A Finsler Space with Conservative Normal Projective Curvature Tensor

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Abstract: In this paper, the conservativeness of normal projective curvature tensor of a Finsler Space is considered. The condition of the conservativeness of normal projective curvature tensor is weaker than the condition characterizing a normal projective symmetric space. It is established that Berwald torsion tensor and Berwald deviation tensor of a Finsler Space with conservative normal projective curvature tensor are necessarily conservative. However, the converse is not necessarily true. Apart from other results, the symmetry of the covariant derivative $B_{hk}$ of the Ricci tensor and the tensor $B_{hi}$ in the indices $h$ and $k$ have been proved.

Keywords: Finsler space, Berwald curvature tensor, Berwald deviation tensor, Ricci tensor, conservativeness of Berwald curvature tensor, Berwald deviation tensor and Ricci tensor, normal projective curvature tensor, conservativeness of normal projective curvature tensor.

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1. Introduction

A vector field in a differentiable manifold is said to be conservative if its divergence is zero. If the divergence of curvature is zero in a neighbourhood of any point $p$ of a differentiable manifold $M^n$ then the bending is conservative i.e. rigid around the point $p$. In 1980, P. N. Pandey established the relation between the normal projective curvature tensor and Berwald curvature tensor. In 1984, P. N. Pandey discussed projective motion in a symmetric and projectively symmetric manifold. The condition characterizing a Finsler space with conservative Berwald curvature tensor is weaker than the condition characterizing a symmetric Finsler Space. In 1997, U. C. De and A. A. Shaikh studied the conservativeness of quasi-
conformal curvature tensor on a K-Contact and Sasakian manifolds. In 2005, C. S. Bagewadi and N. B. Gatti studied conservative curvature tensor on an Einstein manifold. In 2007, several authors including C. S. Bagewadi and D. G. Prakasha made significant contributions towards the spaces with conservative curvature tensors.

In 2008, D. G. Prakash, C. S. Bagewadi and Venkatesha studied the conformally and quasi-confirmly conservative curvature tensors on a trans-Sasakian manifold. In 2012, C. S. Bagewadi and M. C. Bharthi introduced the concept of conservative conditions on curvature tensor of a Kaehler manifold. In 2015, S. Kumari and P. N. Pandey studied a Finsler space with conservative Berwald curvature tensor and obtained some significant results. The aim of the present paper is to study a Finsler Space with conservative normal projective curvature tensor.

2. Preliminaries

Consider an n-dimensional Finsler Space $F_n$ equipped with a metric function $F(x,y)$ satisfying the requisite conditions (H. Rund and P. L. Antonelli). Suppose $g_{ij}$, $G^i_{jk}$, $\pi^i_{jk}$ & $N^i_{jk}$ denote the components of the corresponding metric tensor, Berwald connection coefficients, components of Berwald curvature tensor, normal projective connection coefficients and components of normal projective curvature tensor respectively. $H^i_{jk}$ and $N^i_{jk}$ are positively homogeneous of degree zero in $y$ and are skew symmetric in their last two lower indices.

The Berwald connection coefficients and the normal projective connection coefficients are related as

\begin{align}
(2.1) \quad & (a) \quad \pi^i_{jkh} = \pi^i_{jkh}, \\
& (b) \quad \pi^i_{jkh} y^j = 0, \\
& (c) \quad G_{hbi} = G_{hbi}, \\
& (d) \quad \pi^i_{jkh} y^j = \frac{1}{(n+1)} G^i_{jkr} y^r, \\
& (e) \quad \pi^i_{jkh} = \frac{2}{(n+1)} G_{jkh}, \\
& (f) \quad \pi^i_{jkh} - \pi^i_{jkh} = \frac{1}{(n+1)} \left( \delta^i_k G^k_{jhr} - \delta^i_l G^l_{jhr} \right).
\end{align}

The components of Berwald-Ricci tensor are given by

$$H^i_{jk} = H^i_{jkr}.$$
The components of Berwald curvature tensor, deviation tensor, torsion tensor & Ricci tensor satisfy the following

\[(2.2) \quad \begin{align*}
(a) & \quad H^i_{jk} = H_{jk}^i y^j, \\
(b) & \quad \hat{\partial}_j H_{kh}^i = H_{jh}^i, \\
(c) & \quad H^i_j = H_{ij}^i y^k, \\
(d) & \quad H^i_{jk} = -H_{kj}^i, \\
(e) & \quad H^i_j = (n-1)H, \\
(f) & \quad H_k y^i = (n-1)H, \\
(g) & \quad 3H^i_{kh} = \hat{\partial}_i H^i_h - \hat{\partial}_h H^i_i, \\
(h) & \quad H_{kh} y^i = H_h, \\
(i) & \quad H^i_{rkh} = H_{rk} - H_{kh}.
\end{align*}\]

Bianchi identities for the components of Berwald Curvature tensor are given by \(9-10\)

\[(2.3) \quad \begin{align*}
(a) & \quad H^i_{jk} + H^i_{kj} + H^i_{hjk} = 0, \\
(b) & \quad \begin{align*}
0 & \quad \hat{\partial}_i H^i_j = H^i_{jm} + H^i_{jm} + H^i_{jm} + H_{km} G^i_j + H_{km} G^i_j + H_{km} G^i_j = 0.
\end{align*}
\end{align*}\]

Relation between the components of Berwald curvature tensor and normal projective curvature tensor is given by \(1\)

\[(2.4) \quad N^i_{hjk} = H^i_{hjk} - \hat{\partial}_h y^i.\]

The Berwald covariant differential operator is defined as

\[(2.5) \quad T'_{jk} = \hat{\partial}_k T'_{j} - (\hat{\partial}_i T'_{j}) G^i_k + T'_{ij} G^i_k - T'_{j} G^i_k,\]

where \(T'_{jk}\) are the components of an arbitrary tensor of type \((1, 1)\); \((k)\) denotes the Berwald covariant differential operator.

Relation between the Berwald covariant differential operator and the directional differential operator is given by

\[(2.6) \quad \hat{\partial}_k T'_{jk} - (\hat{\partial}_j T'_{k})_{(k)} = T'_{jk} G^i_{kh} - T'_{j} G^i_{kh},\]

where \(\hat{\partial}_j G^i_{kh} = G^i_{jkh}\) are the components of symmetric tensor and satisfy

\[(2.7) \quad G^i_{jkh} y^j = G^i_{jkh} y^k = G^i_{jkh} y^h = 0.\]
**Definition 2.1:** A Finsler Space $F^n$ with conservative Berwald curvature tensor is characterized by

\[(2.8) \quad H'_{jkh(r)} = 0, \text{ with } H'_{jkh} \neq 0.\]

**Definition 2.2:** A Finsler Space $F^n$ with conservative normal projective curvature tensor is characterized by

\[(2.9) \quad N'_{jkh(r)} = 0, \text{ with } N'_{jkh} \neq 0.\]

Contracting the indices $i$ & $h$ in (2.4), we have

\[(2.10) \quad N'_{ijk} = H'_{ijk}.\]

Transvecting (2.4) with respect to $y^h$, we have

\[(2.11) \quad N'_{ijk} y^h = H'_{ijk}.\]

Contracting the indices $i$ & $j$ in (2.4), we have

\[(2.12) \quad N_{ik} = H_{ik} - \frac{y^j}{(n+1)} \partial_j H'_{ik}.\]

After carrying out some simple calculations, (2.12) reduces to

\[(2.13) \quad N_{ik} = \frac{n}{(n+1)} H_{ik} - \frac{1}{(n+1)} H_{ij} (n-1) \partial_j \partial_k H.\]

First Bianchi identity for the components of normal projective Curvature tensor is given by

\[(2.14) \quad N'_{jkh} + N'_{kjh} + N'_{jkh} = 0.\]

Normal projective curvature tensor is skew symmetric in its last two lower indices, i.e.

\[(2.15) \quad N'_{jkh} = -N'_{jkh}.\]

Relation between the components of projective curvature tensor and normal projective curvature tensor is given by
(2.16) \[ W_{jkh}^i = N_{jkh}^i + (\delta^i_{jkh} - \delta^i_{jkh}) - \delta^j_{jkh} M_{jkh}, \]

where

\[ M_{jkh} = \frac{-1}{n^2-1} (n N_{jkh} + N_{jkh}). \]

**Definition 2.3:** A Finsler Space \( F^n \) with conservative projective curvature tensor is characterized by

(2.18) \[ W_{j(h)}^i = 0, \text{ with } W_{jkh}^i \neq 0. \]

The components of projective curvature tensor, deviation tensor and torsion tensor satisfy the following

(2.19) \[ W_{jkh}^i x^j = W_{jkh}^i, \]

(2.20) \[ W_{jkh}^i x^k = W_{jkh}^i. \]

3. Conservativeness of Berwald Curvature tensor in a Finsler Space with Conservative Normal Projective Curvature Tensor

Let us consider a Finsler Space \( F^n \) with conservative normal projective curvature tensor characterized by (2.9).

Differentiating covariantly (2.4) with respect to \( x^n \), we get

(3.1) \[ N_{jkh}^{i(m)} + H_{jkh}^{i(m)} \left( \frac{y'}{(n+1)} \right) H_{jkh}^{i(m)} = 0. \]

On contracting the indices \( i \& m \) and using (2.9), we get

(3.2) \[ H_{jkh}^{i(r)} \left( \frac{y'}{(n+1)} \right) H_{jkh}^{i(r)} = 0. \]

Transvecting (3.2) by \( y' \), we get

(3.3) \[ H_{jkh}^{i(r)} = 0. \]

Again transvecting (3.3) by \( y^k \), we have

(3.4) \[ H_{jkh}^{i(r)} = 0. \]
Thus, we have the theorem

**Theorem 3.1:** In a Finsler Space with conservative normal projective curvature tensor, Berwald torsion tensor and Berwald deviation tensor are conservative.

On transvecting Bianchi’s second identity (2.3)(b) by \( y^i \) and using (2.2)(a) and (2.7), we have
\[
(3.5) \quad H'_{k(n)} + H'_{m(n)} + H'_{n(k)} = 0.
\]
Contracting the indices \( i \) & \( m \) and using (3.3), we get
\[
(3.6) \quad H_{h(k)} = H_{k(h)}.
\]
Differentiate partially (3.5) with respect to \( y^j \), we get
\[
(3.7) \quad \dot{\partial}_j H_{h(k)} = \dot{\partial}_j H_{k(h)}.
\]
Using the commutation formulae (2.6), we get
\[
(3.8) \quad H_{j(h(k)} = H_{j(k(h)\}}.
\]
Hence, we have the theorem

**Theorem 3.2:** The Ricci tensor \( H_{jk} \) and the vector \( H_k \) of a Finsler space with conservative normal projective curvature tensor satisfy \( H_{j(h(k)} = H_{j(k(h)} \) and \( H_{h(k)} = H_{k(h)} \), equivalently.

For Berwald torsion tensor, the commutation formulae (2.6) is given as
\[
(3.9) \quad \dot{\partial}_j H_{k} = H_{j(k)} G_{j} - H_{j} G_{k} + H_{j} G_{k} - H_{j} G_{j} = 0.
\]
Contracting the indices \( i \) & \( m \), we have
\[
(3.10) \quad H'_{k(n)} + H'_{j(n)} G_{j} - H'_{j} G_{n} - H'_{j} G_{j} = 0.
\]
Rotating cyclically \( j, k \) & \( h \) in (3.10), we have
\[
(3.11) \quad H'_{k} + H'_{j} G_{k} - H'_{j} G_{h} - H'_{j} G_{j} = 0.
\]
Rotating cyclically \( j, k \) & \( h \) in (3.10), we have
\[ H'_{bij(r)} + H'_{jk}G'_{kuv} - H'_{rk}G'_{bij} - H'_{jr}G'_{buk} = 0. \]

Adding (3.10), (3.11) and (3.13) and using (2.2)(d), we get
\[ H'_{k} G'_{kuv} + H'_{ij} G'_{kuv} + H'_{jk} G'_{kuv} = 0. \]

Transvecting (3.13) by \( y' \), we get
\[ H'_{k} G'_{kuv} - H'_{ij} G'_{kuv} = 0. \]
\[ H'_{k} G'_{kuv} = H'_{ij} G'_{kuv}. \]

Thus, we have the following theorem

**Theorem 3.3:** The identities (3.13) and (3.15) hold in a Finsler space with conservative normal projective curvature tensor.

Consider
\[ \left( \partial_{j} H'_{skr(h)} - \partial_{r} H'_{sjr(h)} \right) y' = \left[ \partial_{r} \left( H'_{k(r)} - H'_{l(r)} \right) - \partial_{k} \left( H'_{l(r)} - H'_{j(r)} \right) \right] y' \]
\[ = \partial_{j} H'_{s(kr)} y' - \partial_{r} \dot{H}'_{j(kr)} y' - \partial_{k} \dot{H}'_{l(kr)} y' + \partial_{k} \dot{H}'_{j(kr)} y' \]
\[ = \dot{H}'_{l(kr)} y' - H'_{j(kr)} y' + \dot{H}'_{j(kr)} y' + H'_{j(kr)} y' \]
\[ = H'_{j(kr)} - H'_{j(kr)} + H'_{j(kr)} = 0. \]

Hence we have
\[ \partial_{j} H'_{skr(h)} y' = \dot{H}'_{jkr(h)} y'. \]

For Berwald curvature tensor, the commutation formulae (2.6) is given as
\[ \partial_{j} H'_{jk(h)} = H'_{jkh} G'_{kuv} - H'_{jkh} G'_{uv} - H'_{jkh} G'_{muv} - H'_{jkh} G'_{muv}. \]

Transvecting (3.18) by \( y^m \), we get
\[ y^m \partial_{i} H'_{jk(h)} = y^m \left( \partial_{i} H'_{jk(h)} \right) = 0. \]
\[ y^m \partial_{i} H'_{jk(h)} = y^m \left( \partial_{i} H'_{jk(h)} \right). \]
Contracting the indices $i$ & $j$ in (3.20), we get

\begin{equation}
(3.21) \quad y^\mu \hat{\partial}_j H^r_{skh(m)} = y^\mu \left( \hat{\partial}_j H^r_{shk(m)} \right).
\end{equation}

Thus, we have the following theorem

**Theorem 3.4:** In a Finsler space with conservative normal projective curvature tensor the identities (3.17), (3.20) and (3.21) hold.

From (3.21), we have

\begin{equation}
(3.22) \quad y^r \hat{\partial}_j H^s_{skh(r)} = y^r \left( \hat{\partial}_j H^s_{shk(r)} \right).
\end{equation}

Thus we have

\begin{equation}
(3.23) \quad y^r \hat{\partial}_j H^s_{skh(r)} - y^r \hat{\partial}_h H^s_{skh(r)} = H^s_{skh(r)}.
\end{equation}

Or

\begin{equation}
(3.24) \quad y^r \left( \hat{\partial}_h H^s_{skh(r)} - \hat{\partial}_j H^s_{skh(r)} \right) = H^s_{skh(r)}.
\end{equation}

Transvecting (3.24) by $y^j$, we get

\begin{equation}
(3.25) \quad y^r \left( y^k \hat{\partial}_k H^s_{skh(r)} - y^k \hat{\partial}_h H^s_{skh(r)} \right) = y^r H^s_{skh(r)}.
\end{equation}

Using (3.17) in (3.25), we have
\[ y^k H_{\beta(k)}^\alpha = 0. \]  
(3.27) \[ y^k (H_{\beta(k)} - H_{\gamma(k)}) = 0. \]  
(3.28) \[ y' H_{\beta(k)} = H_{\gamma(k)}. \]

Differentiating partially (3.28) with respect to \( y^k \), we get

\[ y' \partial_k H_{\beta(k)} + H_{\beta(k)} = \partial_k H_{\gamma(k)}. \]  
(3.29)

Interchanging the indices \( j \& k \) in (3.29), we have

\[ y' \partial_j H_{\beta(k)} + H_{\beta(k)} = \partial_j H_{\gamma(k)}. \]  
(3.30)

From (3.29) and (3.30), we have

\[ y' \partial_k H_{\beta(k)} + H_{\beta(k)} - y' \partial_j H_{\beta(k)} - H_{\beta(k)} = \partial_k H_{\gamma(k)} - \partial_j H_{\gamma(k)}. \]  
(3.31)

Using the commutation formulae (2.6) in (3.31), we get

\[ y' \left( \partial_k H_{\beta(k)} - H_{\nu} G_{\beta(k)}^{\nu} y' + H_{\beta(k)} \right) = H_{\beta(k)} - H_{\gamma(k)} - H_{\beta(k)} = H_{\gamma(k)} - H_{\beta(k)}. \]  
(3.32)

\[ y' \left( \partial_j H_{\beta(k)} - H_{\nu} G_{\beta(k)}^{\nu} y' + H_{\beta(k)} \right) = H_{\beta(k)} - H_{\gamma(k)} - H_{\beta(k)} = H_{\gamma(k)} - H_{\beta(k)}. \]  
(3.33)

\[ y' \partial_j H_{\beta(k)} + H_{\nu} G_{\beta(k)}^{\nu} y' - H_{\nu} G_{\gamma(k)}^{\nu} y' + H_{\beta(k)} - y' \partial_j H_{\beta(k)} - H_{\beta(k)} = H_{\beta(k)} - H_{\beta(k)}. \]  
(3.34)

\[ H_{\beta(k)} = H_{\gamma(k)}. \]  
(3.35)

Thus, we conclude

**Theorem 3.5:** In a Finsler space with conservative normal projective curvature tensor, identity (3.24) holds and Ricci tensor \( H_{\gamma} \) of a Finsler space with conservative normal projective curvature tensor satisfy

\[ H_{\gamma(k)} = H_{\beta(k)}. \]  

From (3.2), we have
Using (3.21) in (3.36), we have

\begin{equation}
H'_{\delta h(r)} = -\frac{y^r}{(n+1)} \left( \delta_j H'_{\delta h(r)} \right).
\end{equation}

(3.37)

Using (3.35) in (3.38), we have

\begin{equation}
H'_{\delta h(r)} = 0.
\end{equation}

(3.39)

Thus, we have the theorem

**Theorem 3.6:** In a Finsler space with conservative normal projective curvature tensor, the Berwald curvature tensor is also conservative.

4. Conservativeness of Projective Curvature tensor in a Finsler Space with Conservative Normal Projective Curvature Tensor

Let us consider a Finsler Space $F^n$ with conservative normal projective curvature tensor characterized by (2.9).

Differentiating covariantly (2.13) with respect to $x^h$, we get

\begin{equation}
N_{j(k)} = \frac{n}{(n+1)} H_{j(k)} - \frac{1}{(n+1)} H_{j(k)} + \frac{(n-1)}{(n+1)} \left( \delta_j \delta_k H \right).
\end{equation}

(4.1)

Interchanging the indices $j & k$ in (4.1), we have

\begin{equation}
N_{k(j)} = \frac{n}{(n+1)} H_{k(j)} - \frac{1}{(n+1)} H_{k(j)} + \frac{(n-1)}{(n+1)} \left( \delta_k \delta_j H \right).
\end{equation}

(4.2)

Thus we have

\begin{equation}
N_{j(k)} - N_{k(j)} = H_{j(k)} - H_{k(j)}.
\end{equation}

(4.3)

Using (3.35) in (4.3), we get
(4.4) \[ N_{jk(h)} = N_{kj(h)}. \]

Interchanging the indices \(k \& h\) in (4.1) then subtracting it from (4.1), we have

\[(4.5) \quad N_{jk(h)} - N_{kj(h)} = \frac{1}{(n+1)}\left( H_{hk(j)} - H_{kh(j)} \right) + \frac{(n-1)}{(n+1)}\left( \partial_j \hat{\partial}_h H - \partial_h \hat{\partial}_j H \right) \]

\[= \frac{1}{(n+1)}\left( H_{hk(j)} - H_{kh(j)} \right) + \frac{(n-1)}{(n+1)} \left[ \partial_j \left( \partial_h H \right)_{(j)} - \partial_h \left( \partial_j H \right)_{(j)} \right] \]

\[= \frac{1}{(n+1)}\left( H_{hk(j)} - H_{kh(j)} \right) + \partial_j \left( H_{hk(r)} - H_{kh(r)} \right) \]

\[= \frac{1}{(n+1)} \partial_j \left( H_{hk(r)} - H_{kh(r)} \right) \]

Using (3.35) in (4.5), we have

\[(4.6) \quad N_{jk(h)} - N_{kj(h)} = 0. \]

\[(4.7) \quad N_{jk(h)} = N_{kj(h)}. \]

Interchanging the indices \(j \& h\) in (4.1), we have

\[(4.8) \quad N_{hk(j)} = \frac{n}{(n+1)} H_{hk(j)} - \frac{1}{(n+1)} H_{kh(j)} + \frac{(n-1)}{(n+1)} \left( \partial_j \hat{\partial}_k H \right)_{(j)}. \]

From (4.1) & (4.8), we have

\[(4.9) \quad N_{jk(h)} - N_{hk(j)} = \frac{n}{(n+1)} \left( H_{jk(h)} - H_{kj(h)} \right) + \frac{(n-1)}{(n+1)} \left( \partial_j \hat{\partial}_k H - \partial_k \hat{\partial}_j H \right) \]

\[= \frac{(n-1)}{(n+1)} \left( \partial_j \hat{\partial}_k H - \partial_k \hat{\partial}_j H \right) \]

\[= \frac{n}{(n+1)} \left( \partial_j \hat{\partial}_k H \right)_{(j)} + \partial_k \hat{\partial}_j H - \left( \partial_j H \right)_{(j)} \]
\[
\begin{align*}
= & \frac{(n-1)}{(n+1)} \hat{\partial}_k \left\{ (\hat{\partial}_j H)_{(\beta)} - (\hat{\partial}_h H)_{(\alpha)} \right\} \\
= & \frac{1}{(n+1)} \left\{ \hat{\partial}_k \left( H_{\beta(c)} y' - H_{\beta(c)} y' \right) \right\} \\
= & \frac{1}{(n+1)} \hat{\partial}_k \left( \left( H_{\beta(c)} - H_{\beta(c)} \right) y' \right).
\end{align*}
\]

Using (3.35) in (4.9), we have

\[(4.10) \quad N_{\beta(k)} = N_{h(k)}.
\]

Differentiating covariantly (2.17) with respect to \( x' \), we get

\[(4.11) \quad M_{\beta(h)} = - \frac{1}{n^2 - 1} (n N_{\beta(h)} + N_{h(k)}).
\]

Interchanging the indices \( j \) & \( h \) in (4.11), we have

\[(4.12) \quad M_{\beta(h)} = - \frac{1}{n^2 - 1} (n N_{\beta(h)} + N_{h(k)}).
\]

From (4.11) & (4.12), we have

\[(4.13) \quad M_{\beta(h)} - M_{\beta(h)} = - \frac{n}{n^2 - 1} (N_{\beta(h)} - N_{h(k)}) + \frac{1}{n^2 - 1} (N_{h(k)} + N_{h(h)}).
\]

Using (4.7) & (4.10) in (4.13), we have

\[(4.14) \quad M_{\beta(h)} = M_{\beta(h)}.
\]

Interchanging the indices \( k \) & \( h \) in (4.12), we have

\[(4.15) \quad M_{\beta(h)} = - \frac{1}{n^2 - 1} (n N_{\beta(h)} + N_{h(k)}).
\]

From (4.12) & (4.15), we have

\[(4.16) \quad M_{\beta(h)} - M_{\beta(h)} = - \frac{1}{n^2 - 1} (n N_{\beta(h)} + N_{h(h)}) + \frac{1}{n^2 - 1} (n N_{\beta(h)} + N_{h(h)}) \\
= - \frac{1}{n^2 - 1} \left( n \left( N_{\beta(h)} - N_{h(h)} \right) + \left( N_{\beta(h)} - N_{h(h)} \right) \right).
\]
Using (4.7) & (4.10) in (4.16), we have

\[(4.17)\]  \[M_{h(h)} = M_{h(h)}.\]

Interchanging the indices \(k \& h\) in (4.11), we have

\[(4.18)\]  \[M_{h(k)} = -\frac{1}{n-1}(nN_{h(k)} + N_{h(h)}).\]

From (4.11) & (4.18), we have

\[(4.19)\]  \[M_{h(k)} - M_{h(k)} = -\frac{1}{n-1}(n(N_{h(k)} - N_{h(h)})) + (N_{h(k)} - N_{h(h)}).\]

Using (4.4) in (4.19), we have

\[(4.20)\]  \[M_{h(k)} = M_{h(h)}.\]

Thus, we have the following theorem

**Theorem 4.1:** The identities (4.4), (4.7), (4.10), (4.14), (4.17) and (4.20) hold in a Finsler space with conservative normal projective curvature tensor.

Differentiating covariantly (2.16) with respect to \(x^n\), we get

\[(4.21)\]  \[W^i_{jkh(m)} = N^i_{jkh(m)} + \left(\delta^i_k M_{j(h)\alpha} - \delta^i_j M_{k(h)\alpha}\right) - \left(\delta^i_k M_{j(h)\alpha} - \delta^i_j M_{k(h)\alpha}\right).\]

Contracting the indices \(i \& m\) and using (2.9), we get

\[(4.22)\]  \[W^\gamma_{jkh(r)} = (M_{j(h)\gamma} - M_{k(h)\gamma}) - (M_{j(h)\gamma} - M_{k(h)\gamma}).\]

\[(4.23)\]  \[W^\gamma_{jkh(r)} = (M_{j(h)\gamma} - M_{h(h)\gamma}) - (M_{j(h)\gamma} - M_{k(h)\gamma}).\]

Using (4.14) & (4.14) in (4.23), we get

\[(4.24)\]  \[W^\gamma_{jkh(r)} = 0.\]

Transvecting (4.24) by \(y^i\) & using (2.19), we have

\[(4.25)\]  \[W^\gamma_{k(h)} = 0.\]
Transvecting (4.25) by \( y^j \) & using (2.20), we have

\[
W^r_{h(r)} = 0.
\]

Thus, we have the theorem

**Theorem 4.2:** In a Finsler space with conservative normal projective curvature tensor, the projective curvature tensor, torsion tensor & deviation tensor are also conservative.

### 5. Finsler Space with Conservative Berwald Curvature Tensor

Consider a Finsler Space \( \mathcal{F}^n \) with conservative Berwald curvature tensor defined by (2.8). Then the Berwald torsion tensor and Berwald deviation tensor are also conservative.

Also in a Finsler space with conservative Berwald curvature tensor, the identities \( H_{j(k)}=H_{j(k)} \) and \( H_{h(k)}=H_{k(h)} \) are satisfied.

Thus, we have the theorem

**Theorem:** In a Finsler space with conservative Berwald curvature tensor the identities (3.17), (3.20), (3.21), (3.25) & (3.35) hold.

Consider equation (3.1). On contracting the indices \( i \) & \( m \) and using (2.8), we get

\[
N^r_{j(k)(r)} = -\frac{y}{(n+1)} \left( \delta^r_{h} H_{j(h)(r)} \right).
\]

Using (3.21) in (5.1), we get

\[
N^r_{j(k)(r)} = -\frac{y}{(n+1)} \delta^r_{h} H'_{j(h)(r)}.
\]

\[
N^r_{j(k)(r)} = -\frac{y}{(n+1)} \delta^r_{h} \left( H_{j(h)(r)} - H'_{j(h)(r)} \right).
\]

Using (3.35) in (5.3), we have

\[
N^r_{j(k)(r)} = 0.
\]

Thus, we conclude
Theorem 5.1: In a Finsler space with conservative Berwald curvature tensor, the normal projective curvature tensor is also conservative.

Hence the identities (4.4), (4.7), (4.10), (4.14), (4.17) and (4.20) also hold in a Finsler space with conservative Berwald curvature tensor.

Differentiating covariantly (2.16) with respect to $x^m$, we get

\[ W^r_{\mu \lambda (m)} = N^l_{\mu \nu (m)} \left( \delta^l_j M^r_{\nu (n)} - \delta^l_j M^r_{\mu (n)} \right) - \left( \delta^r_j M^r_{\mu (n)} - \delta^r_j M^r_{\nu (n)} \right). \]

Contracting the indices $i$ & $m$ and using (2.9), we get

\[ W^r_{\lambda \beta (r)} = \left( M^r_{\mu (i) \lambda (k)} - M^r_{\mu (i) \beta (k)} \right) - \left( M^r_{\mu (i) \beta (k)} - M^r_{\mu (i) \beta (k)} \right). \]

Using (4.14) & (4.14) in (5.7), we get

\[ W^r_{\lambda \beta (r)} = 0. \]

Transvecting (5.8) by $y^r$ & using (2.19), we have

\[ W^r_{\lambda (r)} = 0. \]

Transvecting (5.9) by $y^r$ & using (2.20), we have

\[ W^r_{\lambda (r)} = 0. \]

Thus, we have the theorem

Theorem 5.2: In a Finsler space with conservative Berwald curvature tensor, the projective curvature tensor, torsion tensor & deviation tensor are also conservative.

Reference


