

Conformal Change of the Finsler Space with an (α, β) -metric is of Douglas Type

Ramdayal Singh Kushwaha and Renu

Department of Mathematics and Statistics, Banasthali Vidyapith, Jaipur, India

Email: bhuramdayal@gmail.com ; renu3119@gmail.com

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Abstract: Present work studies with the special (α, β) -metric, defined by $L = c_1\alpha + c_2\beta + c_3\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$. Using the Chern and Shen's lemma, we obtained the conditions for the special (α, β) -metric to be the Finsler metric. Also, we have proved that a Douglas space of second kind with this metric is conformally transformed to a Douglas space of second kind.

Keywords: Finsler metric, Conformal change, Douglas space, Douglas space of second kind.

1. Introduction

The theory of Finsler spaces with (α, β) -metric have contributed to the development of Finsler geometry. The notion of (α, β) -metric was introduced by M. Matsumoto¹⁰ as a generalization of Randers metric $F = \alpha + \beta$, where α is a Riemannian metric and β is a one-form. There are various types of (α, β) -metrics, such as Kropina metric, Matsumoto metric, square metric and exponential metric.

S. Báscó and M. Matsumoto¹ proposed the idea of Douglas space as a generalization of Berwald space and the Douglas space of second kind was introduced by I. Y. Lee⁸. It is well known that a Finsler space with (α, β) -metric is a Douglas space^{1,3} if the Douglas tensor D_{pj}^i vanishes identically or the quantities

$$(1.1) \quad D^{ij} = G^i(x, y)y^j - G^j(x, y)y^i,$$

are hp(3), where hp(n) denotes the homogeneous polynomials in y^i of degree n . I. Y. Lee⁸ found the conditions for a Finsler space with a Matsumoto metric to be a Douglas space of second kind.

The conformal theory of Finsler spaces was introduced by M. S. Kneblman⁵ and it was investigated in detail by M. Hashiguchi⁴. Later on, many researchers^{7,12} found conformally invariant tensors in the Finsler space with (α, β) -metric under conformal β -change. Many geometers have studied the conformal transformation of Douglas space with special (α, β) -metrics^{6,11,13,15-17}.

In the present paper, we consider a special (α, β) -metric L , defined by

$$(1.2) \quad L = c_1\alpha + c_2\beta + c_3 \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha},$$

on a manifold M with dimension $n > 2$, and $c_i \{i = 1, 2, 3\}$ are real constants. Firstly, we find the conditions that the above metric to be a Finsler metric. Also, we have proved that a Douglas space of second kind with above metric is conformally transformed to a Douglas space of second kind. Further, we obtained some results which prove that a Douglas space of second kind with certain (α, β) -metrics such as Randers changed Square metric, singular square metric, and $L = \beta + \frac{a\alpha^2 + \beta^2}{\alpha}$ is conformally transformed to a Douglas space of second kind.

2. Preliminaries

Suppose M^n is a differential manifold of dimension n . Suppose $L(\alpha, \beta)$ is a positively homogeneous function of α and β of degree one in M^n , where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one-form. The space $F^n = (M^n, L(\alpha, \beta))$ is known as a Finsler space with an (α, β) -metric and the space $R^n = (M^n, \alpha)$ is the Riemannian space associated with the Finsler space F^n . We will use the following symbols¹⁰ :

$$\begin{aligned} b^i &= a^{ir} b_r, & b^2 &= a^{rs} b_r b_s, \\ 2r_{ij} &= b_{i;j} + b_{j;i}, & 2s_{ij} &= b_{i;j} - b_{j;i}, \\ s_j^i &= a^{ir} s_{rj}, & s_j &= b_r s_j^r. \end{aligned}$$

The geodesics of an n -dimensional Finsler space $F^n = (M^n, L)$ are given by the system of differential equations⁴ :

$$\frac{d^2 x^i}{dt^2} y^j - \frac{d^2 x^j}{dt^2} y^i + 2(G^i y^j - G^j y^i) = 0, \quad y^i = \frac{dx^i}{dt},$$

in a parameter t , where the spray function $G^i(x, y)$ is given by

$$2G^i(x, y) = g^{ij}(y^r \partial_j \partial_r F - \partial_j F),$$

here $\partial_j = \frac{\partial}{\partial x^j}$, $\hat{\partial}_j = \frac{\partial}{\partial y^j}$, and $F = \frac{L^2}{2}$.

It is well known¹ that a Finsler space F^n becomes a Douglas space if and only if the Douglas tensor

$$D_{ijk}^h = G_{ijk}^h - \frac{1}{(n+1)}(G_{ijk}y^h + G_{ij}\delta_k^h + G_{jk}\delta_i^h + G_{ki}\delta_j^h),$$

vanishes identically, where $G_{ij}^h = \dot{\partial}_k G_{ij}^h$ is the $h\nu$ -curvature tensor of the Berwald connection $B\Gamma^{10}$.

Now, differentiating (1.1) with respect to y^p, y^q, y^r and y^s , we have $D_{pqrs}^{ij} = 0$, which are equivalent to

$$(2.1) \quad D_{pqrm}^{im} = (n+1)D_{pqr}^i = 0.$$

Thus, the Finsler space F^n together with the condition $D_{pqrs}^{ij} = 0$ results a Douglas space¹.

Again, differentiating (1.1) by y^m and contracting m and j in the obtained equation, we have

$$(2.2) \quad D_m^{im} = (n+1)G^i - G_m^m y^i.$$

Furthermore, differentiating (2.2) with respect to y^p, y^j and y^q , we get the condition (2.1).

Definition 2.1: A Finsler space F^n is said to be a Douglas space of second kind if $D_m^{im} = (n+1)G^i - G_m^m y^i$, is a $hp(2)$ ⁸.

According to M. Matsumoto⁹, the spray function $G^i(x, y)$ of the Finsler space F^n with an (α, β) -metric is written in the form

$$(2.3) \quad 2G^i = \gamma_{00}^i + 2B^i,$$

where

$$(2.4) \quad B^i = \frac{\alpha L_\beta}{L_\alpha} s_0^i + C^* \left\{ \frac{\beta L_\beta}{\alpha L} y^i - \frac{\alpha L_{\alpha\alpha}}{L_\alpha} \left(\frac{\gamma^i}{\alpha} - \frac{\alpha b^i}{\beta} \right) \right\},$$

$$(2.5) \quad C^* = \frac{\alpha\beta(r_{00}L_\alpha - 2\alpha s_0 L_\beta)}{2(\beta^2 L_\alpha + \alpha \gamma^2 L_{\alpha\alpha})}, \quad \gamma^2 = b^2 \alpha^2 - \beta^2,$$

γ_{jk}^i are Christoffel symbols, the vector B^i is the difference tensor, $\gamma_{00}^i = \gamma_{jk}^i(x)y^j y^k$ is $hp(2)$, and g^{ij} are the inverse of the metric tensors g_{ij} . Differentiating (2.3) w.r.t. y^j and y^k successively. We get $G_j^i = \gamma_{0j}^i + B_j^i$ and $G_{jk}^i = \gamma_{jk}^i + B_{jk}^i$ respectively, where $B_{jk}^i = \dot{\partial}_k B_j^i$, and $B_j^i = \dot{\partial}_j B^i$.

Definition 2.2: A Finsler space F^n with an (α, β) -metric becomes a Douglas space if and only if¹¹

$$(2.6) \quad B^{ij} = B^i y^j - B^j y^i,$$

are $hp(3)$.

Equation (2.4) yields

$$(2.7) \quad B^{ij} = \frac{\alpha L_\beta}{L_\alpha} (s_0^i y^j - s_0^j y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* (b^i y^j - b^j y^i).$$

Again, differentiating (2.7) by y^m and then by contracting m and j , we get

$$(2.8) \quad B_m^{im} = \frac{(n+1)\alpha L_\beta s_0^i}{L_\alpha} + \frac{\alpha\{(n+1)\alpha^2 \Omega L_{\alpha\alpha} b^i + \beta \gamma^2 A y^i\} r_{00}}{2\Omega^2} \\ - \frac{\alpha^2\{(n+1)\alpha^2 \Omega L_\beta L_{\alpha\alpha} b^i + B y^i\} s_0}{L_\alpha \Omega^2} - \frac{\alpha^3 L_{\alpha\alpha} y^i s_0}{\Omega},$$

where

$$\Omega = (\beta^2 L_\alpha + \alpha \gamma^2 L_{\alpha\alpha}),$$

provided that $\Omega \neq 0$,

$$A = \alpha L_\alpha L_{\alpha\alpha\alpha} + 3L_\alpha L_{\alpha\alpha} - 3\alpha(L_{\alpha\alpha})^2,$$

$$(2.9) \quad B = \alpha\beta\gamma^2 L_\alpha L_\beta L_{\alpha\alpha\alpha} + \beta\{(3\gamma^2 - \beta^2)L_\alpha - 4\alpha\gamma^2 L_{\alpha\alpha}\} L_\beta L_{\alpha\alpha} + \Omega L L_{\alpha\alpha}.$$

Summarizing the above:

Theorem 2.3: A Finsler space F^n with an (α, β) -metric to be a Douglas space of second kind if and only if B_m^{im} are $hp(2)$ provided that $\Omega \neq 0$, where B_m^{im} is given by (2.8)⁸.

Let $F^n = (M^n, L)$ and $\bar{F}^n = (M^n, \bar{L})$ be two Finsler spaces on the same underlying manifold M^n . If we have a function $\sigma(x)$ in each coordinate neighbourhoods of M^n such that $\bar{L}(x, y) = e^\sigma L(x, y)$, then F^n is called conformal to $\bar{F}^n$ and the change $L \rightarrow \bar{L}$ of metric is called a conformal change.

A conformal change of (α, β) -metric is given as $(\alpha, \beta) \rightarrow (\bar{\alpha}, \bar{\beta})$, where $\bar{\alpha} = e^\sigma \alpha$ and $\bar{\beta} = e^\sigma \beta$. Therefore, we have

$$(2.10) \quad \bar{a}_{ij} = e^{2\sigma} a_{ij}, \quad \bar{b}_i = e^\sigma b_i, \\ \bar{a}^{ij} = e^{-2\sigma} a^{ij}, \quad \bar{b}^i = e^{-\sigma} b^i,$$

and $b^2 = a^{ij} b_i b_j = \bar{a}^{ij} \bar{b}_i \bar{b}_j$. Hence, the Finsler space with (α, β) -metric with the length b of b_i with respect to the Riemannian metric α is invariant under any conformal change of (α, β) -metric.

From (2.10), the conformal change of Christoffel symbols are given by⁴ :

$$(2.11) \quad \bar{\gamma}_{jk}^i = \gamma_{jk}^i + \delta_j^i \sigma_k + \delta_k^i \sigma_j - \sigma^i a_{jk},$$

where $\sigma_j = \partial_j \sigma$ and $\sigma^i = a^{ij} \sigma_j$.

From (2.10) and (2.11), we have the following identities:

$$\bar{\nabla}_j \bar{b}_i = e^\sigma (\nabla_j b_i + \rho a_{ij} - \sigma_i b_j), \\ \bar{r}_{ij} = e^\sigma \{r_{ij} + \rho a_{ij} - \frac{1}{2} (b_i \sigma_j + b_j \sigma_i)\},$$

$$\begin{aligned}
\bar{s}_{ij} &= e^\sigma \{s_{ij} + \frac{1}{2}(b_i \sigma_j - b_j \sigma_i)\}, \\
\bar{s}_j^i &= e^{-\sigma} \{s_j^i + \frac{1}{2}(b^i \sigma_j - b_j \sigma^i)\}, \\
\bar{s}_j &= s_j + \frac{1}{2}(b^2 \sigma_j - \rho b_j),
\end{aligned}
\tag{2.12}$$

where $\rho = \sigma_r b^r$. From (2.11) and (2.12), we can obtain the following:

$$\begin{aligned}
\bar{\gamma}_{00}^i &= \gamma_{00}^i + 2\sigma_0 y^i - \alpha^2 \sigma^i, \\
\bar{r}_{00} &= e^\sigma (r_{00} + \rho \alpha^2 - \sigma_0 \beta), \\
\bar{s}_0^i &= e^{-\sigma} \{s_0^i + \frac{1}{2}(b^i \sigma_0 - \beta \sigma^i)\}, \\
\bar{s}_0 &= s_0 + \frac{1}{2}(b^2 \sigma_0 - \rho \beta).
\end{aligned}
\tag{2.13}$$

Next, we find the conformal change of B^{ij} given in (2.7). We have $\bar{L}(\alpha, \beta) = e^\sigma L(\alpha, \beta)$, and

$$\bar{L}_{\bar{\alpha}} = L_\alpha, \quad \bar{L}_{\bar{\alpha}\bar{\alpha}} = e^{-\sigma} L_{\alpha\alpha}, \quad \bar{L}_{\bar{\beta}} = L_\beta, \quad \bar{\gamma}^2 = e^{2\sigma} \gamma^2.
\tag{2.14}$$

By using (2.5), (2.13), (2.14) and theorem (2.3), we obtain

$$\bar{C}^* = e^\sigma (C^* + D^*),$$

where

$$D^* = \frac{\alpha\beta(\rho\alpha^2 - \sigma_0\beta)L_\alpha - \alpha(b^2\sigma_0 - \rho\beta)L_\beta}{2(\beta^2L_\alpha + \alpha\gamma^2L_{\alpha\alpha})}.$$

Hence, B^{ij} under the conformal change can be written as:

$$\begin{aligned}
\bar{B}^{ij} &= \frac{\alpha L_\beta}{L_\alpha} (s_0^i y^j - s_0^j y^i) + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} C^* (b^i y^j - b^j y^i) \\
&+ \left(\frac{\alpha \sigma_0 L_\beta}{2L_\alpha} + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} D^* \right) (b^i y^j - b^j y^i) - \frac{\alpha \beta L_\beta}{2L_\alpha} (\sigma^i y^j - \sigma^j y^i), \\
&= B^{ij} + C^{ij},
\end{aligned}$$

where

$$C^{ij} = \left(\frac{\alpha \sigma_0 L_\beta}{2L_\alpha} + \frac{\alpha^2 L_{\alpha\alpha}}{\beta L_\alpha} D^* \right) (b^i y^j - b^j y^i) - \frac{\alpha \beta L_\beta}{2L_\alpha} (\sigma^i y^j - \sigma^j y^i).$$

From (2.9), we have

$$\bar{\Omega} = e^{2\sigma} \Omega, \quad \bar{A} = e^{-\sigma} A, \quad \bar{B} = e^{2\sigma} B.$$

Now, we apply conformal transformation to B_m^{im} , and obtain

$$\bar{B}_m^{im} = B_m^{im} + K_m^{im},$$

where

$$\begin{aligned}
2K_m^{im} &= \frac{(n+1)\alpha L_\beta}{L_\alpha} (\sigma_0 b^i - \beta \sigma^i) + \alpha \left\{ \frac{(n+1)\alpha^2 \Omega L_{\alpha\alpha} b^i + \beta \gamma^2 A y^i}{\Omega^2} \right\} (\rho \alpha^2 - \sigma_0 \beta) \\
&- \left[\frac{\alpha^2 \{(n+1)\alpha^2 \Omega L_\beta L_{\alpha\alpha} b^i + B y^i\}}{L_\alpha \Omega^2} - \frac{\alpha^3 L_{\alpha\alpha} y^i}{\Omega} \right] (b^2 \sigma_0 - \rho \beta).
\end{aligned}
\tag{2.15}$$

According to G. Shanker^{13,14}, we have the following result:

Theorem 2.4: *The necessary and sufficient condition for a conformal change of Douglas space of the second kind to be a Douglas space of second kind, is that K_m^{im} , given by (2.15), are $hp(2)$.*

3. The condition to be a Finsler space

Lets us consider a special (α, β) -metric (1.2), which can be written as

$$L = \alpha\phi(s), \quad s = \frac{\beta}{\alpha},$$

where

$$\phi(s) = c_1 + c_2s + \frac{c_3}{s} + s^2,$$

$$\phi'(s) = c_2 - \frac{c_3}{s^2} + 2s,$$

$$\phi''(s) = \frac{2c_3}{s^3} + 2.$$

Now, $\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''$

$$= c_1 + c_2s + \frac{c_3}{s} + s^2 - sc_2 + \frac{c_3}{s} - 2s^2 + (b^2 - s^2)\left(\frac{2c_3}{s^3} + 2\right)$$

$$= c_1 - 3s^2 + \frac{2b^2c_3}{s^3} + 2b^2 = c_1 - s^2 + \frac{2b^2c_3}{s^3} + 2(b^2 - s^2)$$

$$> c_1 - s^2 + \frac{2b^2c_3}{s^3}, \quad \text{whenever } |s| \leq b < b_0 \quad \text{and} \quad \|\beta\|_\alpha < 1,$$

$$> c_1 - s^2 + \frac{2c_3}{s}, \quad \text{whenever } c_3 \in [0, \infty) \quad \text{and} \quad |s| \leq b < b_0,$$

$$> 0, \quad \text{whenever } c_1 \in [1, \infty).$$

Thus, one can get the following:

Theorem 3.1: *The metric $L = c_1\alpha + c_2\beta + c_3\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ is the Finsler metric if $\|\beta\|_\alpha < 1, |s| \leq b < b_0, c_1 \in [1, \infty), c_3 \in [0, \infty)$, and $c_2 \in \mathbb{R}$.*

4. Conformal change of Douglas space of second kind with Finsler space with special (α, β) -metric

In this section, we extend the study on conformal change of Douglas space of second kind. For a Finsler (α, β) -metric (1.2), we get

$$L_\alpha = \frac{\alpha^2(c_1\beta + 2c_3\alpha) - \beta^3}{\alpha^2\beta}, \quad L_\beta = \frac{\beta^2(c_2\alpha + 2\beta) - c_3\alpha^3}{\alpha\beta^2},$$

$$L_{\alpha\alpha} = \frac{2(c_3\alpha^3 + \beta^3)}{\beta\alpha^3}, \quad L_{\alpha\alpha\alpha} = \frac{-6\beta^2}{\alpha^4}.$$

Hence, from (2.9), we have

$$\Omega = \frac{1}{\beta\alpha^2} \{2b^2c_3\alpha^5 + \beta^3\alpha^2(c_1 + 2b^2) - 3\beta^5\},$$

$$A = \frac{6}{\alpha^5\beta} (c_1c_3\alpha^5 - 5\beta^2c_3\alpha^3 - 2\beta^5),$$

$$\begin{aligned} B = \frac{2}{\beta^3\alpha^6} [& 4b^2c_3^3\alpha^{11} - b^2c_1c_3^2\beta\alpha^{10} + c_3\beta^3\alpha^8\{3b^2(c_1c_2 + 7c_3) + 5c_1c_3\} \\ & + c_1c_3\beta^4\alpha^7(c_1 + 10b^2) - 3c_3\beta^5\alpha^6\{7c_3 + c_2(5b^2 + c_1)\} \\ & - 8c_3\beta^6\alpha^5(3b^2 + c_1) + \beta^7\alpha^4(2b^2c_1 + 15c_2c_3 + c_1^2) \\ & + 3\beta^8\alpha^3(7c_3 - 2b^2c_2) - 2\beta^9\alpha^2(2c_1 + 7b^2) \\ & + 6c_2\beta^{10}\alpha + 15\beta^{11}]. \end{aligned}$$

Thus, $2K_m^{im}$ in (2.15) is

$$(4.1) \quad 2K_m^{im} = \frac{(n+1)\alpha^2\{\alpha(c_2\beta^2 - c_3\alpha^2) + 2\beta^3\}(\sigma_0b^i - \beta\sigma^i)}{\beta\{(c_1\beta + 2c_3\alpha)\alpha^2 - \beta^3\}} + s_1 + s_2,$$

where

$$\begin{aligned} s_1 = & \frac{-2b^i(n+1)\alpha^2(c_3\alpha^3 + \beta^3)}{\beta\{\alpha^2(c_1\beta + 2c_3\alpha) - \beta^3\}\{2b^2c_3\alpha^5 + \beta^3\alpha^2(c_1 + 2b^2) - 3\beta^5\}} \times \\ & [-c_3(\beta\rho + b^2\sigma_0)\alpha^5 - \beta^2\rho c_1\alpha^4 + \beta^2\alpha^3\{\sigma_0(2c_3 + \\ & c_2b^2) - c_2\beta\rho\} - 7cm + \beta^3\alpha^2\{\sigma_0(2b^2 + c_1) - \beta\rho\} - \beta^5\sigma_0], \\ s_2 = & \frac{6\beta y^i}{\{\alpha^2(c_1\beta + 2c_3\alpha) - \beta^3\}\{2b^2c_3\alpha^5 + \beta^3\alpha^2(c_1 + 2b^2) - 3\beta^5\}^2} \times \\ & [\beta\{\alpha^2(c_1\beta + 2c_3\alpha) - \beta^3\}\{\alpha^3c_3(\alpha^2c_1 - 5\beta^2) - 2\beta^5\}(b^2\alpha^2 - \\ & \beta^2)(\rho\alpha^2 - \sigma_0\beta) + \alpha^2(b^2\sigma_0 - \rho\beta)[b^2c_1c_3^2\alpha^{10} - \\ & c_3\alpha^8\beta^2\{b^2(c_1c_2 + 5c_3) + c_1c_3\} - 2b^2\beta^3c_1c_3\alpha^7 + \\ & \alpha^6\beta^4c_3(5b^2c_2 + c_1c_2 + 5c_3) - 5\beta^6\alpha^4c_2c_3 + \\ & 2\beta^7\alpha^3(b^2c_2 - 4c_3) + 4b^2\beta^8\alpha^2 - 2\beta^9\alpha c_2 - 4\beta^{10}]], \end{aligned}$$

which shows that $2K_m^{im}$ is a hp(2).

Hence, we have the following Theorem:

Theorem 4.1: A Douglas space of second kind with metric L given by (1.2) is conformally transformed to a Douglas space of second kind if and only if K_m^{im} , given by the equation (4.1), are hp(2).

From theorem (4.1), we can easily obtain the conditions that the Douglas space of second kind for certain Finsler spaces with (α, β) -metric is conformally transformed to a Douglas space of second kind. We have the following cases:

Case(i) If $c_1 = 1, c_2 = 3, c_3 = 0$, the special (α, β) -metric reduces to the form

$$(4.2) \quad L = \alpha + 3\beta + \frac{\beta^2}{\alpha} = \frac{(\alpha+\beta)^2}{\alpha} + \beta,$$

which is the Randers changed square metric. Put $c_1 = 1, c_2 = 3$, and $c_3 = 0$ in the equation (4.1). Then, we get $2K_m^{im}$ as given below:

$$(4.3) \quad 2K_m^{im} = \frac{(n+1)(3\alpha+2\beta)\alpha^2(\sigma_0 b^i - \beta \sigma^i)}{(\alpha^2 - \beta^2)} + s_1 + s_2,$$

where

$$s_1 = \frac{-2b^i(n+1)\alpha^2\{-\rho\alpha^4 + 3\alpha^3(b^2\sigma_0 - \rho\beta) + \beta\alpha^2(2b^2\sigma_0 - \rho\beta + \sigma_0) - \sigma_0\beta^3\}}{\{\alpha^2(1+2b^2) - 3\beta^2\}(\alpha^2 - \beta^2)},$$

$$s_2 = \frac{-12\beta y^i}{(\alpha^2 - \beta^2)\{\alpha^2(1+2b^2) - 3\beta^2\}^2} [(\alpha^2 - \beta^2)(b^2\alpha^2 - \beta^2)(\rho\alpha^2 - \sigma_0\beta) - \alpha^2(b^2\sigma_0 - \rho\beta)\{3\alpha^3b^2 + 2b^2\beta\alpha^2 - 3\beta^2\alpha - 2\beta^3\}],$$

which shows that $2K_m^{im}$ is a hp(2).

Hence, we have the following Corollary:

Corollary 4.2: A Douglas space of second kind with Randers changed square metric is conformally transformed to a Douglas space of second kind if and only if K_m^{im} , given by the equation (4.3), are hp(2).

Case(ii) If $c_1 = a^2, c_2 = 2a, c_3 = 0$, the special (α, β) -metric reduces to the form

$$(4.4) \quad L = a^2\alpha + 2a\beta + \frac{\beta^2}{\alpha} = \frac{(a\alpha+\beta)^2}{\alpha},$$

Put $c_1 = a^2, c_2 = 2a$, and $c_3 = 0$ in the equation (4.1). Then, we get $2K_m^{im}$ as given below:

$$(4.5) \quad 2K_m^{im} = \frac{2(n+1)\alpha^2(\sigma_0 b^i - \beta \sigma^i)}{(a\alpha - \beta)} + s_1 + s_2,$$

where

$$s_1 = \frac{2b^i(n+1)\alpha^2\{a\rho\alpha^3 - \alpha^2(2b^2\sigma_0 - \rho\beta) - a\sigma_0\beta\alpha + \sigma_0\beta^2\}}{\{\alpha^2(a^2+2b^2) - 3\beta^2\}(a\alpha - \beta)},$$

$$s_2 = \frac{-12\beta y^i(b^2\alpha^2 - \beta^2)\{(a\alpha - \beta)(\rho\alpha^2 - \sigma_0\beta) - 2\alpha^2(b^2\sigma_0 - \rho\beta)\}}{(a\alpha - \beta)\{\alpha^2(a^2+2b^2) - 3\beta^2\}^2},$$

which shows that $2K_m^{im}$ is a hp(2).

Hence, we have the following Corollary:

Corollary 4.3: A Douglas space of second kind with singular square metric is conformally transformed to a Douglas space of second kind if and only if K_m^{im} , given by the equation (4.5), are hp(2).

Case (iii) If $c_1 = a, c_2 = 1, c_3 = 0$, the special (α, β) -metric reduces to the form

$$(4.6) \quad L = a\alpha + \beta + \frac{\beta^2}{\alpha} = \beta + \frac{a\alpha^2 + \beta^2}{\alpha}.$$

Put $c_1 = a, c_2 = 1$, and $c_3 = 0$ in the equation (4.1). Then, we get $2K_m^{im}$ as given below:

$$(4.7) \quad 2K_m^{im} = \frac{(n+1)(\alpha+2\beta)\alpha^2(\sigma_0 b^i - \beta \sigma^i)}{(a\alpha^2 - \beta^2)} + s_1 + s_2,$$

where

$$s_1 = \frac{-2b^i(n+1)\alpha^2\{-a\rho\alpha^4 + \alpha^3(b^2\sigma_0 - \rho\beta) + \beta\alpha^2(2b^2\sigma_0 - \rho\beta + a\sigma_0) - \sigma_0\beta^3\}}{\{\alpha^2(a+2b^2) - 3\beta^2\}(a\alpha^2 - \beta^2)},$$

$$s_2 = \frac{-12\beta y^i}{(a\alpha^2 - \beta^2)\{\alpha^2(a+2b^2) - 3\beta^2\}^2} [(a\alpha^2 - \beta^2)(b^2\alpha^2 - \beta^2)(\rho\alpha^2 - \sigma_0\beta) \\ - \alpha^2(b^2\sigma_0 - \rho\beta)\{b^2\alpha^3 + 2b^2\beta\alpha^2 - \beta^2\alpha - 2\beta^3\}],$$

which shows that $2K_m^{im}$ is a hp(2).

Hence, we have the following Corollary:

Corollary 4.4: A Douglas space of second kind with metric L given by (4.6) is conformally transformed to a Douglas space of second kind if and only if K_m^{im} , given by the equation (4.7), are hp(2).

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