

Lifts on Contangent Bundle

J.P. Srivastava and Rakesh Kumar

P.G. Department of Mathematics, University of Jammu, Jammu, India

(Received April 8, 2002)

Abstract: The present paper deals with the lifts of almost complex structure and almost product structure on cotangent bundle.

1. Introduction

Let M^n be an n -dimensional differentiable manifold of class C^∞ . Let $\{U\}$ be its coordinate neighbourhood with coordinates $\{x^i\}$ and ${}^cT(M^n)$ be its cotangent space. Clearly, ${}^cT(M^n)$ is the space involving covariant vectors in M^n . Let $\pi: {}^cT(M^n) \rightarrow M^n$ be the projection map such that $\pi^{-1}(x^i) = (x^i, p_i)$, where x^i is a point in M^n and p is a 1-form on M^n such that p_i is the i^{th} component of p at the point x^i . ${}^cT(M^n)$ is a $2n$ -dimensional manifold.

Let U and U' be two coordinate neighbourhoods in M^n and their corresponding neighbourhoods in ${}^cT(M^n)$ be $\pi^{-1}(U)$ and $\pi^{-1}(U')$ respectively. If $U \cap U' \neq \emptyset$, then $\pi^{-1}(U) \cap \pi^{-1}(U')$ is also non-empty. Also, we denote the coordinates in ${}^cT(M^n)$ by $(x^i, x^{\bar{i}})$ such that $x^{\bar{i}} = p_i$ [Dombroski¹].

Let the transformation of coordinates in M^n be $x^{i'} = x^{i'}(x^j)$, where the indices j and i' vary over the range 1 to n . Then the corresponding law of transformation in ${}^cT(M^n)$ is

$$(1.1) \quad x^{i'} = x^{i'}(x^j), \quad p_{i'} = \left(\frac{\partial x^i}{\partial x^{i'}} \right) p_i.$$

Then (1.1) is called the extension of the transformation $x^j \rightarrow x^{i'}$ in M^n to ${}^cT(M^n)$. If we write $p_i = x^{\bar{i}} = x^{n+i}$, then equation (1.1) can also be written as $x^{A'} = x^{A'}(x^B)$, where $A, B, A', B', \dots$ run over the range 1 to $2n$.

Let V be an infinitesimal transformation of M^n . The vector field V with component $(V, \bar{V})$, where $\bar{V} = p_i$, is in ${}^cT(M^n)$. This extension is called the complete lift of the vector field V on M^n .

The 1-form on ${}^cT(M^n)$ with components $(p_1, p_2, \dots, p_n, 0, 0, \dots, 0)$ is called the basic 1-form on ${}^cT(M^n)$, where $p = p_i dx^i$ is a 1-form on M^n . Let L_V denotes the Lie differentiation with respect to the vector field V . It is clear that

$$(1.2) \quad L_V pS = V^E \partial_E pS + p_E \partial_S V^E = -p_m \partial_S V^m + p_m \partial_S V^m = 0.$$

Consider a tensor field F of type (1,1) defined in ${}^cT(M^n)$ and having the local expression as follows:

$$(p_a F_{h'}^{a'}) \frac{\partial X^{h'}}{\partial x^a} = p_a F_i^a$$

or

$$(1.3) \quad (p_a F_{h'}^{a'}) \begin{pmatrix} \frac{\partial x^{h'}}{\partial x^i} & 0 \\ \frac{\partial x^{i'}}{\partial x^i} \frac{\partial^2 x^h}{\partial x^{i'} \partial x^{h'}} p^h & \frac{\partial x^i}{\partial x^{h'}} \end{pmatrix} = (p_a F_i^a, 0),$$

which shows that the local covector field, which has components $(p_a F_i^a, 0)$ with respect to induced coordinates in each $\pi^{-1}(U)$ determines globally in ${}^cT(M^n)$ a covector field. This covector field is denoted by γF , which has components

$$(1.4) \quad \gamma F.(p_a F_i^a, 0),$$

with respect to induced coordinates (x^h, p_i) in $\pi^{-1}(U)$, F_i^h being the components of F relative to U .

The Component of F^c in $\pi^{-1}(U)$ are given by

$$(1.5) \quad \tilde{F}_B^A = \begin{pmatrix} F_i^h & 0 \\ p_a (\partial_i F_h^a - \partial_h F_i^a) & F_n^i \end{pmatrix},$$

with respect to induced coordinates (X^h, p_i) in $\pi^{-1}(U)$, we call F^c , the complete lift of the tensor field F of type (1,1) in M^n .

If $X, Y \in \mathfrak{S}_0^1(M^n)$, $\omega, \theta \in \mathfrak{S}_1^0(M^n)$ and $F, G \in \mathfrak{S}_1^1(M^n)$, then

$$(1.6) \quad \begin{cases} [X^c, \omega^V] &= (L_v \omega)^V, \\ [X^c, \gamma F] &= \gamma(L_X F), \\ F^c X^c &= (FX)^c + \gamma(L_X F). \end{cases}$$

For a vector field $\tilde{X}$ and tensor field $\tilde{F}$ of type (1,1) given in ${}^cT(M^n)$, the Lie derivative $L_{\tilde{X}}\tilde{F}$ of $\tilde{F}$ with respect to $\tilde{X}$ is defined by

$$(1.7) \quad (L_{\tilde{X}}\tilde{F})\tilde{Y} = [\tilde{X}, \tilde{F}\tilde{Y}] - \tilde{F}[\tilde{X}, \tilde{Y}],$$

where $[X, Y]$ is the Lie-bracket of X and Y .

2. Almost Complex Structure of Cotangent Bundle

The construction of lifts of tensor field gives the existence of certain types of structure in ${}^cT(M^n)$.

Definition: A vector field X is said to be contravariant almost analytic if it satisfies

$$(2.1) \quad L_X F = 0.$$

Lemma: For an almost complex manifold the Lie product $[X^c, \gamma F]$ vanishes and $F^c X^c = (FX)^c$ and $L_X \tilde{F}_B^A = 0$.

Definition: A 1-form p is said to be the stable, if it satisfies

$$(2.2) \quad L_X P_S = 0,$$

for an almost complex manifold.

The vertical lift of the 1-form ω in M^n is denoted by ω^v . The ω^v has components of the form

$$(2.3) \quad \omega^v = \begin{pmatrix} 0 \\ \omega \end{pmatrix},$$

with respect to the induced co-ordinates in ${}^cT(M^n)$. Then we have

$$(2.4) \quad [\omega^v, \theta^v] = 0,$$

where $\omega, \theta \in \mathfrak{S}_1^0(M^n)$.

Lemma: The basic 1-form is Lie stable if

$$(2.5) \quad X(\omega(y)) = \omega([x, y]) \quad \text{or} \quad [X^c, \omega^v] = 0.$$

Proof is obvious.

The torsion of two tensor field F, G of type $(1,1)$ plays an important role. If $F, G \in \mathfrak{S}_1^1(M^n)$, the torsion $N_{F,G}$ of F, G is the tensor field $N_{F,G}$ of type $(1,2)$ defined by

$$(2.6) \quad \begin{aligned} 2N_{F,G}(X, Y) = & [FX, GY] + [GX, FY] + FG[X, Y] + GF[X, Y] \\ & - F[X, GY] - F[GX, Y] - G[X, FY] - G[FX, Y]. \end{aligned}$$

If $F = G$, we obtain

$$(2.7) \quad N_{F,F}(X, Y) = [FX, FY] + F^2[X, Y] - F[X, FY] - F[FX, Y].$$

Let $F \in \mathfrak{S}_1^1(M^n)$ and $F^2 = a^2 I$, where I is the Identity tensor field. If $a = i$, then F is an almost complex structure on M^n . If $F^3 + F = 0$, then F is called an almost F -structure on M^n . It is well known that F is integrable if and only if $N_{F,F} = 0$.

Theorem 2.1: *If $F, G \in \mathfrak{S}_1^1(M^n)$, then*

$$(2.8) \quad F^c G^c + G^c F^c = (FG + GF)^c + Q^V,$$

where

$$(2.9) \quad Q = (L_{GX} F) + (L_X G)F + (L_{FX} G) + (L_X F)G - L_X(GF) - L_X(FG).$$

Proof: We have

$$(2.10) \quad \begin{aligned} QY = & [GX, FY] - F[GX, Y] + [X, GFY] - G[X, FY] - [X, FGY] + FG[X, Y] \\ & + [FX, GY] - G[FX, Y] + [X, FGY] - F[X, GY] - [X, GFY] + GF[X, Y], \end{aligned}$$

for any $Y \in \mathfrak{S}_0^1(M^n)$ from which it follows that

$$QY = 2N_{F,G}(X, Y).$$

But we know

$$N^V X^c = (N_X)^c.$$

So it becomes

$$2N_X(Y) = 2N(X, Y) = QY.$$

Hence $Q^V = 2N^V X^c$. Since the action of $F^c G^c + G^c F^c$ and $(FG + GF)^c + 2N^V$ on X^c are same. So, we have:

Theorem 2.2: *Let $F, G \in \mathfrak{S}_1^1(M^n)$. Then*

$$(2.11) \quad F^c G^c + G^c F^c = (FG + GF)^c + (2N)^V.$$

If we replace F by G in the above theorem, then we get

$$(2.12) \quad (F^c)^2 = (F^2) + (N_{F,F})^V.$$

Theorem 2.3: *The almost complex structure $F^C + \frac{1}{2}(NF)^\nu$ on ${}^cT(M^n)$ is integrable if and only if F is integrable.*

Proof: If F is integrable, then $N = 0$ and so

$$F^C + \frac{1}{2}(NF)^\nu = F^C$$

is also integrable.

Conversely, suppose that $F^C + \frac{1}{2}(NF)^\nu$ is integrable. Then the Nijenhuis tensor of $F^C + \frac{1}{2}(NF)^\nu$ in ${}^cT(M^n)$ is zero. We can show that Nijenhuis tensor $\tilde{N}$ of $F^C + \frac{1}{2}(NF)^\nu$ satisfies

$$(2.13) \quad \tilde{N}(X^C, Y^C) = \{N(X, Y)\}^C + P^\nu,$$

where P is the tensor field of type (1,1) in M^n given by

$$\begin{aligned} 2P(Z) = & N(Y, [X, Z]) - N(X, [Y, Z]) + N(X, F[FY, Z]) - N(Y, F[FX, Z]) \\ & + N([FY, X], FZ) - N([FX, Y], FZ) + [Y, N(X, Z)] - [X, N(Y, Z)] \\ & + F[FY, N(X, Z)] - (F[FX, N(Y, Z)]) + \frac{1}{2}N(Y, N(X, Z)) - \frac{1}{2}N(X, N(Y, Z)). \end{aligned}$$

Since $\tilde{N}$ is zero, we get

$$\{N(X, Y)\}^C + P^\nu = 0.$$

But this shows that the vector $\{N(X, Y)\}^C$ is vertical. Since the complete lift of a non-zero vector field can not be vertical, it follows that $N(X, Y) = 0$. This holds for all $X, Y \in \mathfrak{X}_0^1(M^n)$ and so $N = 0$. Hence F is integrable.

It is important to note that the expression for $2P(Z)$ is not linear in X and Y . If we write $Q(X, Y, Z)$ for $p(Z)$, we find that

$$(2.14) \quad Q(fX, gY, Z) = fgQ(X, Y, Z) + (f(Zg) + g(Zf))N(X, Y).$$

Theorem 2.4: *Let M^n be a manifold with almost complex structure F . Then the Nijenhuis tensor $N_{F,F}$ in cotangent bundle ${}^cT(M^n)$ satisfy the following identity:*

$$(2.15) \quad T(FX, FY) - F(T(FX, Y)) - F(T(X, FY)) - T(X, Y) = -\frac{1}{2}N(X, Y).$$

Proof: This is a direct calculation of theorem (2.3).

Theorem 2.5: *A vector field is an infinitesimal automorphism of an almost complex structure F on a manifold M^n if and only if*

$$[X, FY] = F([X, Y]),$$

for all vector fields Y on M^n .

An immediate consequence of the above Theorem 2.4, we can state the theorem:

Theorem 2.6: *A vector field $\tilde{X}$ is an infinitesimal automorphism of an almost complex structure F^c on a cotangent bundle ${}^cT(M^n)$ if and only if*

$$(2.16) \quad [X^c, F^c Y^c] = F^c([X^c, Y^c]),$$

for all vector fields Y^c , which is vertical on ${}^cT(M^n)$.

Proof: Let X^c and Y^c be any vector fields on ${}^cT(M^n)$. Then we have

$$\begin{aligned} [X^c, F^c Y^c] &= L_X C(F^c Y^c) = (L_X C F^c) Y^c + F^c (L_X C Y^c) \\ &= (L_X C F^c) Y^c + F^c ([X^c, Y^c]). \end{aligned}$$

Hence $L_X C F^c = 0$ if and only if $[X^c, F^c Y^c] = F^c([X^c, Y^c])$ for all Y^c .

Theorem 2.7: *Let F be an almost complex structure on M^n with N , the Nijenhuis tensor of F , then*

$$(2.17) \quad F^c + \frac{1}{2}(NF)^V$$

is an almost complex structure on ${}^cT(M^n)$.

Proof: we have

$$\begin{aligned} \{F^c + \frac{1}{2}(NF)^V\}^2 &= (F^c)^2 + \frac{1}{2}F^c(NF)^V + \frac{1}{2}(NF)^V F^c \\ &= (F^c)^2 + \frac{1}{2}(NF^2)^V + \frac{1}{2}(NF^2)^V. \end{aligned}$$

Since $F^2 = -I$, we get

$$\{F^c + \frac{1}{2}(NF)^V\}^2 = (F^c)^2 + N^V = (-I)^V = -\tilde{I}.$$

3. Almost Product structure in Cotangent Bundle

Let $F \in \mathfrak{S}_1^1(M^n)$ and $F^2 = a^2 I$, where I is the Identity tensor field. Then F is an almost product structure on M^n when $a = 1$. If $F^3 - F = 0$, then F is called an almost para F -structure on M^n .

Theorem 3.1: If F is an almost product structure on M^n and $N = N_{F,F}$, then

$$(3.1) \quad N^\vee F^C = (NF)^\vee,$$

and

$$(3.2) \quad (NF)^\vee F^C = N^\vee.$$

Proof : We know

$$N(Z, FY) = N(FZ, Y).$$

Replace Z by FZ in above equation

$$N(FZ, FY) = N(Z, Y),$$

for all $Z, Y \in \mathfrak{S}_0^1(M^n)$.

Yano and Patterson⁴ establish a connection between the complete lift of two tensor fields $F, G \in \mathfrak{S}_1^1(M^n)$ and the torsions of F and G .

Theorem 3.2: Let F be an almost product structure on M^n with the Nijenhuis tensor N . Then

$$(3.3) \quad F^C - \frac{1}{2}(NF)^\vee$$

is also an almost product structure on ${}^C T(M^n)$.

Proof: Using above theorem, we have

$$\begin{aligned} \{F^C - \frac{1}{2}(NF)^\vee\}^2 &= (F^C)^2 - \frac{1}{2}F^C(NF)^\vee F^C \\ &= (F^C)^2 - \frac{1}{2}(NF^2)^\vee - \frac{1}{2}(NF^2)^\vee. \end{aligned}$$

Since $F^2 = I$, we get

$$\{F^C - \frac{1}{2}(NF)^\vee\}^2 = (F^C)^2 - (N)^\vee = (I)^\vee = \tilde{I}.$$

References

1. P. Dombroski: On the geometry of Tangent bundle, *J. Reino. Angew. Math*, **210** (1962) 75-83.
2. Ram Nivas and S.K. Srivastava: On Horizontal and complete lifts from a manifold with $(7,-1)$ structures to cotangent bundle, *J. Tensor Soc. India*, **14** (1996) 42-48.
3. J.P. Srivastava: Quadratic structure on cross-section in the cotangent bundle, *J. Tensor Soc. India*, **13** (1995) 31-36.
4. K. Yano and E.M. Patterson: Vertical and complete lifts from a manifold to its cotangent bundle, *J. Math. Soc. Japan*, **19** (1) (1967) 91-113.
5. K. Yano and S. Ishihara: Tangent and cotangent bundles, *Marcel Dekker, Inc, New York*, 1973.

