

Applications of Differential Subordination to the Starlikeness and Convexity Conditions of Certain Integral Operator

P. K. Banerji and G. M. Shenan

Department of Mathematics, Faculty of Science, J. N. V. University, Jodhpur 342 005

(Received June 14, 2002)

Abstract : In the present paper, using the concept of the differential subordination, some conditions are established for certain integral operator, which is analytic in the unit disc, to be starlike convex, α -convex or be in other subclasses of analytic functions.

1. Introduction

Let $\mathcal{S}$ denotes the class of function $f(z)$, normalized by

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which is analytic and univalent in the open disc $\mathcal{U} = \{z : |z| < 1\}$. A function $f(z) \in \mathcal{S}$ is said to be *starlike* (with respect to the origin) in $\mathcal{U}$, denoted by $\mathcal{S}^*$, if and only if

$$(1.2) \quad \operatorname{Re} \left[\frac{zf'(z)}{f(z)} \right] > 0, \quad z \in \mathcal{U}$$

and is called *convex* in $\mathcal{U}$, denoted by K , if and only if

$$(1.3) \quad \operatorname{Re} \left[1 + \frac{zf''(z)}{f'(z)} \right] > 0, \quad z \in \mathcal{U}.$$

Mocanu¹ studied linear combinations of representations of convex and starlike functions and defined the class of α -convex functions which satisfy

$$(1.4) \quad \operatorname{Re} \left[\alpha \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} + (1-\alpha) \frac{zf'(z)}{f(z)} \right] > 0, \quad z \in \mathcal{U}, \quad -\infty < \alpha < \infty.$$

It is shown in Miller *et. al.*,² that, f is starlike for α real, and convex for $\alpha \geq 1$.

For the function $f(z) \in \mathcal{S}$, the integral operator $I_c[f(z)]$ is defined by

$$(1.5) \quad I_c[f(z)] = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt, \quad c > -1$$

$$(1.6) \quad = z + \sum_{n=2}^{\infty} b_n z^n.$$

The operator I_c for $c \in \mathbb{N}$, was introduced by Bernardi³. In particular, the operator I_1 was studied by Libera⁴ and others.

Let $f(z)$ and $g(z)$ be analytic in the unit disc $\mathcal{U}$, then the function $f(z)$ is said to be subordinate to $g(z)$, written as $f(z) \prec g(z)$, if there exists a function $\omega(z)$ analytic in $\mathcal{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathcal{U}$, such that $f(z) = g(\omega(z))$, $z \in \mathcal{U}$. In the present paper, instead of (1.4), we consider the class using the notation of subordination in the following form

$$(1.7) \quad \alpha \left[1 + \frac{zf''(z)}{f'(z)} \right] + (1-\alpha) \frac{zf'(z)}{f(z)} \prec 1 + \lambda z, \quad z \in \mathcal{U}$$

for $\alpha, \lambda > 0$, $f(z) \in \mathcal{S}$. We have obtained certain conditions for the operator $I_c[f(z)]$ to be starlike convex or be in some other classes. Following Lemma [cf. Jack⁵] will be useful in our analysis.

Lemma 1. Let ω be non-constant and analytic in $\mathcal{U}$ with $\omega(0) = 0$. If ω attains its maximum value on the circle $|z| = r$ at z_0 , then $z_0 \omega'(z_0) = K \omega(z_0)$, $K \geq 1$.

2. Main Results

Some theorems constitute this section which are justify our proposed inverse. For first two theorems, we state and prove a lemma.

Lemma 2. Let $P(z) = 1 + P_1(z) + P_2(z^2) + \dots$ be analytic in $\mathcal{U}$ and satisfy the condition

$$(2.1) \quad P(z) + \frac{1}{c+1} zP'(z) \prec 1 + \lambda z$$

for some $c > -1$ and $\lambda > 0$, then

$$(2.2) \quad P(z) \prec 1 + \left(\frac{c+1}{c+2} \right) \lambda z$$

Proof. Let us set

$$(2.3) \quad P(z) = 1 + \left(\frac{c+1}{c+2} \right) \lambda \omega(z)$$

then $\omega(z)$ is analytic in $\mathcal{U}$ with $\omega(0) = 0$. We are to show that $|\omega(z)| < 1$, $z \in \mathcal{U}$. If not by the Lemma 1, there exists $z_0 \in \mathcal{U}$ such that $|\omega(z_0)| = 1$ and $z_0 \omega'(z_0) = K \omega(z_0)$, $K \geq 1$.

Thus, we have

$$\begin{aligned}
 & \left| P(z_0) + \frac{1}{c+1} z P'(z_0) - 1 \right| \\
 &= \left| 1 + \left(\frac{c+1}{c+2} \right) \lambda \omega(z_0) + \left(\frac{c+1}{c+2} \right) \lambda \left(\frac{1}{c+1} \right) K \omega(z_0) - 1 \right| \\
 &= \left(\frac{c+1}{c+2} \right) \lambda \left| 1 + \frac{K}{c+1} \right| \\
 &\geq \left(\frac{c+1}{c+2} \right) \left(1 + \frac{1}{c+1} \right) \lambda = \lambda
 \end{aligned}$$

which contradicts (2.1) and thus $|\omega(z)| < 1$, $z \in \mathcal{U}$, and this suggest that (2.2) is obtained from (2.3)

Theorem 1. Let $f(z) \in \mathcal{S}$, $z \in \mathcal{U}$ satisfies the condition

$$(2.4) \quad \frac{f(z)}{z} \prec 1 + \lambda z, \quad \lambda > 0$$

then

$$(2.5) \quad \frac{I_c(f(z))}{z} \prec 1 + \left(\frac{c+1}{c+2} \right) \lambda z, \quad c > -1$$

Proof. From (1.5) we can mention

$$(2.6) \quad z \left[I_c(f(z)) \right]' + c I_c(f(z)) = (c+1)f(z)$$

Let us write

$$P(z) = \frac{I_c(f(z))}{z},$$

then $P(z) = 1 + b_1 z + b_2 z^2 + \dots$ is analytic in $\mathcal{U}$ and (2.6) is equivalent to

$$(2.7) \quad P(z) + \frac{1}{c+1} z P'(z) = \frac{f(z)}{z}$$

Now, from (2.4) and the Lemma 2, we obtain

$$P(z) = \frac{I_c(f(z))}{z} \prec 1 + \left(\frac{c+1}{c+2} \right) \lambda z$$

This completes the proof of the theorem.

Following cases may arises from this theorem :

(I) For some $0 \leq \alpha < 1$, if $\lambda = (1 - \alpha) \left(\frac{c+2}{c+1} \right)$, then we may state that, if $f(z) \in \mathcal{S}$ satisfies

$$\frac{f(z)}{z} \prec 1 + (1 - \alpha) \left(\frac{c+2}{c+1} \right) z, \quad \lambda > 0$$

then

$$(2.8) \quad \operatorname{Re} \left[\frac{I_c(f(z))}{z} \right] > \alpha, \quad 0 \leq \alpha < 1$$

(II) Let $\lambda = 1 - \alpha$ ($0 \leq \alpha < 1$), we come to state that, if $f(z) \in \mathcal{S}$, $z \in \mathcal{U}$ satisfy

$$\operatorname{Re} \left[\frac{f(z)}{z} \right] > \alpha$$

then

$$(2.9) \quad \frac{I_c(f(z))}{z} \prec 1 + (1 - \alpha) \left(\frac{c+1}{c+2} \right) z, \quad c > -1, \quad 0 \leq \alpha < 1$$

Theorem 2. Let $f(z) \in \mathcal{S}$, $z \in \mathcal{U}$ satisfy the condition

$$(2.10) \quad f'(z) \prec 1 + \lambda z, \quad \lambda > 0$$

then

$$(2.11) \quad [I_c f(z)]' \prec 1 + \left(\frac{c+1}{c+2} \right) \lambda z$$

Proof. Differentiate (2.6) with respect to z , we have

$$(2.12) \quad z [I_c(f(z))]'' + (c+1) [I_c(f(z))]' = (c+1) f'(z)$$

If we set $[I_c(f(z))]' = P(z)$ then $P(z) = 1 + c_1 z + c_2 z^2 + \dots$ is analytic in $\mathcal{U}$ and (2.12) is equivalent to

$$(2.13) \quad P(z) + \frac{1}{c+1} z P'(z) = f'(z)$$

Finally, using (2.10) and the lemma 2, the theorem is established.

If we set $\lambda = (c+2)/(c+1)$, the above theorem allows $f(z) \in \mathcal{S}$ to satisfy the condition

$$f'(z) \prec 1 + \left(\frac{c+2}{c+1} \right) z$$

and then $I_c(f(z))$ is univalent in $\mathcal{U}$.

Now, let us state and prove another lemma, in order to establish two more theorems.

Lemma 3. Let $P(z) = 1 + P_1 z + P_2 z^2 + \dots$ be analytic in $\mathcal{U}$ and satisfy the condition

$$(2.14) \quad P(z) + \frac{zP'(z)}{P(z) + c} \prec 1 + \lambda z, \quad c > -1, \quad 0 < \lambda \leq \frac{c+1}{c+2}$$

then

$$(2.15) \quad P(z) \prec 1 + \lambda_1 z$$

where

$$(2.16) \quad \lambda_1 = \left(\frac{c+2}{c+1} \right) \lambda$$

Proof. We put $P(z) = 1 + \lambda_1 \omega(z)$ where λ_1 is given by (2.16). Then $\omega(z)$ is analytic in $\mathcal{U}$ with $\omega(0) = 0$, and we have to show that $|\omega(z)| < 1$ ($z \in \mathcal{U}$). If not by lemma 1, there exists $z_0 \in \mathcal{U}$, such that $|\omega(z_0)| = 1$ and $z_0 \omega'(z_0) = K\omega(z_0)$, $K \geq 1$, we then have

$$\begin{aligned} & \left| P(z_0) + \frac{z_0 P'(z_0)}{P(z_0) + c} - 1 \right| \\ &= \frac{|z\lambda_1\omega(z_0) + \lambda_1^2\omega^2(z_0) + (c-1)\lambda\omega(z_0) + \lambda_1 K\omega(z_0)|}{|(1+c) + \lambda_1\omega(z_0)|} \\ &= \frac{\lambda_1 |(c+1+K) + \lambda_1\omega(z_0)|}{|(1+c) + \lambda_1\omega(z_0)|} \\ &\geq \frac{\lambda_1 ||1+c+K| - \lambda_1|}{\lambda_1 + (1+c)} \\ &\geq \frac{\lambda_1 |c+2-\lambda_1|}{1+c+\lambda_1} \geq \lambda \end{aligned}$$

But this contradicts (2.14) and thus, $|\omega(z)| < 1$, $z \in \mathcal{U}$. Thus, the lemma is proved.

Theorem 3. If $f(z) \in \delta$ satisfies the condition

$$(2.17) \quad \frac{zf'(z)}{f(z)} \prec 1 + \lambda z$$

for $0 < \lambda \leq \frac{c+1}{c+2}$, $c > -1$, then

$$(2.18) \quad \frac{z[I_c(f(z))']}{I_c[f(z)]} \prec 1 + \lambda_1 z$$

where λ_1 is given by (2.16)

Proof. From (2.6), we have

$$(2.19) \quad I_c[f(z)] \left[\frac{z \{I_c(f(z))\}'}{I_c(f(z))} + c \right] = (c + 1)f(z)$$

If we put

$$P(z) = \frac{z[I_c(f(z))]' }{I_c(f(z))}$$

then $P(z) = 1 + d_1 z + d_2 z^2 + \dots$ is analytic in $\mathcal{U}$, and thus from (2.19) we obtain

$$(2.20) \quad I_c(f(z)) [P(z) + c] = (c + 1)f(z)$$

Now the logarithmic differentiation of (2.20), with respect to z , gives

$$(2.21) \quad P(z) + \frac{zP'(z)}{P(z) + c} = \frac{zf'(z)}{f(z)}$$

Using (2.18) and (2.21) and the Lemma 3, the required result is established. Following case is interesting.

(I) If $f(z) \in \mathcal{S}$ is starlike in $\mathcal{U}$, then $I_c(f(z))$ is also starlike in $\mathcal{U}$.

Proof. We choose $\lambda = (c + 1)/(c + 2)$, in theorem 3 (above theorem) so that for $\lambda < 1$, the condition (2.5) implies

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < \frac{c + 1}{c + 2} < 1$$

i.e.,

$$f(z) \in \mathcal{S}^*$$

and thus, (2.18) yields

$$\left| \frac{z[I_c(f(z))]'}{I_c(f(z))} - 1 \right| < 1$$

i.e.,

$$I_c(f(z)) \in \mathcal{S}^*$$

Theorem 4. If $f(z) \in \mathcal{S}$ satisfies the condition

$$(2.22) \quad 1 + \frac{zf''(z)}{f'(z)} \prec 1 + \lambda z$$

for $0 < \lambda < (c + 1)/(c + 2)$, $c > -1$

$$(2.23) \quad 1 + \frac{[zI_c(f(z))]'']{I_c(f(z))} \prec 1 + \lambda_1 z$$

where λ_1 is already defined.

Proof. (2.12) yields

$$(2.24) \quad [I_c(f(z))]' \left[\frac{z[I_c(f(z))]''}{[I_c(f(z))]' } + c + 1 \right] = (c + 1)f'(z)$$

If we put

$$(2.25) \quad P(z) = 1 + \frac{z[I_c(f(z))]''}{[I_c(f(z))]' }$$

then $P(z) = 1 + e_1 z + e_2 z^2 + \dots$ is analytic in $\mathcal{U}$, thus (2.24) yields

$$[I_c(f(z))]' [P(z) + c] = (c + 1)f'(z)$$

which on logarithmic differentiation yields

$$(2.26) \quad P(z) + \frac{zP'(z)}{P(z) + c} = \frac{zf'(z)}{f'(z)} + 1$$

The use of (2.22) in (2.26) and using Lemma 3, the theorem established.

Lemma 4. Let $P(z) = 1 + P_1 z + P_2 z^2 + \dots$ be analytic and satisfies the condition.

$$(2.27) \quad P(z) + \frac{\alpha zP'(z)}{P(z)} \prec 1 + \lambda z$$

where $\alpha > 0$ and $0 < \lambda \leq \alpha/2$, then

$$(2.28) \quad P(z) \prec 1 + \frac{2\lambda z}{\alpha}$$

Proof of this lemma is the same as that of lemma 3.

Theorem 5. Let $I_c(f(z)) \in \mathcal{S}$ satisfies (1.7), for some $\alpha > 0$ and $0 < \lambda \leq \alpha/2$, then

$$(2.29) \quad \frac{z[I_c(f(z))]' }{I_c(f(z))} \prec 1 + \frac{2\lambda}{\alpha} z$$

Proof. From (1.7) we obtain

$$(2.30) \quad \alpha \left[1 + \frac{z\{I_c(f(z))\}''}{\{I_c(f(z))\}' } \right] + (1 - \alpha) \frac{z[I_c(f(z))]' }{I_c(f(z))} \prec 1 + \lambda z$$

Now, if we set

$$P(z) = \frac{z[I_c(f(z))]' }{I_c(f(z))}$$

in (2.30), then $P(z) = 1 + d_1 z + d_2 z^2 + \dots$ is analytic in $\mathcal{U}$ and

$$1 + \frac{z[I_c(f(z))]' }{[I_c(f(z))]' } = P(z) + \frac{zP'(z)}{P(z)}$$

Hence, (2.30) is equivalent to

$$(2.31) \quad P(z) + \frac{\alpha z P'(z)}{P(z)} \prec 1 + \lambda z$$

Finally, using lemma 4, we obtain

$$P(z) = \frac{z[I_c(f(z))]' }{I_c(f(z))} \prec 1 + \frac{2\lambda}{\alpha} z$$

which completes the the proof of the theorem.

Following two consequences of this theorem are interesting

(I) Choose $\alpha = 2$ and $\lambda = 1$, we get from the theorem 5 the following : If $I_c(f(z)) \in \mathcal{S}$ is the α -convex function, then $I_c(f(z))$ is the convex function.

(II) Choosing $\lambda = 1/2$ and $\alpha = 1$, the theorem 5 permits us to state : If $I_c(f(z)) \in \mathcal{S}$ is a convex function of order $1/2$ in $\mathcal{U}$, then $I_c(f(z))$ is the starlike function in $\mathcal{U}$.

Acknowledgement

This work is supported by the Research Scheme of C. S. I. R. (India), No.25/0105/99/EMR-II.

References

1. P. T. Mocanu : Une propriété de convexité généralisée dans théorie de représentation conforme, *Math. Cluj*, **11**(34) (1969) 127-133.
2. S. S. Miller, P. T. Mocanu and M. O. Reade : All α -convex functions are univalent and starlike, *Proc. Amer. Math. Soc.*, **37** (1973) 553-554.
3. S. D. Bernardi : Convex and starlike univalent functions, *Trans. Amer. Math. Soc.*, **135** (1969) 429-440.
4. R. L. Libera : Some classes of regular univalent functions, *Proc. Amer. Math. Soc.*, **16** (1965) 755-758.
5. I. S. Jack : Functions starlike and convex of order α , *J. London Math. Soc.*, **2**(3) (1971) 469-474.