

Deformation of the Drop with Surfactant Layer

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Abstract. In the present paper, the effect of adsorbed mono-molecular surfactant film of fluid which covers the surface of the spherical drop is considered. The deformation of the drop is investigated. It is found that for creeping flow, the drop remains spherical, but on including the effect of inertia, the deformation sets in and is accentuated by the presence of the surface layer.

1. Introduction

The effect of dynamic interfacial properties on drop of bubble deformation and orientation at small Reynolds number has been examined by many workers. Taylors and Acrivos¹, Brigness², Rallison³ and Power⁴. Further, when a bubble rises through a liquid containing surface active impurities, the fluid motion near its surface is slowed down or stopped. Savic⁵ first observed the phenomenon and studied the case of spherical drops moving at low Reynolds number, with negligibly small interior viscosity and bearing a rigid spherical cap. The effect of surfactants on drop deformation and breakup was examined by Stone and Leal⁶.

Taylors and Acrivos¹ studied the deformation and drag of falling viscous drop at low Reynolds number, by taking the condition at the surface $r = R = 1 + \zeta(\mu)$ ($\mu = \cos \theta$) of the drop, where (r, θ) represents spherical polar coordinate with pole at the centre of the drop.

The shape of the drop was determined through the normal stress jump condition

$$(1.1) \quad N = \hat{N} + \frac{1}{We} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) \quad \text{at} \quad r = R(\mu) = 1 + \zeta(\mu)$$

Here, N represent the normal stress and caret implies quantities in the interior of the drop; We is the Webber number ($\rho a U^2 / \sigma$) (σ interfacial tension, ρ the density of the exterior fluid, a the radius of the equivalent spherical drop, U the terminal velocity of the drop and R_1 and R_2 the two principal radii of curvature of the drop. The non-dimensionalise velocities by U , distance by a and stress by ρU^2 .

In this paper, we investigate the effect of an adsorbed monomolecular surfactant film of the fluid on the deformation of drop, using Scriven's (1960) boundary conditions.

Scriven⁷ studied the deformation of a thin fluid interface between the two bulk fluids of different viscosities and obtained the following interface condition :

$$(1.2) \quad \frac{dw}{dt} - F = \nabla_s \sigma + (\kappa + \varepsilon) \nabla_s (\nabla_s \cdot \mathbf{w}) + \varepsilon [2K(\mathbf{w} \cdot \mathbf{n} \cdot \mathbf{n}) + \mathbf{n} \times \nabla_s (\mathbf{n} \cdot \nabla_s \times \mathbf{w}) + 2(\mathbf{n} \times \nabla_s \mathbf{n} \times \mathbf{n}) \cdot \nabla_s (\mathbf{n} \cdot \mathbf{w})] + \mathbf{n} [2H\sigma + 2H(\kappa + \varepsilon) \nabla_s \mathbf{w} - 2\varepsilon (\mathbf{n} \times \nabla_s \mathbf{n} \times \mathbf{n}) : \nabla_s \mathbf{w}]$$

where $2H = -\nabla_s \cdot \mathbf{n}$, $2k = -(\mathbf{n} \times \nabla_s \mathbf{n} \times \mathbf{n}) : \nabla_s \mathbf{n}$, κ is the coefficient of surface dilational viscosity, ε is coefficient of surface shear viscosity, $\mathbf{n}$ is unit vector normal to the surface gradient operator⁷. This condition will provide us with the relevant surface conditions on the drop in the next section.

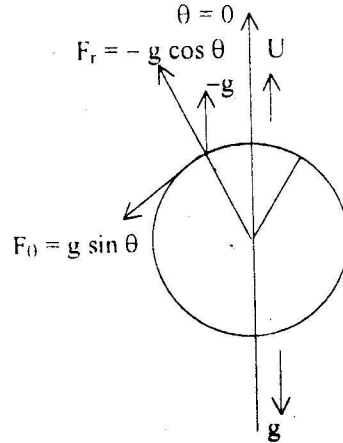
In 1911, Hadmard⁸ and Rybczynski⁹ independently observed that for low Reynolds number, the drop remain exactly spherical. Considering this fact, we shall first determine the creeping flow solution taking the drop should be perfect sphere and the flow to be inertialess ($Re = 0$). In this case, we see that the presence of the surfactant layer, the drop continues to remain spherical.

In section 4, we evaluate the effect of inertia terms on the deformation of the drop.

2. Formulation of the Problem

Consider a surfactant layer with surface shear viscosity ε and surface dilation viscosity κ present on the surface $r = R(\mu) = 1 + \zeta(\mu)$ of the drop. Let (r, θ, ϕ) be a fixed spherical coordinate system with origin at the centre of the drop. The translational motion occurs along the axis of symmetry $\theta = 0$. The non-vanishing components of velocity are radial velocity $u_r(r, \theta)$ and transverse velocity $u_\theta(r, \theta)$ may be expressed in terms of Stokes stream function ψ as

$$(2.1) \quad u_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{1}{r \sin \theta} \frac{\partial \psi}{\partial r}.$$



The equations of motion in non-dimensional form may be expressed in terms of stream function ψ and $\hat{\psi}$ Taylors and Acrivos¹ as follows.

$$(2.2) \quad \frac{1}{Re} D^4 \psi = \frac{1}{r^2} \frac{\partial(\psi, D^2 \psi)}{\partial(r, \mu)} + \frac{2}{r^2} D^2 \psi \left\{ \frac{\mu}{1 - \mu^2} \frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{\partial \psi}{\partial \mu} \right\}$$

for the exterior region, and

$$(2.3) \quad \frac{k}{Re} D^4 \hat{\psi} = \frac{1}{r^2} \frac{\partial(\hat{\psi}, D^2 \hat{\psi})}{\partial(r, \mu)} + \frac{2}{r^2} D^2 \hat{\psi} \left\{ \frac{\mu}{1 - \mu^2} \frac{\partial \hat{\psi}}{\partial r} + \frac{1}{r} \frac{\partial \hat{\psi}}{\partial \mu} \right\}$$

for the region inside the drop, where

$$D^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1 - \mu^2}{r^2} \frac{\partial^2}{\partial \mu^2}, \quad \mu = \cos \theta$$

and $Re = Ua/\nu$ is the Reynolds number.

Next, we shall find the solution of equation (2.2) and (2.3) subject to the following boundary conditions :

(i) *Continuity of normal component of velocity :*

$$(2.4) \quad u_n = \hat{u}_n = 0 \quad \text{at } r = R(\mu)$$

(ii) *Continuity of tangential component of velocity :*

$$(2.5) \quad u_t = \hat{u}_t \quad \text{at } r = R(\mu)$$

(iii) *Surfactant layer condition :* The discontinuity of tangential stress τ and normal stress N at the interface $r = R(\mu)$ are derived from the Scriven's condition (1.2)

$$(2.6) \quad \tau - \hat{\tau} = - \left[(\kappa + \varepsilon) \frac{\partial}{\partial s} \left\{ \frac{1}{R^2 \sin \theta} \frac{\partial(Ru_t \sin \theta)}{\partial s} \right\} + 2\varepsilon \frac{u_t}{R^2} \right]$$

$$(2.7) \quad N - \hat{N} = \frac{2\kappa}{R^3 \sin \theta} \frac{\partial(Ru_t \sin \theta)}{\partial s} + \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

The first of the above conditions replaces the usual tangential stress continuity condition. Second condition (2.7) is not a boundary condition but it is used to determine the boundary shape, replacing the condition (1.1) of Taylor and Acrivos¹

Now, since deformation is assumed to be small, we find it convenient to express these relation in terms of spherical coordinate r and μ as was done by Taylor and Acrivos¹

$$\tan \alpha \approx \frac{1}{R} \frac{dR}{d\theta} \approx - (1 - \mu^2)^{1/2} \frac{d\zeta}{d\mu} \quad \text{as } \max |\zeta| \rightarrow 0,$$

we have following approximations :

$$(2.8) \quad \begin{aligned} \tau &= (\tau_{rr} - \tau_{\theta\theta}) \sin \alpha \cos \alpha + \tau_{r\theta} (\cos^2 \alpha - \sin^2 \alpha) \\ &\rightarrow \tau_{r\theta} - (\tau_{rr} - \tau_{\theta\theta}) (1 - \mu^2)^{1/2} \frac{d\zeta}{d\mu} \end{aligned}$$

$$(2.9) \quad \begin{aligned} N &= \tau_{rr} \cos^2 \alpha + \tau_{\theta\theta} \sin^2 \alpha - 2\tau_{r\theta} \sin \alpha \cos \alpha \\ &\rightarrow \tau_{rr} + 2\tau_{r\theta} (1 - \mu^2)^{1/2} \frac{d\zeta}{d\mu} \end{aligned}$$

$$(2.10) \quad \begin{aligned} u_t &= u_\theta \cos \alpha + u_r \sin \alpha \\ &\rightarrow u_\theta - u_r (1 - \mu^2)^{1/2} \frac{d\zeta}{d\mu} \end{aligned}$$

$$(2.11) \quad \begin{aligned} u_n &= u_r \cos \alpha - u_\theta \sin \alpha \\ &\rightarrow u_r - u_\theta (1 - \mu^2)^{1/2} \frac{d\zeta}{d\mu} \end{aligned}$$

The stress components τ_{rr} , $\hat{\tau}_{rr}$, $\tau_{r\theta}$ and $\hat{\tau}_{r\theta}$ are given by

$$(2.12) \quad \tau_{rr} = -p + \frac{2}{\text{Re}} \frac{\partial u_r}{\partial r}, \quad \hat{\tau}_{rr} = -\hat{p} + \frac{2k}{\text{Re}} \frac{\partial \hat{u}_r}{\partial r}$$

$$(2.13) \quad \tau_{r\theta} = \left[r \frac{\partial}{\partial r} \left(\frac{u_\theta}{r} \right) + \frac{1}{r} \frac{\partial u_r}{\partial \theta} \right] \quad \text{and} \quad \hat{\tau}_{r\theta} = k \left[r \frac{\partial}{\partial r} \left(\frac{\hat{u}_\theta}{r} \right) + \frac{1}{r} \frac{\partial \hat{u}_r}{\partial \theta} \right]$$

where $k = \frac{\hat{\eta}}{\eta}$ is the ratio of viscosities of the interior to that of exterior fluid. Similar expressions may be written down for the interior flow.

(iv) *The uniform stream flow at infinity requires that :*

$$(2.14) \quad \psi \rightarrow \frac{1}{2} r^2 (1 - \mu^2) \quad \text{as} \quad r \rightarrow \infty$$

$$(2.15) \quad \hat{\psi} \text{ is finite at } r = 0$$

3. Creeping Flow Solution

Assuming that Reynolds number is sufficiently small for the neglect of inertia terms in the flow field and taking the drop to be perfectly spherical (i.e. $r = 1$) as in the case of Taylor and Acrivos¹

The boundary conditions (2.4), (2.5) and (2.6) by using (2.1), (2.8), (2.10) and (2.11) when applied at $r = 1$ reduces respectively to

$$(i). (3.1) \quad \psi = \hat{\psi} = 0$$

$$(ii). (3.2) \quad \frac{\partial \psi}{\partial r} = \frac{\partial \hat{\psi}}{\partial r}$$

and

$$(iii). (3.3) \quad \tau_{r\theta} - \hat{\tau}_{r\theta} = \left[(\kappa + \varepsilon) \frac{\partial}{\partial \theta} \left\{ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\frac{\partial \psi}{\partial r} \right) \right\} + \frac{2\varepsilon}{\sin \theta} \frac{\partial \psi}{\partial r} \right].$$

It may be noted that in deriving the last of these equations, the small quantities of second order have been neglected, thereby enabling us to replace the intrinsic coordinates by the polar coordinates.

The solution of the equations (2.2) and (2.3) under the boundary conditions (2.14), (2.15), (3.1), (3.2) and (3.3) are

$$(3.4) \quad \psi_0 = \frac{(1 - \mu^2)}{4} \left[2r^2 - \frac{3\bar{k} + 2}{\bar{k} + 1} r + \frac{\bar{k}}{\bar{k} + 1} \frac{1}{r} \right]$$

and

$$(3.5) \quad \bar{\psi}_0 = (1 - \mu^2) \frac{(r^2 - r^4)}{4(\bar{k} + 1)}$$

Here, $\bar{k} = k + \frac{2}{3} \frac{k}{\eta}$, η is viscosity of fluid outside the spherical drop.

Now, we shall form the equation that governs the deformation in the drop. As mentioned earlier, this will be achieved by equating the value of $(N - \hat{N})$ as obtained from the known solutions ψ_0 and $\hat{\psi}_0$ with its value as given by the Scriven's condition (2.7), after neglecting the square and higher power terms involving the surfactant layer parameter κ and ε . Thus, using (2.1), (2.10) and (3.4), $(N - \hat{N})$ as given by (2.7) approximates to

$$(3.6) \quad N - \hat{N} = \frac{-2\kappa}{\eta(\bar{k} + 1) \text{Re}} \mu + \frac{1}{We} \left\{ 2 - 2\zeta - \frac{d}{d\mu} \left((1 - \mu^2) \frac{d\zeta}{d\mu} \right) \right\},$$

where we have used the approximation

$$\frac{1}{R_1} + \frac{1}{R_2} \rightarrow 2 - 2\zeta - \frac{d}{d\mu} \left((1 - \mu^2) \frac{d\zeta}{d\mu} \right),$$

tenable for small ζ (Landau and Lifschitz¹⁰)

Next, from the relation (2.9)

$$N - \hat{N} = \tau_{rr} - \hat{\tau}_{rr} + 2(\tau_{r\theta} - \hat{\tau}_{r\theta})(1 - \mu^2)^{1/2} \frac{d\zeta}{d\mu}$$

If we make use the condition (3.3), we conclude that the second term on the right hand side above, being of second order in smallness, is negligible and hence

$$(3.7) \quad N - \hat{N} = \tau_{rr} - \hat{\tau}_{rr}.$$

Now by using (2.1), (2.12), (3.4) and (3.5), the difference of stresses τ_{rr} and $\hat{\tau}_{rr}$ corresponding to stream function ψ_0 and $\hat{\psi}_0$ respectively at $r = 1$, is given by

$$(3.8) \quad \tau_{rr} - \hat{\tau}_{rr} = \frac{-2k}{\eta(\bar{k} + 1) Re} \mu + \Pi,$$

where the overall force balance requires that

$$\frac{ga(1 - \gamma)}{U^2} = -\frac{3}{2 Re} \frac{3\bar{k} + 2}{\bar{k} + 1}$$

Next, on using equations (3.6), (3.7) and (3.8), we obtain the following differential equation governing the shape of the drop

$$\begin{aligned} N - \hat{N} &= \frac{-2\kappa}{\eta(\bar{k} + 1) Re} \mu + \Pi \\ &= \frac{1}{We} \left\{ 2 - 2\zeta - \frac{d}{d\mu} \left((1 - \mu^2) \frac{d\zeta}{d\mu} \right) \right\} \frac{-2\kappa}{\eta(\bar{k} + 1) Re} \mu. \end{aligned}$$

or

$$(3.9) \quad \frac{1}{We} \left\{ -(1 - \mu^2) \frac{d^2\zeta}{d\mu^2} + 2\mu \frac{d\zeta}{d\mu} - 2\zeta \right\} + \frac{2}{We} = \Pi$$

Now, we have to determine the solution of equation (3.9) subject to following boundary conditions given by Taylors and Acrivos¹

(i) The volume of the drop is fixed

$$(3.10) \quad \int_{-1}^1 \zeta(\mu) d\mu = 0, \quad \text{for } \max |\zeta(\mu)| < < 1,$$

(ii) The center of mass of the drop is fixed

$$(3.11) \quad \int_{-1}^1 \mu \zeta(\mu) d\mu = 0, \quad \text{for } \max |\zeta(\mu)| < < 1,$$

We see that the governing differential equation (3.9) and the boundary conditions (3.10) and (3.11) are same obtained by Taylors and Acrivos¹ for the drop without surfactant layer; therefore, the solution is same viz. $\zeta = 0$. Hence, we arrive at the interesting result that the drop under creeping flow but with surfactant layer remains underformed.

In the next section, following the approach of Taylors and Acrivos¹ we shall see how does the presence of the surfactant layer affect the deformation when inertial effects are taken into account.

4. Inertial Effects

In this section, we include the inertial effects in our analysis when Reynolds number is small. Thus, in view of the analysis described by Taylors and Acrivos¹ we assume :

$$(4.1) \quad \psi = \psi_0 + \text{Re } \psi_1 + \dots, \dots \quad \text{for } r \gg 1 + \zeta,$$

$$(4.2) \quad \hat{\psi} = \hat{\psi}_0 + \text{Re } \hat{\psi}_1 + \dots, \dots \quad \text{for } 0 < r < 1 + \zeta.$$

Further, the interia terms are approximated by zero-order solution ψ_0 and $\hat{\psi}_0$ in equations (2.2) and (2.3).

$$(4.3) \quad D^4 \psi_1 = 3Q_2(\mu) \frac{3\bar{k} + 2}{\bar{k} + 1} \left\{ \frac{1}{r^2} - \frac{1}{2r^3} \frac{3\bar{k} + 2}{\bar{k} + 1} + \frac{1}{2r^5} \frac{\bar{k}}{\bar{k} + 1} \right\}$$

$$(4.4) \quad D^4 \hat{\psi}_1 = 0$$

where

$$Q_n(\mu) = \int_{-1}^{\mu} P_n(\mu) d\mu,$$

$P_n(\mu)$ being Legendre polynomial of degree n and obtain as in Taylors and Acrivos¹

$$\psi_1 = \left[-\frac{1}{8} \frac{3\bar{k} + 2}{\bar{k} + 1} \left(r^2 - \frac{1}{r} \right) + C_1 \left(r - \frac{1}{r} \right) \right] Q_1(\mu) \\ + \left[\frac{1}{8} \frac{3\bar{k} + 2}{\bar{k} + 1} \left\{ r^2 - \frac{r}{2} \frac{3\bar{k} + 2}{\bar{k} + 1} + \frac{\bar{k}}{\bar{k} + 1} - \frac{1}{2r} \frac{\bar{k}}{\bar{k} + 1} \right\} + C_2 \left\{ 1 - \frac{1}{r^2} \right\} \right]$$

and

$$(4.6) \quad \hat{\psi}_1 = \hat{C}_1 (r^4 - r^2) Q_1(\mu) + \hat{C}_2 (r^5 - r^3) Q_2(\mu)$$

Next application of boundary conditions (3.2) and (3.3), yields the values of the constant as

$$C_1 = \frac{1}{16} \left(\frac{3\bar{k} + 2}{\bar{k} + 1} \right)^2,$$

$$C_2 = \frac{-(3\bar{k} + 2) \{ \bar{k} (5\bar{k} + 6) + (\bar{k} + 1)(4\lambda + (8/3)\bar{\lambda}) \}}{16(\bar{k} + 1)^2 (5 + 5\bar{k} - 4\lambda + (8/3)\bar{\lambda})},$$

$$\hat{C}_1 = -\frac{1}{16} \frac{3\bar{k} + 2}{(\bar{k} + 1)^2},$$

$$\hat{C}_2 = \frac{(3\bar{k} + 2)(4\bar{k} + 5)}{16(\bar{k} + 1)^2 \left(5 + 5\bar{k} + 4\lambda + \frac{8}{3}\bar{\lambda} \right)}$$

where $\lambda = \varepsilon/\eta$ and $\bar{\lambda} = \kappa/\eta$.

Thus,

$$(4.7) \quad \psi_1 = -\frac{Q_1(\mu)}{8} \frac{3\bar{k} + 2}{\bar{k} + 1} \left[r^2 - \frac{r(3\bar{k} + 2)}{2(\bar{k} + 1)} + \frac{1}{2r} \left(\frac{\bar{k}}{\bar{k} + 1} \right) \right] \\ + \frac{Q_2(\mu)}{8} \frac{3\bar{k} + 2}{\bar{k} + 1} \left[r^2 - \frac{r(3\bar{k} + 2)}{2(\bar{k} + 1)} \right] \\ + \frac{\{\bar{k}(5\bar{k} + 4) + (\bar{k} - 1)(4\lambda + (8/3)\bar{\lambda})\}}{2(\bar{k} + 1)^2 (5 + 5\bar{k} + 4\lambda + (8/3)\bar{\lambda})} \\ \text{and} \quad - \frac{1}{2r} \left(\frac{\bar{k}}{\bar{k} + 1} \right) + \frac{1}{r^2} \frac{\{\bar{k}(5\bar{k} + 6) + (\bar{k} + 1)(4\lambda + (8/3)\bar{\lambda})\}}{2(\bar{k} + 1) \left(5 + 5\bar{k} + 4\lambda + \frac{8}{3}\bar{\lambda} \right)} \right]$$

$$(4.8) \quad \hat{\psi}_1 = \frac{1}{16} \frac{3\bar{k} + 2}{(\bar{k} + 1)^2} \left[(r^2 - r^4) Q_1(\mu) + \frac{(4\bar{k} + 5)}{\left(5 + 5\bar{k} + 4\lambda + \frac{8}{3}\bar{\lambda} \right)} (r^5 - r^3) \right]$$

Here, we note when the surfactant layer is absent (i.e. $\lambda = \bar{\lambda} = 0$) then the above solution (4.7) and (4.8) are exactly same as obtained by Taylors and Acrivos¹

5. The Deformation of the Drop

For determine the deformation of the drop for small values of Webber number We , we, first determine the pressure distribution and normal stresses at surface $r = 1$ of the drop by taking

$$(5.1) \quad p = p_0 + \text{Re } p_1 + \dots, \dots,$$

and

$$(5.2) \quad \tau_{rr} = \tau_{rr}^{(0)} + \text{Re } \tau_{rr}^{(1)} + \dots, \dots,$$

where p_0 and $\tau_{rr}^{(1)}$ are pressure and normal stress corresponding to stream function ψ_0 and, p_1 and $\tau_{rr}^{(1)}$ corresponding to stream function ψ_1 for the region outside the drop.

Similar expansions may also be taken for interior of the drop. Evaluating pressure terms through the small Reynolds number approximations of Navier-Stokes equations and employing equations (2.12), we get

$$(5.3) \quad p_0 = -\frac{ga\mu}{U^2} r - \frac{3\bar{k} + 2}{\bar{k} + 1} \frac{\mu}{2r^2 \text{Re}}, \\ \tau_{rr}^{(0)} = \frac{3\bar{k} + 2}{\bar{k} + 1} \frac{3\mu}{2r^2 \text{Re}} - \frac{\bar{k}}{\bar{k} + 1} \frac{3\mu}{r^4 \text{Re}} + \frac{ga}{U^2} \mu r,$$

$$\begin{aligned}
\text{Re } p_1 &= -\frac{P_1(\mu)}{16} \left(\frac{3\bar{k} + 2}{\bar{k} + 1} \right)^2 + \frac{P_2(\mu)}{16} \frac{(3\bar{k} + 2)}{(\bar{k} + 1)^2} \\
&\quad \times \frac{\{15\bar{k}^2 + 27\bar{k} + 4\bar{k}(3\bar{\lambda} + 2\bar{\lambda}) + 16(\bar{\lambda} + (2/3)\bar{\lambda}) + 10\}}{\left(5 + 5\bar{k} + 4\bar{\lambda} + \frac{8}{3}\bar{\lambda}\right)} \\
&\quad + \frac{P_2(\mu)}{12(\bar{k} + 1)^2} + \Pi_1 \\
(5.4) \quad \text{Re } \tau_{rr}^{(1)} &= \frac{P_1(\mu)}{16} \frac{(3\bar{k} + 2)}{(\bar{k} + 1)^2} \cdot 3(\bar{k} + 2) - \frac{P_2(\mu)}{16} \frac{(3\bar{k} + 2)}{(\bar{k} + 1)^2} \\
&\quad \times \frac{\{15\bar{k}^2 + 43\bar{k} + 4\bar{k}(3\bar{\lambda} + 2\bar{\lambda}) + 16(\bar{\lambda} + (2/3)\bar{\lambda}) + 30\}}{\left(5 + 5\bar{k} + 4\bar{\lambda} + \frac{8}{3}\bar{\lambda}\right)} \\
&\quad - \frac{P_2(\mu)}{12(\bar{k} + 1)^2} + \Pi_1
\end{aligned}$$

where Π_1 is a constant and g is gravitational force. Similarly for the region inside the drop

$$\begin{aligned}
\hat{p}_0 &= -\frac{ga\gamma}{U^2} r\mu + \frac{5k}{\bar{k} + 1} \frac{\mu r}{Re} + \Pi, \\
(5.5) \quad \hat{\tau}_{rr}^{(0)} &= \frac{ga\gamma}{U^2} \mu r - \frac{3k}{\bar{k} + 1} \frac{\mu r}{Re} - \Pi, \\
\text{Re } \hat{p}_1 &= \frac{k}{16} \frac{(3\bar{k} + 2)}{(\bar{k} + 1)^2} \left[10P_1(\mu) - \frac{7(4\bar{k} + 5)}{\left(5 + 5\bar{k} + 4\bar{\lambda} + \frac{8}{3}\bar{\lambda}\right)} P_2(\mu) \right] \\
&\quad + \frac{\gamma P_2(\mu)}{12(\bar{k} + 1)^2} + \hat{\Pi}_1 \\
\text{Re } \hat{\tau}_{rr}^{(1)} &= -\frac{k}{16} \frac{(3\bar{k} + 2)}{(\bar{k} + 1)^2} \left[6P_1(\mu) - \frac{3(4\bar{k} + 5)}{\left(5 + 5\bar{k} + 4\bar{\lambda} + \frac{8}{3}\bar{\lambda}\right)} P_2(\mu) \right] \\
&\quad - \frac{\gamma P_2(\mu)}{12(\bar{k} + 1)^2} - \hat{\Pi}_1
\end{aligned}$$

Since the total drag on the top by Taylors and Acrivos¹ is

$$F_D = \frac{\text{drag}}{a^2 \rho U^2} = \frac{2\pi}{Re} \left(\frac{3\bar{k} + 2}{\bar{k} + 1} \right) + \frac{\pi}{4} \left(\frac{3\bar{k} + 2}{\bar{k} + 1} \right)^2 + \dots \dots ,$$

we obtain from equation (3.6), (3.7), (5.2), (5.4), (5.5) and (5.6),

$$\begin{aligned}
 N - \bar{N} &= -\frac{2\kappa}{\eta(\bar{k} + 1)} \frac{\mu}{\text{Re}} \\
 &\quad - P_2(\mu) \frac{(3\bar{k} + 2)}{16(\bar{k} + 1)^2} \left[\frac{\{27\bar{k}^2 + 58\bar{k} + 4(3\bar{k} + 4)\lambda + (2/3)\bar{\lambda} + 30\}}{(5 + 5\bar{k} + 4\lambda + (2/3)\bar{\lambda})} \right] \\
 &\quad + (\gamma - 1) \frac{P_2(\mu)}{12(\bar{k} + 1)^2} + \hat{\Pi}_1 + \Pi_1 + \Pi \\
 &= -\frac{2\kappa}{\eta(\bar{k} + 1)} \frac{\mu}{\text{Re}} + \frac{2}{We} - \frac{1}{We} \left\{ 2\zeta + \frac{d}{d\mu} \left((1 - \mu^2) \frac{d\zeta}{d\mu} \right) \right\} \\
 (5.7) \quad (1 - \mu^2) \frac{d^2\zeta}{d\mu^2} - 2\mu \frac{d\zeta}{d\mu} + 2\zeta \\
 &= We P_2(\mu) \left[\frac{(3\bar{k} + 2)}{16(\bar{k} + 1)^2} \left\{ \frac{\{27\bar{k}^2 + 58\bar{k} + 4(3\bar{k} + 4)\lambda + (2/3)\bar{\lambda} + 30\}}{(5 + 5\bar{k} + 4\lambda + (8/3)\bar{\lambda})} \right\} \right. \\
 &\quad \left. - (\gamma - 1) \frac{P_2(\mu)}{12(\bar{k} + 1)^2} \right] + We(\Pi_1 - \hat{\Pi}_1)
 \end{aligned}$$

Now, because the condition (3.10) and (3.11), solution of the equation is

$$\begin{aligned}
 (5.8) \quad \zeta &= -\frac{We P_2(\mu)}{4(\bar{k} + 1)^2} \\
 &\quad \times \left[\left\{ \frac{\{81\bar{k}^3 + 228\bar{k}^2 + 206\bar{k} + 4\lambda(9\bar{k}^2 + 18\bar{k} + 8) + 2\bar{\lambda}(\bar{k} + 2/3) + 60\}}{16(5 + 5\bar{k} + 4\lambda + (8/3)\bar{\lambda})} \right\} + \frac{(1 - \gamma)}{12} \right]
 \end{aligned}$$

The above expression is seen to reduce to that obtained by Taylors and Acrivos¹ for $\lambda = \bar{\lambda} = 0$. Further, the effect of surface shear viscosity and surfacedilational viscosity is seen to accentuate the deformation.

Further, it may be evaluated from (5.8) that for a very viscous drop ($k \rightarrow \infty$), the deformation ζ remains unaffected at the value $-0.25We P_2(\mu)$. But for the case of a small gas bubble ($k \rightarrow 0, \gamma \rightarrow 0$), we have :

$$\zeta = -[0.21 - (0.05\lambda + 0.07\bar{\lambda})] We p_2(\mu).$$

References

1. T. D. Taylor and A. Acrivos, On the deformation and drag of a falling viscous drop at low Reynolds number, *J. Fluid Mech.*, **18** (1964) 466.
2. A. S. Bringnell, The deformation of a liquid drop at small Reynolds number, *Quart. J. Mech. Appl. Maths.*, **26** (1973) 99.

