

On Non-invariant Hypersurfaces of Trans-Sasakian Manifolds

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Abstract : Non-invariant hypersurfaces of a trans-Sasakian manifold are studied. It is proved that the fundamental 2-form of the induced (f, g, u, v, λ) -structure on the non-invariant hypersurface of a trans-Sasakian manifold is closed. A sufficient condition for certain vector field on the hypersurface of a Kenmotsu manifold to be harmonic is given. A necessary and sufficient condition for a totally umbilical non-invariant hypersurface of a trans-Sasakian manifold to be totally geodesic is proved. It is also proved that on a totally geodesic non-invariant hypersurface with (f, g, u, v, λ) -structure of an α -Sasakian manifold, v is Killing. Sufficient conditions for the induced (f, g, u, v, λ) -structure on the hypersurface to be normal and quasi-normal have also been obtained.

1. Introduction

In S. Tanno¹ classified connected almost, contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c . He showed that they can be divided into three classes :

- (1) Homogeneous normal contact Riemannian manifolds with $c > 0$,
- (2) Global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and
- (3) A warped product space $\mathbb{R} \times_f \mathbb{C}^n$ if $c < 0$. It is known that the manifold of class (1) are characterized by admitting a Sasakian structure.

Kenmotsu² characterized the differential geometric properties of the manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian².

In the Gray-Hervella³ classification of almost Hermitian manifolds, there appears a class, $\mathcal{W}_4$, of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds. An almost contact metric structure on a manifold $\bar{M}$ is called a *Trans-Sasakian*

*Structure*⁴, if the product manifold $\bar{M} \times \mathbb{R}$ belongs to the class $\mathcal{W}_4$. The class $C_6 \oplus C_5$ Marrero⁵ coincides with the class of trans-Sasakian structures of type (α, β) . We note that trans-Sasakian structures of type $(0, 0)$ are cosymplectic Blair⁶, trans-Sasakian structures of type $(0, \beta)$ are β -Kenmotsu, Janssens and Vanhecke⁷ and trans-Sasakian structures of type $(\alpha, 0)$ are α -Sasakian.

On the other hand, (f, g, u, v, λ) -structure on a manifold were introduced by Yano and Okumura⁸. It is well known that on a non-invariant hypersurface of an almost contact metric manifold there always exists a (f, g, u, v, λ) -structure. Motivated by this fact, in this paper non-invariant hypersurfaces of a Trans-Sasakian manifold are studied. The paper is organized as follows. Section 2 is devoted to preliminaries. In section 3, some properties of non-invariant hypersurfaces of a trans-sasakian manifold are given. It is proved that the fundamental 2-form of the induced (f, g, u, v, λ) -structure on the non-invariant hypersurface of a trans-Sasakian manifold is closed. A sufficient condition for certain vector field on the hypersurface of a Kenmotsu manifold to be harmonic is obtained. In section 4, we find a necessary and sufficient condition for a totally umbilical non-invariant hypersurface of a trans-Sasakian manifold to be totally geodesic. For a non-invariant hypersurface with the induced (f, g, u, v, λ) -structure of a trans-Sasakian manifold, the second fundamental form of the immersion is also calculated, provided f is parallel. It is also proved that on a totally geodesic non-invariant hypersurface with structure of an α -Sasakian manifold, v is Killing. In the last section sufficient conditions for the induced (f, g, u, v, λ) -structure to be normal and quasi-normal are found.

Trans-Sasakian Manifolds

Let $\bar{M}$ be an almost contact metric manifold⁶ with an almost contact metric structure (ϕ, ξ, η, g) , that is, ϕ is a $(1, 1)$ tensor field, ξ is a vector field; η is 1-form and g is a compatible Riemannian metric such that

$$(1) \quad \phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0,$$

$$(2) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(3) \quad g(X, \phi Y) = -g(\phi X, Y) \quad g(X, \xi) = \eta(X)$$

for all $X, Y \in T\bar{M}$

An almost contact metric structure (ϕ, ξ, η, g) on $\bar{M}$ is called a *trans-Sasakian structure* (Oubina,⁴ if $(\bar{M} \times \mathbb{R}, J, G)$ belong to the class $\mathcal{W}_4$ of Gray-Hervella classification of almost Hermitian manifolds³, when J is the almost complex structure on $\bar{M} \times \mathbb{R}$ defined by

$$J\left(X, \frac{ad}{dt}\right) = \left(\phi X, -a\xi, \eta(X) \frac{d}{dt}\right)$$

for all vector fields X on $\bar{M}$ and smooth functions a on $\bar{M} \times \mathbb{R}$ and G is the product metric on $\bar{M} \times \mathbb{R}$. This may be expressed by the condition (Blair and Oubina)⁹

$$(4) \quad (\bar{\nabla}_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X)$$

for some smooth functions α and β on $\bar{M}$, and we say that the trans-Sasakian structure is of type (α, β) . From the equation (4) it follows that⁹

$$(5) \quad \bar{\nabla}_X \xi = -\alpha \phi X + \beta(X - \eta(X)\xi)$$

The class $C_6 \oplus C_5$ Marrero⁵ coincides with the class of trans-Sasakian structures of types (α, β) . We note that trans-Sasakian structures of type $(0, 0)$ are cosymplectic⁶, trans-Sasakian structure of type $(0, \beta)$ are β -Kenmotsu⁷ and trans-Sasakian structures of type $(\alpha, 0)$ are α -Sasakian⁷.

Let M be a hypersurface of a Riemannian manifold $\bar{M}$, with a Riemannian metric g . Then Gauss and Weingarten formulae are given respectively by

$$(6) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X, Y)N, \quad (X, Y \in TM)$$

$$(7) \quad \bar{\nabla}_X N = -HX$$

where $\bar{\nabla}, \nabla$ are respectively the Riemannian and induced Riemannian connection in $\bar{M}$ and M ; N is the unit normal vector in the normal bundle $T^\perp M$ and h is the second fundamental form related to H by

$$(8) \quad h(X, Y) = g(HX, Y)$$

Hypersurface M is known to be *totally geodesic* in $\bar{M}$ if $h = 0$, and *totally umbilical* in $\bar{M}$ if $H = \gamma I$.

If $\bar{M}$ is an almost contact metric manifold and M its hypersurface, then defining

$$(9) \quad \phi X = fX + u(X)N$$

$$(10) \quad \phi N = -U$$

$$(11) \quad \xi = V + \lambda N$$

$$(12) \quad \eta(X) = v(X)$$

for $X \in TM$, we get an induced (f, g, u, v, λ) -structure⁸ on the non-invariant hypersurface such that

$$(13) \quad f^2 = -I + u \otimes U + v \otimes V$$

$$(14) \quad fU = -\lambda V, \quad fV = \lambda U$$

$$(15) \quad u \circ f = \lambda v \quad v \circ f = -\lambda u$$

$$(16) \quad u(U) = 1 - \lambda^2, \quad u(V) = v(U) = 0, \quad v(V) = 1 - \lambda^2$$

$$(17) \quad g(fX, fY) = g(X, Y) - u(X)u(Y) - v(X)v(Y)$$

$$(18) \quad g(X, fY) = -g(fX, Y), \quad g(X, U) = u(X), \quad g(X, V) = v(X)$$

for all $X, Y \in TM$, where $\lambda = \eta(N)$

3. Some Properties of Non-invariant Hypersurfaces

First, we state the following lemma.

Lemma 3.1 : *Let M be a non-invariant hypersurface with (f, g, u, v, λ) -structure of an almost contact metric manifold M . Then*

$$(19) \quad (\bar{\nabla}_X \phi)Y = ((\nabla_X f)Y - u(Y)HX + h(X, Y)U) + ((\nabla_X u)Y + h(X, fY))N$$

$$(20) \quad \bar{\nabla}_X \xi = (\nabla_X V - \lambda HX) + (h(X, V) + X\lambda)N$$

$$(21) \quad (\bar{\nabla}_X \phi)N = (-\nabla_X U + fHX)$$

$$(22) \quad (\bar{\nabla}_X \eta)Y = (\nabla_X v)Y - \lambda h(X, Y)$$

for all $X, Y \in TM$,

The proof is straightforward and hence omitted.

Theorem 3.2 : *Let M be non-invariant hypersurface with (f, g, u, v, λ) -structure of a trans-Sasakian manifold M . Then we have :*

$$(23) \quad (\nabla_X f)Y = \alpha(g(X, Y)V - v(Y)X + \beta(g(fX, Y)V - v(Y)fX) + u(Y)HX - h(X, Y)U$$

$$(24) \quad (\nabla_X u)Y = \alpha\lambda g(X, Y) + \beta(\lambda g(fX, Y) - u(X)v(Y)) - h(X, fY)$$

$$(25) \quad \nabla_X V = \lambda HX - \alpha fX + \beta(X - v(X)V)$$

$$(26) \quad h(X, V) = -\alpha u(X) - \beta\lambda v(X) - X\lambda$$

$$(27) \quad \nabla_X U = fHX + \alpha\lambda X + \beta(\lambda fX - u(X)V)$$

$$(28) \quad (\nabla_X v)Y = \lambda h(X, Y) - \alpha g(fX, Y) + \beta(g(X, Y) - v(X)v(Y))$$

for all $X, Y \in TM$.

Proof : Using (4), (9) and (11) in (19) we obtain

$$\begin{aligned} & ((\nabla_X f)Y - u(Y)HX + h(X, Y)U) + ((\nabla_X u)Y + h(X, fY))N \\ &= \alpha(g(X, Y)V - v(Y)X) + \beta(g(fX, Y)V - v(Y)fX) \\ &+ \alpha\lambda g(X, Y) + \beta(\lambda g(fX, Y) - u(X)v(Y)) \end{aligned}$$

Equating tangential and normal parts in the above equation, we get (23) and (24) respectively. Using (5) and (11) in (20) we have

$$\begin{aligned} & (\nabla_X V - \lambda HX) + (h(X, V) + X\lambda)N \\ &= -\alpha fX + \beta(X - v(X)V) - (\alpha u(X) + \beta\lambda v(X))N \end{aligned}$$

and equating tangential and normal parts we get (25) and (26) respectively. Using (4), (10) and (11) in (21) and equating tangential parts we get (27). In last, (28) follows from (22).

Theorem 3.3 : *If M is a non-invariant hypersurface with (f, g, u, v, λ) -structure of a trans-Sasakian manifold, then the 2-form F on M given by :*

$$F(X, Y) \equiv g(X, fY)$$

is closed.

Proof : From (23), we get

$$\begin{aligned} (\nabla_X F)(Y, Z) &= -\alpha(g(X, Y)v(Z) - g(X, Z)v(Y)) \\ &- \beta(g(fX, Y)v(Z) - g(fX, Z)v(Y)) \\ &+ h(X, Y)u(Z) - h(X, Z)u(Y) \end{aligned}$$

which gives

$$(\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y) = 0$$

that is, $dF = 0$.

Theorem 3.4 : *Let M be a non-invariant hypersurface with (f, g, u, v, λ) -structure of a Kenmotsu manifold. If $H = -\lambda I$, then V is harmonic.*

Proof : From (28) we get

$$(\nabla_X v)Y - (\nabla_Y v)X = 2\alpha g(X, fY)$$

Moreover, if $H = -\lambda I$ then from (25) we obtain

$$\nabla_V V = \lambda^2 V - \alpha\lambda U + \beta\lambda^2 V$$

Now, if the ambient manifold is Kenmotsu, then $\alpha = 0$ and $\beta = 1$; and the above two equations we get

$$(\nabla_X v)Y - (\nabla_Y v)X = 0 \quad \text{and} \quad \nabla_V V = 0$$

Thus, V is harmonic.

4. Totally Geodesic Non-invariant Hypersurfaces

First, we find a necessary and sufficient condition for a totally umbilical non-invariant hypersurface of a trans-Sasakian manifold to be totally geodesic.

Theorem 4.1 : *Let M be a totally umbilical non-invariant hypersurface with (f, g, u, v, λ) -structure of a trans-Sasakian manifold. Then it is totally geodesic if and only if*

$$(29) \quad \frac{\alpha}{\lambda} u + \beta v + d(\log \lambda) = 0$$

Proof : If M is totally umbilical, putting $H = \gamma I$, the equation (26) gives

$$(30) \quad \alpha u(X) + \beta \lambda v(X) + X\lambda + \gamma v(X) = 0$$

Then, M is totally geodesic, that is $\gamma = 0$, if and only if (29) is true.

Theorem 4.2 : *Let M be a non-invariant hypersurface with (f, g, u, v, λ) -structure of a trans-Sasakian manifold. If f is parallel, then we get*

$$(31) \quad h(X, Y) = \frac{1}{(1 - \lambda^2)^2} (\mu u(X) u(Y) - \alpha(1 - \lambda^2)(u(X) v(Y) + u(Y) v(X)) - \beta \lambda (1 - \lambda^2) v(X) v(Y))$$

$$(32) \quad d(\log \lambda) = 0$$

where $\mu = h(U, U)$. Consequently, M is totally geodesic if and only if

$$(33) \quad \alpha u + \lambda \beta v = 0$$

Proof : If f is parallel, then (23) gives

$$(1 - \lambda^2) h(X, Y) = u(Y) u(HX) - \alpha u(X) v(Y) - \beta \lambda v(X) v(Y)$$

From here we get

$$\begin{aligned} (1 - \lambda^2) h(X, U) &= (1 - \lambda^2) u(HX) \\ &= \mu u(X) - \alpha(1 - \lambda^2) v(X) \end{aligned}$$

that is,

$$(35) \quad (1 - \lambda^2) u(HX) = \mu u(HX) - \alpha(1 - \lambda^2) v(X)$$

From (34) and (35), eliminating $u(H, X)$, we get (31). The equation (32) follows from (31) and (26). The last part is obvious.

Theorem 4.3 : *On a totally geodesic non-invariant hypersurface with (f, g, u, v, λ) -structure of an α -Sasakian manifold, v is Killing.*

Proof : If M is a non-invariant hypersurface with (f, g, u, v, λ) -structure of a trans-Sasakian manifold, then in view of (28) we have

$$(\nabla_X v)Y + (\nabla_Y v)X = 2\lambda h(X, Y) + 2\beta(g(X, Y) - v(X)v(Y)).$$

This equation implies the above Proposition.

5. Normal and Quasi-normal (f, g, u, v, λ) -structure

In this section we find sufficient condition for the induced (f, g, u, v, λ) -structure on the non-invariant hypersurface of a β -Kenmotsu manifold to be normal and quasi-normal.

Let M be a non-invariant hypersurface with (f, g, u, v, λ) -structure of a trans-Sasakian manifold. In view of (23), (24) and (28), we get :

$$\begin{aligned} S(X, Y) &\equiv [f, f](X, Y) + du(X, Y)U + dv(X, Y)V \\ &= \alpha v(X)fY - \alpha v(Y)fX - \beta(u(X)v(Y) - v(X)u(Y))U \\ &\quad + u(X)(fH - Hf)Y - u(Y)(fH - Hf)X \end{aligned}$$

In view of the above equation we can state the following two theorems.

Theorem 5.1 : *Let M be a non-invariant hypersurface with (f, g, u, v, λ) -structure of a β -Kenmotsu manifold. If f commutes with H and $u \otimes v = v \otimes u$, then the induced (f, g, u, v, λ) -structure on M is normal.*

Theorem 5.2 : *The induced (f, g, u, v, λ) -structure on a non-invariant hypersurface of a cosymplectic manifold is normal if f commutes with H .*

Now, we recall the definition of a quasi-normal (f, g, u, v, λ) -structure.

Definition 5.3 : *The (f, g, u, v, λ) -structure is called quasi-normal¹¹ if*

$$g(S(X, Y), Z) - ((dF)(fX, Y, Z) - (dF)(fY, X, Z)) = 0$$

In view of Theorem 3.3 and Definition 5.3, we can state the following.

Theorem 5.4 : *Let M be a non-invariant hypersurface with (f, g, u, v, λ) -structure of a trans-Sasakian manifold. If (f, g, u, v, λ) -structure on M is normal then it is also quasi-normal.*

In view of Theorem 5.1, 5.2 and 5.4 we get following two corollaries.

Corollary 5.5 : *Let M be a non-invariant hypersurface with (f, g, u, v, λ) -structure of α β -Kenmotsu manifold. The induced (f, g, u, v, λ) -structure on M is quasi-normal if f commutes with H and $u \otimes v = v \otimes u$.*

Corollary 5.6 : *Let M be a non-invariant hypersurface with (f, g, u, v, λ) -structure of a cosymplectic manifold. Then the induced (f, g, u, v, λ) -structure on M is quasi-normal if $fH = Hf$.*

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