$

where $\lambda > 0$,

and

$$\mathcal{C}_\lambda = \left\{ f \in A : \frac{zf''(z)}{f'(z)} < \lambda \right\},$$

where $\lambda > 0$.

From the definition of $\mathcal{S}_\lambda^*$ and $\mathcal{C}_\lambda$ it is worthy to note that

$$f(z) \in \mathcal{C}_\lambda \Leftrightarrow zf'(z) \in \mathcal{S}_\lambda^*.$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{P}_\gamma^\tau(\beta)$ if it satisfies the following condition

$$\left| \frac{(1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - 1}{2\tau(1-\beta) + (1-\gamma)\frac{f(z)}{z} + \gamma f'(z) - 1} \right| < 1,$$

where $0 \leq \gamma < 1, \beta < 1, \tau \in \mathbb{C} \setminus \{0\}$ and $z \in \mathbb{U}$. The class $\mathcal{P}_\gamma^\tau(\beta)$ was investigated in¹¹. The study of Poisson distribution series is a current and interesting topic of research in G.F.T. In fact, by using Poisson distribution series, Porwal¹² introduced a new power series which is named as the Poisson distribution series and gave a beautiful application of it on univalent function. This paper unlocks a new and fascinating direction of research in analytic and harmonic univalent functions. After the appearance of this article various authors introduced hypergeometric distribution series¹³, hypergeometric-type distribution series¹⁴, confluent hypergeometric distribution series¹⁵, Binomial distribution series¹⁶, Pascal distribution series¹⁷, generalized distribution series¹⁸, Mittag-Leffler type Poisson distribution series¹⁹, Borel distribution series²⁰ and obtained some necessary and sufficient conditions on univalent functions.

Motivated by the aforementioned work, we find the necessary and sufficient condition for the generalized Poisson distribution series belonging to the classes $k-\mathcal{UCV}(\alpha)$, $k-\mathcal{S}_p(\alpha)$, $\mathcal{S}_\lambda^*$ and $\mathcal{C}_\lambda$. Finally, we attain some inclusion relations between these classes with $\mathcal{P}_\gamma^r(\beta)$.

Now, we first introduce generalized Poisson distribution series for this purpose we will miss the description of Poisson distribution series which was first introduced by Porwal¹² and subsequently studied by Altinkaya and Yalcin²¹, Frasin and Gharaibeh²², Magesh *et al.*²³ and Murugusundaramoorthy *et al.*²⁴.

The Poisson distribution series is defined by

$$(1.2) \quad k(m, z) = z + \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} e^{-m} z^n.$$

Now, we define

$$k(\mu, m, z) = (1-\mu)k(m, z) + \mu z k'(m, z).$$

Hence

$$(1.3) \quad k(\mu, m, z) = z + \sum_{n=2}^{\infty} (1-\mu + \mu n) \frac{m^{n-1}}{(n-1)!} e^{-m} z^n.$$

It is easy to verify that for $\mu = 0$ the series (1.3) reduce to the series (1.2).

The convolution of two functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=0}^{\infty} b_n z^n,$$

is given as the power series $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$.

Now, we introduce a linear operator $\Omega(\mu, m): \mathcal{A} \rightarrow \mathcal{A}$ defined by

$$(1.4) \quad \Omega(\mu, m) f = k(\mu, m, z) * f(z)$$

$$= z + \sum_{n=2}^{\infty} (1 - \mu + \mu n) \frac{m^{n-1}}{(n-1)!} e^{-m} a_n z^n.$$

2. Preliminary Results

The following lemmas are used to prove our main results.

Lemma¹ 2.1: A function $f \in \mathcal{A}$ is said to be in the class $k - \mathcal{UCV}(\alpha)$, if it holds the condition

$$(2.1) \quad \sum_{n=2}^{\infty} n [n(1+k) - (k+\alpha)] |a_n| \leq 1 - \alpha.$$

Lemma¹ 2.2: A function $f \in \mathcal{A}$ is said to be in the class $k - \mathcal{S}_p(\alpha)$, if it holds the following inequality

$$(2.2) \quad \sum_{n=2}^{\infty} [n(1+k) - (k+\alpha)] |a_n| \leq 1 - \alpha.$$

Remark 2.1: The functions of the form

$$(2.3) \quad f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n,$$

are also necessary for conditions (2.1) and (2.2).

Lemma² 2.3: A function $f \in \mathcal{A}$ be of the form (1.1). It for some $k(0 \leq k < \infty)$, the condition

$$\sum_{n=2}^{\infty} n(n-1) |a_n| \leq \frac{1}{k+2},$$

satisfies, then $f \in k - \mathcal{UCV}$. The result is sharp.

Lemma¹⁰ 2.4: Let $f \in \mathcal{A}$ be of the form (1.1). If

$$(2.4) \quad \sum_{n=2}^{\infty} (\lambda + n - 1) |a_n| \leq \lambda, \quad (\lambda > 0),$$

then $f \in \mathcal{S}_\lambda^*$.

Lemma¹⁰ 2.5: Let $f \in \mathcal{A}$ be of the form (1.1). If

$$(2.5) \quad \sum_{n=2}^{\infty} n(\lambda + n - 1) |a_n| \leq \lambda, \quad (\lambda > 0),$$

then $f \in \mathcal{C}_\lambda$.

Remark 2. It should be worthy to note that the function $f(z)$ is of the form (2.3) however also requires conditions (2.4) and (2.5).

Lemma¹¹ 2.6: If $f \in \mathcal{P}_\gamma^\tau(\beta)$ is of the form (1.1) then

$$|a_n| \leq \frac{2|\tau|(1-\beta)}{1+\gamma(n-1)}.$$

3. Main Results

In our first theorem, we find a sufficient condition for the series (1.3) belonging to the class $k - \mathcal{UCV}(\alpha)$.

Theorem 3.1: If $m > 0$, $0 \leq \alpha < 1$, $0 \leq \mu \leq 1$ and the inequality

$$(3.1) \quad \begin{aligned} & e^m [\mu(k+1)m^3 + \{(k+1)(1+5\mu) - \mu(k+\alpha)\}m^2 \\ & + \{2(k+1)(1+2\mu) - 2\mu(k+\alpha) + 1 - \alpha\}m] \leq 1 - \alpha \end{aligned}$$

is satisfied then $k(\mu, m, z) \in k - \mathcal{UCV}(\alpha)$.

Proof: To prove that $k(\mu, m, z) \in k - \mathcal{UCV}(\alpha)$ we must have to prove that by Lemma 2.1

$$\sum_{n=2}^{\infty} n \{n(k+1) - (k+\alpha)\} \frac{(\mu n + 1 - \mu)m^{n-1}}{(n-1)!} e^{-m} \leq 1 - \alpha.$$

Now

$$\begin{aligned}
& \sum_{n=2}^{\infty} n \{n(k+1) - (k+\alpha)\} (\mu n + 1 - \mu) \frac{m^{n-1}}{(n-1)!} e^{-m} \\
&= e^{-m} \left[\sum_{n=2}^{\infty} \{ \mu(k+1)(n-1)(n-2)(n-3) \right. \\
&\quad + \{ (k+1)(1+5\mu) - \mu(k+\alpha) \} (n-1)(n-2) \\
&\quad + \{ 2(k+1)(1+2\mu) - 2\mu(k+\alpha) + 1 - \alpha \} (n-1) \\
&\quad \left. + 1 - \alpha \} \frac{m^{n-1}}{(n-1)!} \right] = e^{-m} \sum_{n=2}^{\infty} \{ \mu(k+1) \frac{m^{n-1}}{(n-4)!} \\
&\quad + \{ (k+1)(1+5\mu) - \mu(k+\alpha) \} \frac{m^{n-1}}{(n-3)!} \\
&\quad + \{ 2(k+1)(1+2\mu) - 2\mu(k+\alpha) + 1 - \alpha \} \frac{m^{n-1}}{(n-2)!} \\
&\quad + (1-\alpha) \frac{m^{n-1}}{(n-1)!} \} \\
&= e^{-m} \left[\mu(k+1)m^3e^m + \{ (k+1)(1+5\mu) - \mu(k+\alpha) \} m^2e^m \right. \\
&\quad + \{ 2(k+1)(1+2\mu) - 2\mu(k+\alpha) + 1 - \alpha \} me^m \\
&\quad \left. + (1-\alpha)(e^m - 1) \right]
\end{aligned}$$

$\leq 1 - \alpha$ by the giving premise.

Thus, the proof is complete.

Theorem 3.2: If $m > 0$, $0 \leq \alpha < 1$, $0 \leq \mu \leq 1$ and the inequality

$$(3.2) \quad e^m \left[\mu(k+1)m^2 + \{ (k+1)(1+2\mu) - \mu(k+\alpha) \} m \right] \leq 1 - \alpha$$

is satisfied then $k(\mu, m, z) \in k - \mathcal{S}_p(\alpha)$.

Proof: To prove that $k(\mu, m, z) \in k - \mathcal{S}_p(\alpha)$, from Lemma 2.2 it is sufficient to prove that

$$\sum_{n=2}^{\infty} \{n(k+1) - (k+\alpha)\} (1 - \mu + \mu n) \frac{m^{n-1}}{(n-1)!} e^{-m} \leq 1 - \alpha.$$

Now

$$\begin{aligned}
 & \sum_{n=2}^{\infty} \{n(k+1) - (k+\alpha)\} (1-\mu + \mu n) \frac{m^{n-1}}{(n-1)!} e^{-m} \\
 &= e^{-m} \sum_{n=2}^{\infty} \{ \mu(k+1)(n-1)(n-2) + \{(k+1)(1+2\mu) - \mu(k+\alpha)\} \\
 & \quad (n-1) + (1-\alpha) \} \frac{m^{n-1}}{(n-1)!} \\
 &= e^{-m} \left[\frac{\mu(k+1)m^2 e^m + \{(k+1)(1+2\mu) - \mu(k+\alpha)\} e^m}{me^m + (1-\alpha)(e^m - 1)} \right] \leq 1-\alpha
 \end{aligned}$$

which is true by the assumption.

Theorem 3.3: If $m > 0$, $0 \leq \mu \leq 1$ and the inequality $\mu m^3 + (1+4\mu)m^2 + 2(1+\mu)m \leq \frac{1}{k+2}$ is satisfied then $k(\mu, m, z) \in k - \mathcal{UCV}$.

Proof: To prove $k(\mu, m, z) \in k - \mathcal{UCV}$, from Lemma 2.3, it is enough to prove that

$$\sum_{n=2}^{\infty} n(n-1)(1-\mu + \mu n) \frac{m^{n-1}}{(n-1)!} e^{-m} \leq \frac{1}{k+2}.$$

Now

$$\begin{aligned}
 & \sum_{n=2}^{\infty} n(n-1)(1-\mu + \mu n) \frac{m^{n-1}}{(n-1)!} e^{-m} \\
 &= e^{-m} \sum_{n=2}^{\infty} n(1-\mu + \mu n) \frac{m^{n-1}}{(n-2)!} \\
 &= e^{-m} \left\{ \sum_{n=2}^{\infty} \mu(n-2)(n-3) + (1+4\mu)(n-2) + 2(1+\mu) \right\} \frac{m^{n-1}}{(n-2)!} \\
 &= e^{-m} \left[\mu \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-4)!} + (1+4\mu) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-3)!} + 2(1+\mu) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} \right]
 \end{aligned}$$

$$\begin{aligned}
&= e^{-m} \left[\mu m^3 e^m + (1+4\mu) m^2 e^m + 2(1+\mu) m e^m \right] \\
&= \mu m^3 + (1+4\mu) m^2 + 2(1+\mu) m \leq \frac{1}{k+2}.
\end{aligned}$$

which is true by the assumption.

Theorem 3.4: If $m > 0$, $\lambda > 0$, $0 \leq \mu \leq 1$ and the inequality

$$(3.3) \quad e^m \left[\mu m^2 + (\mu\lambda + 1 + \mu) m \right] \leq \lambda$$

is satisfied then $k(\mu, m, z) \in \mathcal{S}_\lambda^*$.

Proof: To show that $k(\mu, m, z) \in \mathcal{S}_\lambda^*$ from Lemma 2.4, it is enough to show that

$$\sum_{n=2}^{\infty} (\lambda + n - 1) (1 - \mu + \mu n) \frac{m^{n-1}}{(n-1)!} e^{-m} \leq \lambda.$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty} (\lambda + n - 1) (1 - \mu + \mu n) \frac{m^{n-1}}{(n-1)!} e^{-m} \\
&= e^{-m} \sum_{n=2}^{\infty} \{ \mu(n-1)(n-2) + (\mu\lambda + 1 + \mu)(n-1) + \lambda \} \frac{m^{n-1}}{(n-1)!} \\
&= e^{-m} \sum_{n=2}^{\infty} \left\{ \frac{\mu m^{n-1}}{(n-3)!} + (\mu\lambda + 1 + \mu) \frac{m^{n-1}}{(n-2)!} + \frac{\lambda m^{n-1}}{(n-1)!} \right\} \\
&= e^{-m} \left[\mu m^2 e^m + (\mu\lambda + 1 + \mu) m e^m + \lambda (e^m - 1) \right] \leq \lambda.
\end{aligned}$$

which is true by the assumption.

Theorem 3.5: If $m > 0$, $\lambda > 0$, $0 \leq \mu \leq 1$ and the inequality

$$e^m \left[\mu m^3 + (\lambda\mu + 1 + 4\mu) m^2 + \{ \lambda(1+2\mu) + 2(1+\mu) \} m^2 \right] \leq \lambda$$

is satisfied then $k(\mu, m, z) \in \mathcal{C}_\lambda$.

Proof. The proof of the above theory is similar to the Theorem 3.4, so we leave out the explanation.

4. Inclusion Relations

Theorem 4.1: If $m > 0$, $0 \leq \alpha < 1$, $0 \leq \mu \leq 1$, $f \in \mathcal{P}_\gamma^r(\beta)$, $0 < \gamma \leq 1$, $0 < \gamma \leq 1$, and the inequality $0 \leq \beta < 1$

$$\frac{2|\tau|(1-\beta)}{\gamma} \left[\frac{\mu(k+1)m^2 + \{(k+1)(1+2\mu) - \mu(k+\alpha)\}}{m + (1-\alpha)(1-e^{-m})} \right] \leq 1 - \alpha$$

is satisfied then $\Omega(\mu, m) f(z) \in k - \mathcal{UCV}(\alpha)$.

Since

$$\Omega(\mu, m) f(z) = z + \sum_{n=2}^{\infty} \frac{(1-\mu+\mu n)m^{n-1}}{(n-1)!} e^{-m} a_n z^n.$$

Proof: To prove that $\Omega(\mu, m) f(z) \in k - \mathcal{UCV}(\alpha)$, from Lemma 2.1 it is enough to show that

$$\sum_{n=2}^{\infty} n \{n(k+1) - (k+\alpha)\} (1-\mu+\mu n) \frac{m^{n-1}}{(n-1)!} e^{-m} |a_n| \leq 1 - \alpha.$$

Now, by applying Lemma 2.6 and using result $1 + \gamma(n-1) \geq \gamma n$, we have

$$\begin{aligned} & \sum_{n=2}^{\infty} n [n(k+1) - (k+\alpha)] (1-\mu+\mu n) \frac{m^{n-1}}{(n-1)!} e^{-m} |a_n| \\ & \leq e^{-m} \sum_{n=2}^{\infty} n [n(k+1) - (k+\alpha)] (1-\mu+\mu n) \frac{m^{n-1}}{(n-1)!} \frac{2|\tau|(1-\beta)}{1+\gamma(n-1)} \\ & \leq \frac{2|\tau|(1-\beta)e^{-m}}{\gamma} \sum_{n=2}^{\infty} [n(k+1) - (k+\alpha)] (1-\mu+\mu n) \frac{m^{n-1}}{(n-1)!} \\ & \leq \frac{2|\tau|(1-\beta)e^{-m}}{\gamma} \sum_{n=2}^{\infty} [\mu(k+1)(n-1)(n-2)] \end{aligned}$$

$$\begin{aligned}
& + \left\{ (k+1)(1+2\mu) - \mu(k+\alpha) \right\} (n-1) + (1-\alpha) \Big] \frac{m^{n-1}}{(n-1)!} \\
& = \frac{2|\tau|(1-\beta)e^{-m}}{\gamma} \left\{ \mu(k+1)m^2e^m + \left[(k+1)(1+2\mu) - \mu(k+\alpha) \right] \right. \\
& \quad \left. me^m + (1-\alpha)(e^m - 1) \right\} \leq 1 - \alpha.
\end{aligned}$$

which is true by the assumption.

Theorem 4.2: If $m > 0$, $0 \leq \alpha < 1$, $0 \leq \mu \leq 1$, $f \in \mathcal{P}_\gamma^r(\beta)$, $0 \leq \gamma \leq 1$, $0 \leq \beta \leq 1$ and the inequality

$$\begin{aligned}
& \frac{2|\tau|(1-\beta)}{\gamma} \left[\mu(k+1)m + \left\{ (k+1) - \mu(k+\alpha) \right\} (1 - e^{-m}) \right. \\
& \quad \left. - \frac{(1-\mu)(k+\alpha)}{m} (1 - e^{-m} - me^{-m}) \right] \leq 1 - \alpha.
\end{aligned}$$

is satisfied then $\Omega(\mu, m) f(z) \in k - \mathcal{S}_p(\alpha)$.

Proof: Evidence of the above theory is very similar to that of Theorem 4.1, so we leave out the relevant details.

Theorem 4.3: If $m > 0$, $0 \leq \mu \leq 1$, $f \in \mathcal{P}_\gamma^r(\beta)$, $0 < \gamma \leq 1$, $0 \leq \beta < 1$ and the inequality

$$\frac{2|\tau|(1-\beta)}{\gamma} \left\{ \mu m^2 + (1+\mu)m \right\} \leq \frac{1}{k+2}$$

is satisfied then $\Omega(\mu, m) f \in k - \mathcal{UCV}$.

Proof: To prove that $\Omega(\mu, m) f \in k - \mathcal{UCV}$ from Lemma 2.3 it is sufficient to prove that

$$\sum_{n=2}^{\infty} n(n-1)(1-\mu+\mu n) \frac{m^{n-1}}{(n-1)!} e^{-m} |a_n| \leq \frac{1}{k+2}.$$

Now

$$\begin{aligned}
 & \sum_{n=2}^{\infty} n(n-1)(1-\mu+\mu n) \frac{m^{n-1}}{(n-1)!} e^{-m} |a_n| \\
 & \leq \frac{2|\tau|(1-\beta)e^{-m}}{\gamma} \sum_{n=2}^{\infty} (1-\mu+\mu n) \frac{m^{n-1}}{(n-2)!} \\
 & = \frac{2|\tau|(1-\beta)e^{-m}}{\gamma} \sum_{n=2}^{\infty} \{\mu(n-2)+1+\mu\} \frac{m^{n-1}}{(n-2)!} \\
 & = \frac{2|\tau|(1-\beta)e^{-m}}{\gamma} \sum_{n=2}^{\infty} \left\{ \frac{\mu m^{n-1}}{(n-3)!} + (1+\mu) \frac{m^{n-1}}{(n-2)!} \right\} \\
 & = \frac{2|\tau|(1-\beta)e^{-m}}{\gamma} \{\mu m^2 e^m + (1+\mu)m e^m\} = \frac{2|\tau|(1-\beta)}{\gamma} \{\mu m^2 + (1+\mu)m\} \\
 & \leq \frac{1}{k+2}.
 \end{aligned}$$

which is true by the assumption.

The proof of Theorems 4.4-4.5 are much akin to the proof of Theorem 4.1. Therefore, we only state the results.

Theorem 4.4: If $m > 0$, $\lambda > 0$, $f \in \mathcal{P}_{\gamma}^{\tau}(\beta)$, $0 < \gamma \leq 1$, $0 \leq \beta < 1$ and the inequality

$$\frac{2|\tau|(1-\beta)}{\gamma} \{\mu m^2 + (\mu\lambda + 1 + \mu)m + \lambda(1 - e^{-m})\} \leq \lambda$$

is satisfied then $\Omega(\mu, m) f \in \mathcal{C}_{\lambda}$.

Theorem 4.5: If $m > 0$, $\lambda > 0$, $f \in \mathcal{P}_{\gamma}^{\tau}(\beta)$, $0 < \gamma \leq 1$, $0 \leq \beta < 1$ and the inequality

$$\frac{2|\tau|(1-\beta)}{\gamma} \left[\mu m + (\mu\lambda + 1 + \mu)(1 - e^{-m}) + \frac{(\lambda-1)(1-\mu)}{m} (1 - e^{-m} - m e^{-m}) \right] \leq \lambda$$

is satisfied then $\Omega(\mu, m) f \in \mathcal{S}_{\lambda}^*$.

5. A Class Preserving Integral Operator

In this section, we find some sufficient conditions with a particular integral operator $G(\mu, m, z)$ which is given as

$$G(\mu, m, z) = \int_0^z \frac{k(\mu, m, t)}{t} dt.$$

Theorem 5.1: If $m > 0$, $0 \leq \alpha < 1$, $0 \leq \mu \leq 1$ and the condition (3.2) is then $G(\mu, m, z) \in k - \mathcal{UCV}(\alpha)$.

Proof: To prove that $G(\mu, m, z) \in k - \mathcal{UCV}(\alpha)$, from Lemma 2.1, it is enough to show that

$$\sum_{n=2}^{\infty} n \{n(k+1) - (k+\alpha)\} (1-\mu+\mu n) \frac{m^{n-1}}{(n-1)!} e^{-m} \frac{1}{n} \leq 1-\alpha$$

Now

$$\begin{aligned} & \sum_{n=2}^{\infty} n \{n(k+1) - (k+\alpha)\} (1-\mu+\mu n) \frac{m^{n-1}}{n!} e^{-m} \\ &= e^{-m} \sum_{n=2}^{\infty} \{n(k+1) - (k+\alpha)\} (1-\mu+\mu n) \frac{m^{n-1}}{(n-1)!} \\ &= e^{-m} \sum_{n=2}^{\infty} \{\mu(k+1)(n-1)(n-2) + \{(k+1)(1+2\mu) - \mu(k+\alpha)\} \\ & \quad (n-1) + (1-\alpha)\} \frac{m^{n-1}}{(n-1)!} = e^{-m} \left[\mu(k+1) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-3)!} \right. \\ & \quad \left. + \{(k+1)(1+2\mu) - \mu(k+2)\} \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-2)!} + (1-\alpha) \sum_{n=2}^{\infty} \frac{m^{n-1}}{(n-1)!} \right] \\ &= e^{-m} \left[\mu(k+1) m^2 e^m + \{(k+1)(1+2\mu) - \mu(k+2)\} \right. \\ & \quad \left. m e^m + (1-\alpha)(e^m - 1) \right] \leq 1-\alpha \end{aligned}$$

which is true by the assumption.

Theorem 5.2: If $m > 0$, $0 \leq \alpha < 1$, $0 \leq \mu \leq 1$ and the inequality

$$\left[\mu(k+1)m + \{(k+1) - \mu(k+\alpha)\}(1-e^{-m}) - \frac{(1-\mu)(k+\alpha)}{m}(1-e^{-m} - me^{-m}) \right] \leq 1-\alpha,$$

is satisfied then $G(\mu, m, z) \in k - \mathcal{S}_p(\alpha)$.

Proof: Evidence of the above theory is very similar to that of Theorem 5.1. Therefore, we leave out the relevant details.

Theorem 5.3: If $m > 0$, $0 \leq \mu \leq 1$ and the condition

$$\mu m^2 + (1+\mu)m \leq \frac{1}{k+2}$$

is satisfied then $G(\mu, m, z) \in k - \mathcal{UCV}$

Proof: To prove that $G(\mu, m, z) \in k - \mathcal{UCV}$ from Lemma 2.3, is enough to show that

$$\sum_{n=2}^{\infty} n(n-1)(1-\mu+\mu n) \frac{m^{n-1}}{n!} e^{-m} \leq \frac{1}{k+2}.$$

Now

$$\begin{aligned} & \sum_{n=2}^{\infty} n(n-1)(1-\mu+\mu n) \frac{m^{n-1}}{n!} e^{-m} \\ &= e^{-m} \sum_{n=2}^{\infty} (\mu n + 1 - \mu) \frac{m^{n-1}}{(n-2)!} \\ &= e^{-m} \sum_{n=2}^{\infty} \{\mu(n-2) + 1 + \mu\} \frac{m^{n-1}}{(n-2)!} \\ &= e^{-m} \left[\mu m^2 e^m + (1+\mu)m e^m \right] \leq \frac{1}{k+2}. \end{aligned}$$

which is true by the assumption.

Theorem compliance evidence is very similar to Theorem 5.1, so we only set out the results.

Theorem 5.4: If $m > 0$, $\lambda \geq 0$, $0 \leq \mu \leq 1$ and the condition (3.3) is hold then $G(\mu, m, z) \in \mathcal{C}_\lambda$.

Theorem 5.5: If $m > 0$, $\lambda \geq 0$, $0 \leq \mu \leq 1$ and the inequality

$$\mu m + (\mu \lambda + 1 - \mu)(1 - e^{-m}) + \frac{(\lambda - 1)(1 - \mu)}{m}(1 - e^{-m} - me^{-m}) \leq \lambda$$

is hold then $G(\mu, m, z) \in \mathcal{S}_\lambda^*$.

Remark 5.1: If we substitute $\mu = 0$ in Theorems 3.1-5.5, then we get the corresponding results of Srivastava and Porwal²⁵.

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