

Corrigendum to a Necessary and Sufficient Condition for Quasi-projectivity and Quasi-injectivity

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Abstract : An interesting pair of dual concept in the theory of modules is quasi-projective and quasi-injective module. The aim of this paper is to give a detailed corrected version of two propositions of Y. Miyashita¹.

1. Introduction

Quasi-projective and Quasi-injective modules were studied by many authors viz. Y. Miyashita¹, L. E. T. Wu and J. P. Jans², A. Koehler³, C. Faith and Y. Utumi⁴, C. Faith⁵, R. E. Johanson and E. T. Wong⁶, K. M. Rangaswami and N. Vanaja⁷ etc. Y. Miyashita¹ presented a very short proof of necessary and sufficient condition for quasi-projectivity and quasi-injectivity. However, the proof given in the paper of Y. Miyashita¹ has certain flaws, which have appeared due to cutting short of the proof. For an example; at many places the composite mapping do not seem to be having proper domain and range. Further in part (ii) $\Rightarrow$ (i) of proposition 2.1 mapping v in the last line is assumed to be from M to M/A which should be inject v from B to B/A . Further in the statement (ii) of proposition 2.2 the composition $\phi \cdot \psi$ is not identify on A but is injection map i from A into M . In this paper we give detailed corrected version of the proofs of the two propositions of Y. Miyashita¹ for quasi-projective and quasi-injective modules.

2. Preliminaries

Throughout this paper R denotes a ring with unity and all the modules considered are unitary left R -modules. The symbol " $\rightarrow$ " means an epimorphism and the symbol " $\hookrightarrow$ " means a monomorphism. We use the notation $(x) f$ to denote the image of an element x under f . For any R -submodule A of M , we denote by $v_{M/A}$ and i_A the projection of M onto M/A and the injection of A into M respectively.

Proposition 1 : The following are equivalent :

(i) M is R -quasi-projective.

(ii) For any submodules A, B of M and any epimorphism $\phi : A \rightarrow M/B$, there exists a homomorphism $\psi : M \rightarrow A$ with $\psi \cdot \phi = \nu_{M/B}$.

Proof : (i) $\Rightarrow$ (ii) Let $\phi : A \rightarrow M/B$ be given by (a) $\phi = m + B$ ($a \in A$ and $m \in M$) then ϕ induces an isomorphism $\bar{\phi} : A/\text{Ker } \phi \cong M/B$ given by $(a + \text{Ker } \phi) \bar{\phi} = (a) \phi = m + B$. Further, we have a homomorphism

$$\bar{\phi}^* = \nu_{M/B} (: M \rightarrow M/B) \cdot \bar{\phi}^{-1} (M/B \rightarrow A/\text{Ker } \phi)$$

$$\cdot \bar{i}_A (: A/\text{Ker } \phi \rightarrow M/\text{Ker } \phi) : M \rightarrow M/\text{Ker } \phi$$

where $\bar{i}_A$ is induced by the natural monomorphism $i_A : A \rightarrow M$.

Then by (i), there exists a homomorphism $\psi : M \rightarrow A$ such that the following diagram is commutative :

$$\begin{array}{ccccc}
 & & M & & \\
 & & \downarrow \nu_{M/B} & & \downarrow \phi^* \\
 & & M/B & & \\
 & & \downarrow \bar{\phi}^{-1} & & \\
 & & A/\text{Ker } \phi & & \\
 & & \downarrow \bar{i}_A & & \\
 M & \xrightarrow{\psi} & M/\text{Ker } \phi & \xrightarrow{\nu_{M/\text{Ker } \phi}} & 0
 \end{array}$$

We observe that

$$(m) \phi^* = a + \text{Ker } \phi$$

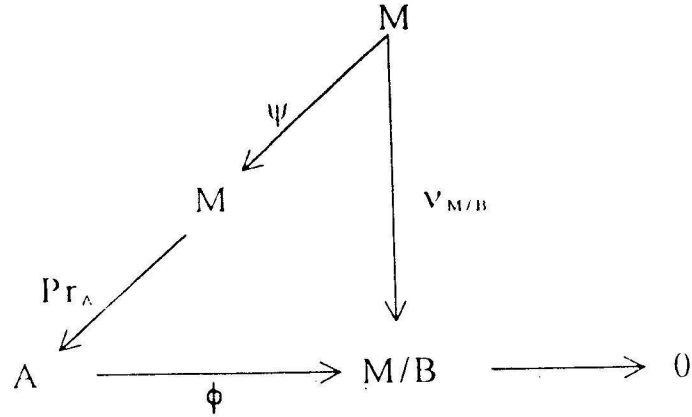
where $a \in A$ is such that $(a) \phi = m + B$ and $(a + \text{Ker } \phi) \bar{\phi} = m + B$.

$$\text{Now } (m) \phi^* = a + \text{Ker } \phi$$

$$\Rightarrow (m) \psi \cdot \nu_{M/\text{Ker } \phi} = a + \text{Ker } \phi.$$

So we must have $(m) \psi = a$.

Then we get the following diagram :



with $\psi \cdot \text{Pr}_A \cdot \phi = \nu_{M/B}$.

Here $\psi : M \rightarrow M$ by virtue of definition can be treated as

$$\psi : M \rightarrow A.$$

Thus in fact $\psi \cdot \phi = \nu_{M/B}$.

(ii) $\Rightarrow$ (i) Let $\theta : M \rightarrow M/A$ be any mapping. We can find a submodule B of M containing A such that $\phi : M \rightarrow B/A$ given by $(m) \phi = (m) \theta$ is onto and $\theta = \phi \cdot \bar{i}_{B/A}$ where $\bar{i}_{B/A} : B/A \rightarrow M/A$ is induced the natural injection $i_B : B \rightarrow M$.

Now ϕ induces an isomorphism

$$\bar{\phi} : M/\text{Ker } \phi \cong B/A$$

given by

$$(m + \text{Ker } \phi) \bar{\phi} = (m) \phi = b + A \text{ (say).}$$

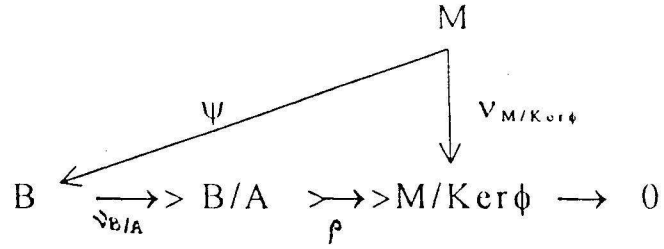
Therefore,

$$\bar{\phi}^{-1} = \rho \text{ (say) : } B/A \rightarrow M/\text{Ker } \phi$$

is such that

$$(b + A) \rho = m + \text{Ker } \phi.$$

Again by (ii) there exists a homomorphism $\psi : M \rightarrow B$ such that the following diagram is commutative.



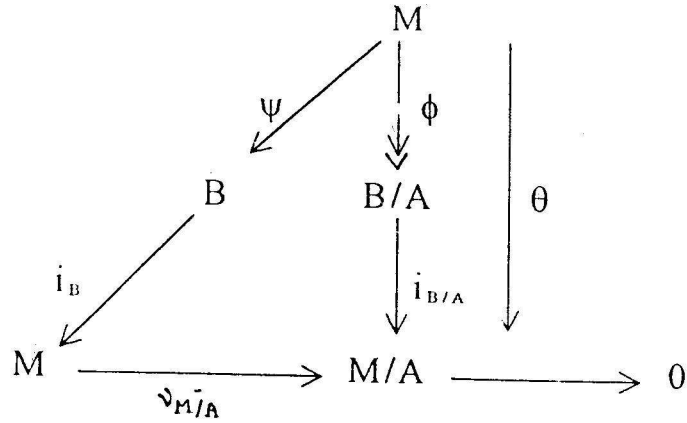
i.e. $\psi \cdot v_{B/A} \cdot \rho = v_{M/Ker \phi} \cdot \rho$

But then $(m) \psi \cdot v_{B/A} \cdot \rho = (m) v_{M/Ker \phi} \cdot \rho$

i.e. $(m) \psi \cdot v_{B/A} \cdot \rho = m + Ker \phi \quad \forall m \in M.$

Therefore, we conclude $(m) \psi = b \in B$, where $b \in B$ is such that $(m)\phi = b + A$.

Now consider the following diagram :



For any $m \in M$ we have

$$(m) \phi \cdot i_{B/A} = (b + A) \cdot i_{B/A}$$

i.e. $(m) \theta = (b + A).$

Again, we have

$$\begin{aligned}
 (m) \psi \cdot i_B \cdot v_{M/A} &= (b) i_B \cdot v_{M/A} \\
 &= (b) v_{M/A} \\
 &= (b + A)
 \end{aligned}$$

which proves the commutativity of the above diagram and completes the proof.

Proposition 2 : The followings are equivalent :

- (i) M is R -quasi-injective.
- (ii) For any submodules A, B of M and any monomorphism $\phi : A \rightarrow M/B$, there exists a homomorphism $\psi : M/B \rightarrow M$ with $\phi \cdot \psi = i_A$.

Proof : (i) $\Rightarrow$ (ii) Let $C \supseteq B$ be a submodule of M such that $\phi_1 : A \rightarrow C/B$ be one onto and $(a)\phi_1 = (a)\phi \ \forall a \in A$.

Then we have a homomorphism

$$\phi^* = v_{C/B} (: C \rightarrow C/B) \cdot \phi_1^{-1} (: C/B \rightarrow A) \cdot i_A (A \rightarrow M) : C \rightarrow M.$$

So by (i), there exists a homomorphism $\psi : M \rightarrow M$ such that the following diagram is commutative :

$$\begin{array}{ccccc}
 0 & \longrightarrow & C & \xrightarrow{i_C} & M \\
 & & \downarrow v_{C/B} & & \searrow \psi \\
 & & C/B & & \\
 & \phi^* \downarrow & \downarrow \phi_1 \cdot i_1 & & \\
 & & A & & \\
 & & \downarrow i_A & & \\
 & & M & &
 \end{array}$$

i.e. $i_C \cdot \psi = \phi^*.$

Since $B \subseteq C$ for any $b \in B$, we have

$$\begin{aligned}
 (b)\psi &= (b)i_C \cdot \psi \\
 &= (b)\phi^* \\
 &= (b)v_{C/B} \cdot \phi_1^{-1} \cdot i_A \\
 &= 0.
 \end{aligned}$$

Thus, $B_\psi = 0$ and so ψ induces a homomorphism

$$\bar{\psi} : M/B \rightarrow M \text{ given by } (m + B) \bar{\psi} = (m) \psi.$$

We show that $\bar{\psi}$ is the required map ψ .

Since $\phi_1^{-1} : C/B \rightarrow A$ is given by $(c + B) \phi_1^{-1} = a$ where $a \in A$ is such that $(a) \phi = c + B$, we have

$$\begin{aligned} (a) \phi \cdot \bar{\psi} &= (c + B) \bar{\psi} \\ &= (c) \psi \\ &= (c) i_C \cdot \psi \\ &= (c) \phi^* \\ &= (c) v_{C/B} \cdot \phi_1^{-1} \cdot i_A \\ &= (c + B) \phi_1^{-1} \cdot i_A \\ &= (a) i_A. \end{aligned}$$

Thus, $\phi \cdot \bar{\psi} = i_A$.

(ii) $\Rightarrow$ (i) $\phi : A \rightarrow M$ be any mapping where A is any submodule of M then there exists a monomorphism

$$\sigma : A\phi \rightarrow M/\text{Ker } \phi$$

given by

$$(a\phi) \sigma = a + \text{Ker } \phi.$$

So by (ii), there exists a homomorphism $\psi : M/\text{Ker } \phi \rightarrow M$ such that the following diagram is commutative :

$$\begin{array}{ccccc} 0 & \longrightarrow & A\phi & \xrightarrow{\sigma} & M/\text{Ker } \phi \\ & & \downarrow i_{A\phi} & \searrow \psi & \\ & & M & & \end{array}$$

i.e. $\sigma \cdot \psi = i_{A\phi}$

Finally, we show that $v_{M/Ker \phi} \cdot \psi$ is the required map.

We have

$$\begin{aligned}
 (a) i_A v_{M/Ker \phi} \cdot \psi &= (a) v_{M/Ker \phi} \cdot \psi \\
 &= (a + Ker \phi) \psi \\
 &= (a \phi) \sigma \cdot \psi \\
 &= (a \phi) i_{A \phi} \\
 &= (a) \phi.
 \end{aligned}$$

This shows that M is R -quasi-injective and completes the proof.

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References

1. Y. Miyashita : Quasi-projective modules, perfect modules and a theorem for modular lattices, *J. Fac. Sci. Hokkaido University (I)*, **19** (1966) 86-110.
2. L. E. T. Wu and J. P. Jans : On Quasi-projectives, *Illinois J. Math.*, **11** (1967) 439-448.
3. A. Koehler : Quasi-projective and Quasi-injective modules, *Pac. J. Math.*, **36**(3) (1971) 713-720.
4. C. Faith and Y. Utumi : Quasi-injective modules and their endomorphism rings, *Arch. Math.*, **15** (1954) 166-174.
5. C. Faith : Lectures on injective modules and quotient rings, Lecture notes from Rutgers University, New Brunswick, N. J. (1965).
6. R. E. Johnson and E. T. Wong : Quasi-injective modules and irreducible rings, *J. London Math. Soc.*, **36** (1961) 260-268.
7. K. M. Rangawami and N. Vanaja : Quasi-projective abelian and module categories, *Pac. J. Math.*, **43** (1972) 221-238.

