Conformal $\beta$-Change of Finsler Metric

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Abstract: The purpose of the present paper is to find the necessary and sufficient conditions under which a conformal $\beta$-change of Finsler metric becomes a projective change. We have also found a condition under which a conformal $\beta$-change of Finsler metric leads a Douglas space into a Douglas space.

Keywords: Finsler Space, Finsler metric, conformal $\beta$-change, projective change, Douglas space.

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1. Introduction

Let $F^n=(M^n, L)$ be an $n$-dimensional Finsler space on the differentiable manifold $M^n$, equipped with the fundamental function $L(x,y)$. B. N. Prasad and Bindu Kumari¹ and C. Shibata² have considered the $\beta$-change of Finsler metric given by

$$L'(x,y) = f(L, \beta),$$

where $f$ is positively homogeneous function of degree one in $L$ and $\beta$, where $\beta$ given by

$$\beta(x,y) = b_i(x)y^i$$

is a one-form on $M^n$.

The conformal theory of Finsler space was initiated by M. S. Knebelman³ in 1929 and has been investigated in detail by many authors (Hashiguchi⁴, Izumi⁵,⁶ and Kitayama⁷). The conformal change is defined as
\[ L^*(x, y) \rightarrow e^{\sigma(x)} L(x, y), \]

where \( \sigma(x) \) is a function of position only and known as conformal factor.

In this paper we have combined the above two changes and have introduced another Finsler metric defined as

\[ (1.1) \quad \overline{L}(x, y) = e^{\sigma} f(L, \beta), \]

where \( \sigma(x) \) is a function of \( x \) and \( \beta(x, y) = b_i(x) y^i \) is a 1-form on \( M^n \).

This conformal change of \((L, \beta)\)-metric will be called as conformal \( \beta \)-change of Finsler metric. When \( \sigma = 0 \), it reduces to a \( \beta \)-change. When \( \sigma \) is constant, it becomes a homothetic \( \beta \)-change. When \( f(L, \beta) \) has special forms as \( L + \beta, \frac{L^2}{L - \beta}, \frac{L^2}{\beta^m} (m \neq 0, -1) \), we get conformal Randers change, conformal Matsumoto change, conformal Kropina change, conformal generalized Kropina change of Finsler metric respectively. The Finsler space equipped with the metric \( \overline{L} \) given by (1.1) will be denoted by \( \overline{F}^n \). Throughout the paper the quantities corresponding to \( \overline{F}^n \) will be denoted by putting bar on the top of them.

The fundamental quantities of \( F^n \) are given by

\[ g_{ij} = \frac{1}{2} \partial^2 L^2, \quad \ell = \frac{\partial L}{\partial y^i} \quad \text{and} \quad h_{ij} = L \frac{\partial^2 L^2}{\partial y^i \partial y^j} = g_{ij} - l_i l_j. \]

We shall denote the partial derivatives with respect to \( x^i \) and \( y^i \) by \( \partial_i \) and \( \partial_i^\prime \) respectively and write

\[ L_i = \partial_i L, \quad L_{ij} = \partial_i ^\prime \partial_j L, \quad L_{ijk} = \partial_k ^\prime \partial_j ^\prime \partial_i L. \]

Then \( L_i = l_i, \quad L^{-1} h_{ij} = L_{ij} \). The geodesics of \( F^n \) are given by the system of differential equations

\[ \frac{d^2 x^i}{ds^2} + 2G^i \left( x, \frac{dx}{ds} \right) = 0, \]

where \( G^i(x, y) \) are positively homogeneous of degree two in \( y^i \) and are given by

\[ 2G^i = g^{ij} \left( y^r \partial_j ^\prime \partial_r F - \partial_j F \right), \quad F = \frac{L^2}{2}. \]
where \( g^{ij} \) are the inverse of \( g_{ij} \).

Berwald connection \( B\Gamma=(G^i_{jk},G^i_j,0) \) of Finsler space is given by:

\[
G^i_j = \frac{\partial G^i_j}{\partial y^j}, \quad G^i_{jk} = \frac{\partial G^i_j}{\partial y^k}.
\]

The Cartan’s connection \( (F^i_{jk},G^i_j,C^i_{jk}) \) is constructed from the metric function \( L \) with the help of following axioms:

1. Cartan’s connection \( C\Gamma \) is \( v \)-metrical.
2. Cartan’s connection \( C\Gamma \) is \( h \)-metrical.
3. The \( (v) \)\( v \)-torsion tensor field \( S \) of Cartan’s connection vanishes.
4. The \( (h) \)\( h \)-torsion tensor field \( T \) of Cartan’s connection vanishes.
5. The deflection tensor field \( D \) of Cartan’s connection vanishes.

The \( h \) - and \( v \) - covariant derivatives with respect to Cartan’s connection are denoted by \( |_k \) and \( |_k \) respectively. It is clear that the \( h \)-covariant derivative of \( L \) with respect to \( B\Gamma \) and \( C\Gamma \) is the same and vanishes identically. Further-more, the \( h \)-covariant derivatives of \( L, L_{ij} \) with respect to \( C\Gamma \) are also zero. We shall write

\[
2r_{ij} = b_{ij} + b_{ji}, \quad 2s_{ij} = b_{ij} - b_{ji}.
\]

### 2. Difference Tensor of Conformal \( \beta \)-Change

The conformal \( \beta \)-change of Finsler metric \( L \) is given by

\[
\bar{L}(x,y)=e^{\sigma f} L(x,y),
\]

where \( f \) is positively homogeneous function of degree one in \( L \) and \( \beta \). Homogeneity of \( f \) gives

\[
L f_1 + \beta f_2 = f,
\]

where subscripts “1” and “2” denote the partial derivatives with respect to \( L \) and \( \beta \) respectively.

Differentiating above equations with respect to \( L \) and \( \beta \) respectively, we get
\[ L_{f_{12}} + \beta f_{22} = 0 \quad \text{and} \quad L_{f_{11}} + \beta f_{21} = 0. \]

Hence, we have
\[ \frac{f_{11}}{\beta^2} - \frac{f_{12}}{L \beta} = \frac{f_{22}}{L^2}, \]
which gives
\[ f_{11} = \beta^2 \omega, \quad f_{12} = -L \beta \omega, \quad f_{22} = L^2 \omega, \]
where Weierstrass function \( \omega \) is positively homogeneous of degree-3 in \( L \) and \( \beta \). Therefore
\[ L \omega_1 + \beta \omega_2 + 3 \omega = 0, \]
where \( \omega_1 \) and \( \omega_2 \) are positively homogeneous of degree – 4 in \( L \) and \( \beta \).

Throughout the paper we frequently use the above equations without quoting them. Also we have assumed that \( f \) is not a linear function of \( L \) and \( \beta \) so that \( \omega \neq 0 \). We now put

\[ 2.1 \quad \overline{G}^i = G^i + D^i. \]

Then \( \overline{G}^i_j = G^i_j + D^i_j \) and \( \overline{G}^i_{jk} = G^i_{jk} + D^i_{jk} \), where \( D^i_j = \hat{\partial}_j D^i \) and \( D^i_{jk} = \hat{\partial}_k D^i_j \).

The tensors \( D^i \), \( D^i_j \) and \( D^i_{jk} \) are positively homogeneous in \( y^i \) of degree two, one and zero respectively. To find \( D^i \) we deal with equation \( L_{ij \, lk} = 0 \), i.e.,

\[ 2.2 \quad \hat{\partial}_k L_{ij} - L_{ij \, ,k}, G^r_{k} - L_{ij} F_{ik}^r - L_{ik} F_{jk}^r = 0. \]

Since \( \hat{\partial}_i \beta = b_1 \), from (1.1), we have

\[ 2.3 \quad \overline{L}_i = e^i \left( f_i L_i + f_2 b_i \right), \]

\[ \overline{L}_{ij} = e^{[i} \left[ f_i L_{ij} + \beta^2 \omega L_i L_j - L \beta \omega (L_i b_j + L_j b_i) + L^2 b_i b_j \right]. \]
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$$\bar{L}_{ijk} = e^\sigma \left[ f_i L_{ijk} + \beta^2 \omega \left( L_i L_{jk} + L_j L_{ik} + L_k L_{ij} \right) - L \beta \omega \left( b_i L_{jk} + b_j L_{ik} + b_k L_{ij} \right) \right],$$

where $\sigma_k = \frac{\partial \sigma}{\partial x^k}$.

Since $\bar{L}_{ij|k} = 0$ in $\bar{F}^n$, after using (2.1), we have

$$\partial_k \bar{L}_{ij} - \bar{L}_{ijr} \tilde{G}^r_k - \bar{L}_{ij} \bar{F}^r_{ik} - L_{ir} \bar{F}^r_{jk} = 0.$$ 

Substituting in the above equation the values of $\partial_k \bar{L}_{ij}$, $\bar{L}_{ir}$ and $\bar{L}_{ijk}$ from (2.3) in (2.4) and then contracting the equation thus obtained with $\gamma^k$, we get

$$2 \bar{L}_{ijr} D^r + f_{ij} \bar{D}^r + \bar{L}_{ijr} D^r - \omega (L^2 b_j - L \beta b_j) (r_{i0} + s_{i0})
- \omega (L^2 b_j - L \beta b_j) (r_{j0} + s_{j0}) - \left\{ -L \beta \omega L_{ij} + \beta (2 \omega + \beta \omega_2) L_1 L_j \right\} r_{00}
+ \left\{ f_i L_{ij} + \beta^2 \omega L_i L_j - L \beta \omega (L_i b_j + L_j b_i) + L^2 \omega b_i b_j \right\} r_0 = 0,$$
where \( 0' \) stands for contraction with \( y^k, \) viz., \( r_{j0} = r_{jk} y^k, \) \( r_{00} = r_{jk} y^j y^k, \) \( \sigma_0 = \sigma_i y^i \)
and we have used the fact that \( D'_{jk} y^k = D'_{jk} = D'_{ij} \), where \( D'_{jk} = F'_{jk} - F'_{jk}. \)
Next, we deal with \( L_{ij} = 0, \) that is,

\[
\partial_j \bar{L}_{ij} - \bar{L}_{ij} \bar{G}'_{ij} - \bar{L}_{r} \bar{F}'_{ij} = 0.
\]

Putting the values of \( \partial_j \bar{L}_{ij}, \) \( \bar{L}_{ir} \) and \( \bar{L}_r \) from (2.3) in (2.6) we get,

\[
f_{2} b_{ij} = \left\{ f_i L_{ir} + \beta^2 \omega L_{ir} - L \beta \omega (L_i b_r + L_r b_i) + L^2 \omega b_r b_i \right\} D'_j \\
+ \left\{ f_i L_{jr} + \beta^2 \omega L_{jr} - L \beta \omega (L_j b_r + L_r b_j) + L^2 \omega b_r b_j \right\} D'_i \\
- \left( L^2 \omega b_i - L \beta \omega L_i \right) (r_{0j} + s_{0j}) - \left( L^2 \omega b_j - L \beta \omega L_j \right) (r_{0i} + s_{0i}) \\
- (f_i L_{jr} + f_j L_{ir} + f_{2j} b_i) \sigma_j - (f_i L_{ir} + f_j L_{jr} + f_{2i} b_j) \sigma_i + 2(f_i L_{ir} + f_{2j} b_j) D'_{ij},
\]

hence after using \( 2 r_{ij} = b_{ijl} + b_{jli} \) and \( 2 s_{ij} = b_{ijl} - b_{jli}, \) we get

\[
2 f_{2} r_{ij} = \left\{ f_i L_{ir} + \beta^2 \omega L_{ir} - L \beta \omega (L_i b_r + L_r b_i) + L^2 \omega b_r b_i \right\} D'_j \\
+ \left\{ f_i L_{jr} + \beta^2 \omega L_{jr} - L \beta \omega (L_j b_r + L_r b_j) + L^2 \omega b_r b_j \right\} D'_i \\
- \left( L^2 \omega b_i - L \beta \omega L_i \right) (r_{0j} + s_{0j}) - \left( L^2 \omega b_j - L \beta \omega L_j \right) (r_{0i} + s_{0i}) \\
- (f_i L_{jr} + f_{2j} b_i) \sigma_j - (f_i L_{ir} + f_{2i} b_j) \sigma_i + 2(f_i L_{jr} + f_{2j} b_j) D'_{ij},
\]

\[
2 f_{2} s_{ij} = \left\{ f_i L_{ir} + \beta^2 \omega L_{ir} - L \beta \omega (L_i b_r + L_r b_i) + L^2 \omega b_r b_i \right\} D'_j \\
+ \left\{ f_i L_{jr} + \beta^2 \omega L_{jr} - L \beta \omega (L_j b_r + L_r b_j) + L^2 \omega b_r b_j \right\} D'_i \\
- \left( L^2 \omega b_i - L \beta \omega L_i \right) (r_{0j} + s_{0j}) + \left( L^2 \omega b_j - L \beta \omega L_j \right) (r_{0i} + s_{0i}) \\
- (f_i L_{jr} + f_{2j} b_i) \sigma_j + (f_i L_{ir} + f_{2i} b_j) \sigma_i + 2(f_i L_{jr} + f_{2j} b_j) D'_{ij}.
\]

Subtracting (2.7) from (2.5) and contracting the resulting equation with \( y^i, \) we get

\[
2 f_{2} r_{ij} = \left\{ f_i L_{ir} + \beta^2 \omega L_{ir} - L \beta \omega (L_i b_r + L_r b_i) + L^2 \omega b_r b_i \right\} D'_j \\
+ \left\{ f_i L_{jr} + \beta^2 \omega L_{jr} - L \beta \omega (L_j b_r + L_r b_j) + L^2 \omega b_r b_j \right\} D'_i \\
- \left( L^2 \omega b_i - L \beta \omega L_i \right) (r_{0j} + s_{0j}) - \left( L^2 \omega b_j - L \beta \omega L_j \right) (r_{0i} + s_{0i}) \\
- (f_i L_{jr} + f_{2j} b_i) \sigma_j - (f_i L_{ir} + f_{2i} b_j) \sigma_i + 2(f_i L_{jr} + f_{2j} b_j) D'_{ij},
\]

Contracting (2.9) with \( y^i, \) we get
$$\{f_{i}L_{i}+f_{2}b_{i}\}D'=\frac{1}{2}(f_{2}r_{00}+f_{o}).$$  \hspace{1cm} (2.10)$$

Subtracting (2.8) from (2.5) and contracting the resulting equation with \(y^{j}\), we get

$$\left\{f_{i}L_{i}+\beta^{2}\omega L_{i}L_{r}-L\beta\omega(L_{i}b_{r}+L_{r}b_{i})+L^{2}\omega b_{r}b_{i}\right\}D'$$

$$=f_{2}s_{0}+\frac{1}{2}(L^{2}\omega b_{i}-L\beta\omega L_{i})r_{00}+L\beta\omega(L_{i}\beta-Lb_{i})y^{k}\sigma_{k}$$

$$+\frac{1}{2}(f_{2}L_{i}+f_{2}b_{i})\sigma_{0}-\frac{1}{2}f\sigma_{i}.$$  \hspace{1cm} (2.11)

In view of \(LL_{ir}=g_{ir}-L_{i}L_{r}\), equation (2.11) can be written as

$$\left\{f_{i}L_{i}g_{ir}D'+\left(-\frac{f_{i}}{L}+\beta^{2}\omega\right)L_{i}L_{r}\beta\omega b_{i}\right\}L_{r}D'+\left(L^{2}\omega b_{i}-L\beta\omega L_{i}\right)b_{r}D'$$

$$=f_{2}s_{0}+\frac{1}{2}(L^{2}\omega b_{i}-L\beta\omega L_{i})r_{00}+\frac{1}{2}(f_{2}L_{i}+f_{2}b_{i})\sigma_{0}-\frac{1}{2}f\sigma_{i}.$$  \hspace{1cm} (2.12)

Contracting (2.12) by \(b^{i}=g^{ij}b_{j}\), we get

$$\left\{-\frac{f_{i}L_{i}+\beta^{2}\omega}{L^{2}}-L\beta\omega L_{r}\right\}L_{r}D'+\left(-\frac{f_{i}}{L}+\beta^{2}\omega\right)L_{i}D'+\left(L^{2}\omega b_{i}-L\beta\omega L_{i}\right)\sigma_{0}-\frac{1}{2}f\sigma_{i}.$$  \hspace{1cm} (2.13)

where \(\Delta = b^{2} - \frac{\beta^{2}}{L^{2}}\) and \(\sigma_{1} = \sigma_{1}b^{i}\).

The equation (2.10) and (2.13) are algebraic equations in \(L_{r}D'\) and \(b_{r}D'\), whose solution is given by

$$\frac{(f_{i}f_{2}\beta+L^{2}\omega f\Delta)r_{00}+2f_{i}f_{2}L^{2}s_{00}+\left\{\beta(f_{i}+L^{3}\omega\Delta)\right\}\sigma_{0}-ff_{2}L\sigma_{1}}{2f\left(f_{i}+L^{3}\omega\Delta\right)}$$  \hspace{1cm} (2.14)

and
(2.15) \[ L_r D' = \frac{L f_1 f_2 r_{00} - f_2^2 L^2 s_0 + L \left[ f (f_1 + L^3 \omega \Delta) - \left( L f_2 b^2 + \beta f_1 f_2 \right) \right] \sigma_0 - ff_2 L^2 \sigma_i}{2 f (f_1 + L^3 \omega \Delta)}. \]

Contracting (2.12) by \( g^{ij} \) and putting the values of \( L_r D' \) and \( b_r D' \), we get

\[
D^i = \left\{ \frac{\left( f_1 f_2 - L \beta \omega f \right) (f_1 r_{00} - 2 L f_2 s_0)}{2 f f_1 (f_1 + L^3 \omega \Delta)} + \sigma_0 + \frac{\left( f_1 f_2 - L \beta \omega f \right)}{2 f f_1 (f_1 + L^3 \omega \Delta)} \right\} y^i \\
+ \left\{ \frac{L^3 \omega (f_1 r_{00} - 2 L f_2 s_0)}{2 f_1 (f_1 + L^3 \omega \Delta)} + \frac{L f_2}{2 f_1} \sigma_0 + \frac{L f \sigma_1 - \left( L f_2 b^2 + \beta f_1 \right) \sigma_0}{2 f_1 (f_1 + L^3 \omega \Delta)} \right\} b^i \\
- \frac{L f}{2 f_1} \sigma_j g^{ij} + \frac{L f_2}{f_1} s_0^i,
\]

where \( l^i = y^i L^{-1} \).

**Proposition 2.1:** The difference tensor \( D' = \overline{G}' - G' \) of conformal \( \beta \)-change of Finsler metric is given by (2.16).

### 3. Projective Change of Finsler Metric

The Finsler space \( F^n \) is said to be projective to Finsler space \( \overline{F}^n \) if every geodesic of \( F^n \) is transformed to a geodesic of \( \overline{F}^n \) and vice-versa. It is well known that the change \( L \rightarrow \overline{L} \) is projective iff \( \overline{G}' = G' + P(x, y)y^i \), where \( P(x, y) \) is a homogeneous scalar function of degree one in \( y^i \), called projective factor\(^{10} \). Thus from (2.1) it follow that \( L \rightarrow \overline{L} \) is projective iff \( D' = Py^i \). Now we consider that the changes \( L \rightarrow \overline{L} \) is projective. Then from equation (2.16), we have
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(3.1)

$$P_{yi} = \left\{ \frac{(f_{i} f_{j} - L \beta \omega f)(f_{i} r_{\omega \omega} - 2 L f_{j} s_{0})}{2 f f_{i}(f_{i} + L^3 \omega \Delta)} + \frac{(f_{i} f_{j} - L \beta \omega f)(Lf \sigma_{i} - (L f_{j} b^{2} + \beta f_{j}) \sigma_{0})}{2 f f_{i}(f_{i} + L^3 \omega \Delta)} \right\} y^{i}$$

$$+ \left\{ \frac{L^{3} \omega(f_{i} r_{\omega \omega} - 2 L f_{j} s_{0})}{2 f f_{i}(f_{i} + L^3 \omega \Delta)} + \frac{L^{3} \omega(L f \sigma_{i} - (L f_{j} b^{2} + \beta f_{j}) \sigma_{0})}{2 f f_{i}(f_{i} + L^3 \omega \Delta)} \right\} b^{i}$$

$$- \frac{Lf_{j}}{2 f f_{i}} g^{ij} + \frac{L f_{j}}{f_{i}} s_{0}.$$ 

Contracting (3.1) with $y_{i}(=g_{ij}y^{j})$ and using the fact that $s_{0}^{i} y_{i} = 0$ and $y_{i} y^{i} = L^{2}$, we get

(3.2) 

$$P = \frac{f_{i} f_{j} f_{i} r_{\omega \omega} - 2 L f_{j} s_{0}}{2 f f_{i}(f_{i} + L^3 \omega \Delta)} + \frac{L f \sigma_{i} + \{ f f_{i}(f_{i} + L^3 \omega \Delta) - f_{j} f f_{j}(L f_{j} b^{2} + \beta f_{j}) \sigma_{0} \}}{2 f f_{i}(f_{i} + L^3 \omega \Delta)}.$$ 

Putting the value of $P$ from (3.2) in (3.1), we get

(3.3)

$$\left\{ L f \sigma_{i} - (L f_{j} b^{2} + \beta f_{j}) \sigma_{0} \right\} \left( L f \omega y^{i} - L^{3} \omega b^{i} \right)$$

$$+ \left\{ L f f_{i} f f_{j} b^{i} \sigma_{0} \right\} \left( L f \sigma_{i} - (L f_{j} b^{2} + \beta f_{j}) \sigma_{0} \right) \left( L f \omega y^{i} - L^{3} \omega b^{i} \right)$$

$$= - L f f_{j} g^{ij} + 2 f f_{j}(f_{j} + L^3 \omega \Delta)s_{0}.$$ 

Transvecting (3.3) by $b_{i}$, we get

(3.4) 

$$r_{\omega \omega} = - \frac{2 L f_{j} s_{0} + L f \sigma_{i} - (L f_{j} b^{2} + \beta f_{j}) \sigma_{0}}{L^3 \omega \Delta}.$$ 

Substituting the value of $r_{\omega \omega}$ from (3.4) in (3.2), we get

(3.5)

$$P = \frac{-2 f f_{j} s_{0} + L f f_{i} f f_{j} - f_{j} f f_{j}(L f_{j} b^{2} + \beta f_{j}) \sigma_{0} + f L^3 \omega \Delta \sigma_{0}}{2 f L^3 \omega \Delta}.$$ 

Substituting the value of $r_{\omega \omega}$ from (3.4) in (3.3), we get
The equations (3.4) and (3.6) give the necessary conditions under which the change $L \rightarrow \bar{L}$ becomes a projective change.

Conversely, if conditions (3.4) and (3.6) are satisfied, then putting the values of $r_{00}^i$ and $s_0^i$ from (3.4) and (3.6) respectively in (2.16), we get

$$D' = \frac{-2f_2^2 s_0 + Lf_2 f_1 f_0 + fL^3 \omega \Delta \sigma_0}{2Lf_2} \frac{\sigma_1 - (Lf_2 b^2 + \beta f_1)\sigma_0}{2Lf_2^4 \omega \Delta}$$

i.e. $D' = P_y'$, where P is given by (3.5). Thus $\overline{F}^n$ is projective to $F^n$.

**Theorem 3.1:** The conformal $\beta$-change of Finsler metric is projective iff (3.4) and (3.6) hold good, the projective factor P is given by (3.5).

When $\sigma = 0$, the change (1.1) is simply a $\beta$-change of original metric and the condition (3.4) reduces to

$$r_{00} = \frac{-2Lf_2 s_0}{L^3 \omega \Delta}$$

where as the condition (3.6) reduces to

$$s_0' = \left(b' - \frac{\beta L^2 y'}{L^2} \right) \frac{s_0}{\Delta}.$$  

Thus we get

**Corollary 3.1:** The $\beta$-change of Finsler metric is projective iff (3.7) and (3.8) hold good.

This result has been investigated in\textsuperscript{12}.
4. Douglas Space

The Finsler space $F^n$ is called a Douglas space iff $G^i y^j - G^j y^i$ is homogeneous polynomial of degree three in $y^i$. We shall write $hp(r)$ to denote a homogeneous polynomial in $y^i$ of degree $r$. If we write $B^{ij} = D^i y^j - D^j y^i$, then from (2.16), we get

\[
B^{ij} = \left[ \frac{L^3 \omega (f_i r_{00} - 2 L f_i s_0)}{2 f_i (f_i + L^3 \omega \Delta)} + \frac{L f_i}{2 f_i} \sigma_0 + \frac{L^3 \omega \left[ L f \sigma_1 - (L f_i b^j + \beta f_i) \sigma_0 \right]}{2 f_i (f_i + L^3 \omega \Delta)} \right] (b^i y^j - b^j y^i) + \frac{L f_i}{f_i} (s^i_0 y^j - s^j_0 y^i).
\]

If a Douglas space is transformed to a Douglas space by a conformal $\beta$-change of Finsler metric (2.1) then $B^{ij}$ must be $hp(3)$ and vice-versa.

**Theorem 4.1:** The conformal $\beta$-change of Finsler metric leads a Douglas space into a Douglas space iff $B^{ij}$ given by (4.1) is $hp(3)$.

When $\sigma = 0$, the change (1.1) is simply a $\beta$-change of original metric and the condition (4.1) reduces to

\[
B^{ij} = \left[ \frac{L^3 \omega (f_i r_{00} - 2 L f_i s_0)}{2 f_i (f_i + L^3 \omega \Delta)} \right] (b^i y^j - b^j y^i) + \frac{L f_i}{f_i} (s^i_0 y^j - s^j_0 y^i).
\]

Thus we get

**Corollary 4.1:** The $\beta$-change of Finsler metric leads a Douglas space into a Douglas space iff $B^{ij}$ given by (4.2) is $hp(3)$.

This result has been investigated in\textsuperscript{12}.

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