On Trans-Sasakian Manifolds

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(Received February 27, 2016)

Abstract: In the present paper, pseudo projectively flat and M-projectively flat trans-Sasakian manifolds are studied. It is proved that a trans-Sasakian manifold cannot be pseudo projectively flat unless \((2n-1)(d\beta - \xi\eta) + d\alpha \psi = 0\).

If trans-Sasakian manifold is pseudo projectively flat then scalar curvature \(r = \frac{-2n(2n+1)a}{(1-a)2nb-a^3}(\alpha^2 - \beta^2 - \xi\beta)\), where \(\alpha\) and \(\beta\) are related by \((2n-1)(d\beta - \xi\eta) + d\alpha \psi = 0\). It is also proved that a trans Sasakian manifold cannot be M-projectively flat unless \((2n-1)\text{grad}\beta - \phi(\text{grad} \alpha) = (2n-1)(\xi\beta)\). If trans-Sasakian manifold is M-projectively flat then scalar curvature \(r = 2n(2n+1)(\alpha^2 - \beta^2 - \xi\beta)\), where \(\alpha\) and \(\beta\) are related by \((2n-1)\text{grad}\beta - \phi(\text{grad} \alpha) = (2n-1)(\xi\beta)\).

Keywords: Trans-Sasakian manifold, pseudo projectively flat manifold, M-projective curvature tensor.

2010 Mathematics Subject Classification: 53C50.

1. Introduction

J. A. Oubina\(^1\) introduced a manifold which generalizes both \(\alpha\)-Sasakian and \(\beta\)-Kenmotsu manifolds. Such manifold was called a trans-Sasakian manifold of type \((\alpha, \beta)\). Trans-Sasakian manifolds of type \((0, 0)\), \((\alpha, 0)\) and \((0, \beta)\) are called cosymplectic\(^2\), \(\alpha\)-Sasakian\(^3,4\) and \(\beta\)-Kenmotsu\(^5,6\) respectively. Concept of a nearly trans-Sasakian manifold was introduced by C. Gherghe\(^6\). Thus, Sasakian, Kenmotsu and cosymplectic manifold are particular cases of trans-Sasakian manifolds. J. C. Marrero\(^7\) constructed three dimensional trans-Sasakian manifold. R. Prasad and V. Srivastava\(^8\) obtained certain results on trans-Sasakian manifolds. Jeong- Sik kim et al.\(^9\) studied a generalized Ricci-recurrent trans-Sasakian manifold.
In 2002, B. Prasad defined a tensor field on a Riemannian manifold of dimension greater than 2 and he called it pseudo projective curvature tensor. This tensor is a generalisation of projective curvature tensor. Such tensor was studied on a φ-recurrent Kenmotsu manifold by Venkatesha and C. S. Bagewadi. C. S. Bagewadi, Venkatesha and N. S. Basavarajappa studied such tensor on LP-Sasakian manifold while H. G. Nagaraja and G. Somashekhara studied such tensor on a Sasakian manifold.

The pseudo projective curvature tensor on an \( n \)-dimensional Riemannian manifold is given by

\[
P(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}b[S(Y,Z)X - S(X,Z)Y] \\
- \frac{r}{2n+1}\left(\frac{a}{2n} + b\right)[g(Y,Z)X - g(X,Z)Y],
\]

where \( R, S, r \) and \( g \) are Riemannian curvature tensor, Ricci tensor, scalar curvature and metric of the Riemannian manifold respectively.

In 1971, G. P. Pokhariyal and R. S. Mishra introduced a new curvature tensor in \( n \)-dimensional manifold denoted by \( W \) and defined by

\[
W(X,Y)Z = R(X,Y)Z - \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y] \\
+ g(Y,Z)QX - g(X,Z)QY.
\]

Such a tensor field \( W \) is known as \( M \)-projective curvature tensor. \( M \)-projective curvature tensor has been studied by J. P. Singh, S. K. Chaubey and R. H. Ojha, R. N. Singh and S. K. Pandey and many others. In this paper we study pseudo-projectively flat and \( M \)-projectively flat trans-Sasakian manifolds.

### 2. Preliminaries

Let \( M \) be a \((2n+1)\)-dimensional almost contact metric manifold with almost contact metric structure \((\phi, \xi, \eta, g)\) where \( \phi \) is a \((1,1)\) tensor field, \( \xi \) is a vector field, \( \eta \) is a 1-form and \( g \) is a compatible Riemannian metric on \( M \) such that

\[
\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi \xi = 0.
\]
(2.2) \[ g(\phi X,\phi Y) = g(X,Y) - \eta(X)\eta(Y). \]

(2.3) \[ g(\phi X,Y) = g(X,\phi Y), \quad g(X,\xi) = \eta(X), \]

where \( X,Y \in TM \).

An almost contact metric manifold is said to be contact manifold if

(2.4) \[ d\eta(X,Y) = \varphi(X,Y) = g(X,\phi Y), \]

\( \varphi(X,Y) \) is being called fundamental 2–form on \( M \).

An almost contact metric manifold \( M \) is called trans-Sasakian manifold if

(2.5) \[ (\nabla_X\phi)Y = \alpha\{g(X,Y)\xi - \eta(Y)X\} + \beta\{g(\phi X,Y)\xi - \eta(Y)\phi X\}, \]

where \( \nabla \) is Levi-Civita connection of Riemannian metric \( g \) and \( \alpha \) & \( \beta \) are smooth functions on \( M \). From equations (2.1), (2.2) and (2.3), we get

(2.6) \[ (\nabla_X\phi)\xi = -\alpha\phi X + \beta[X - \eta(X)\xi], \]

(2.7) \[ (\nabla_X\eta)Y = -\alpha g(\phi X,Y) + \beta g(\phi X,\phi Y). \]

In a \((2n+1)\)–dimensional trans-Sasakian manifold, we have

(2.8) \[ R(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] + (Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2X - (X\beta)\phi^2Y, \]

(2.9) \[ R(\xi,Y)X = (\alpha^2 - \beta^2)[g(X,Y)\xi - \eta(X)Y] + (X\alpha)\phi Y + 2\alpha\beta[g(\phi X,Y)\xi - \eta(X)\phi Y] + g(\phi X,Y)(\text{grad} \alpha) + (X\beta)[Y - \eta(Y)\xi] - g(\phi X,\phi Y)(\text{grad} \beta), \]
\begin{align}
(2.10) \quad \eta(R(\xi, Y)X) &= g(R(\xi, Y)X, \xi) \\
&= (\alpha^2 - \beta^2 - \xi\beta)[g(X, Y) - \eta(X)\eta(Y)], \\
(2.11) \quad 2\alpha\beta + \xi\alpha &= 0, \\
(2.12) \quad S(X, \xi) &= (2n(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - (2n-1)X\beta - (\phi X)\alpha \\
\text{and} \quad (2.13) \quad Q\xi &= (2n(\alpha^2 - \beta^2) - \xi\beta)\xi - (2n-1)(\text{grad} \beta) + \phi(\text{grad} \alpha).
\end{align}

An almost contact metric manifold \( M \) is said to be \( \eta \)-Einstein if its Ricci-tensor \( S \) is of the form

\[ S(X, Y) = a g(X, Y) + b \eta(X)\eta(Y), \]

where \( a \) and \( b \) are smooth functions on \( M \). An \( \eta \)-Einstein manifold becomes Einstein manifold if \( b = 0 \), i.e.

\[ S(X, Y) = a g(X, Y). \]

If \( \{e_1, e_2, \ldots, e_{2n}, e_{2n+1} = \xi\} \) be a local ortho-normal basis of tangent space in a \((2n+1)\)-dimensional almost contact manifold \( M \), then we have

\[
\sum_{i=1}^{2n+1} g(X, Y) = 2n+1.
\]

\[
\sum_{i=1}^{2n+1} g(e_i, Y)S(X, e_i) = \sum_{i=1}^{2n+1} R(e_i, Y, X, e_i) = S(X, Y).
\]

3. \textbf{Pseudo Projectively Flat and M-Projectively Flat Manifolds}

Let \( M \) be a \((2n+1)\)-dimensional Pseudo projectively flat manifold, then from equation (1.1) we have

\[ aR(X, Y)Z = b[S(X, Z)Y - S(Y, Z)X] \]
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+ \frac{r}{2n+1}\left(\frac{a}{2n}+b\right)\left[g(Y,Z)X-g(X,Z)Y\right].

Contracting equation (3.1) with U, we get

(3.2) \quad g(R(X,Y)Z,U) = \frac{b}{a}\left[S(X,Z)g(Y,U)-S(Y,Z)g(X,U)\right]
\quad + \frac{r}{(2n+1)a}\left(\frac{a}{2n}+b\right)\left[g(Y,Z)g(X,U)
\quad - g(X,Z)g(Y,U)\right].

Let \{e_1,e_2,\ldots,e_{2n},e_{2n+1}=\xi\} be a local ortho-normal basis of tangent space.
Putting \(Y=Z=e_i\), in equation (3.2), we get

(3.3) \quad S(X,U) = \frac{r}{2n+1}g(X,U),

for \(a \neq b\).

Hence a pseudo projectively flat manifold is Einstein manifold if \(a \neq b\).

If the manifold is \(M\)- projectively flat then from equation (1.2), we have

(3.4) \quad R(X,Y)Z = \frac{1}{4n}\left[S(Y,Z)X-S(X,Z)Y
\quad + g(Y,Z)QX-g(X,Z)QY\right].

Contracting equation (3.4) with \(U\), we get

(3.5) \quad g(R(X,Y)Z,U) = \frac{1}{4n}\left[S(Y,Z)g(X,U)-S(X,Z)g(Y,U)
\quad + g(Y,Z)S(X,U)-g(X,Z)S(Y,U)\right].

Let \{e_1,e_2,\ldots,e_{2n},e_{2n+1}=\xi\} be a local ortho-normal basis of tangent space.
Putting \(Y=Z=e_i\), in equation (3.5), we get
Hence an $M$-projectively flat manifold is Einstein manifold.

4. Pseudo Projectively Flat Trans-Sasakian Manifold

If a trans-Sasakian manifold is Pseudo projectively flat then from equation (1.1), we have
\begin{equation}
(4.1) \quad aR(X,Y)Z = b\left[ S(X,Z)Y - S(Y,Z)X \right] + \frac{r}{2n+1}\left( \frac{a}{2n} + b \right)\left[ g(Y,Z)X - g(X,Z)Y \right].
\end{equation}

Contracting equation (4.1) with $U$, we get
\begin{equation}
(4.2) \quad g(R(X,Y)Z,U) = \frac{b}{a}\left[ S(X,Z)g(Y,U) - S(Y,Z)g(X,U) \right] + \frac{r}{(2n+1)a}\left( \frac{a}{2n} + b \right)\left[ g(Y,Z)g(X,U) - g(X,Z)g(Y,U) \right].
\end{equation}

Putting $U = \xi$ and using equation (4.2), we get
\begin{equation}
(4.3) \quad g(R(X,Y)Z,\xi) = \frac{b}{a}\left[ S(X,Z)\eta(Y) - S(Y,Z)\eta(X) \right] + \frac{r}{(2n+1)a}\left( \frac{a}{2n} + b \right)\left[ g(Y,Z)\eta(X) - g(X,Z)\eta(Y) \right].
\end{equation}

Putting $X = \xi$ and using equations (2.3), (2.10) and (2.12), we get
\begin{equation}
(4.4) \quad S(Y,Z) = \left[ \frac{ra}{2n+1}\left( \frac{a}{2n} + b \right) - \frac{a}{b}\left( \xi^2 - \beta^2 - \xi\beta \right) \right](Y,Z)
+ \left[ \frac{a}{b}\left( \xi^2 - \beta^2 - \xi\beta \right) - (2n)\left( \xi^2 - \beta^2 \right) - \xi\beta \right]
- \frac{ra}{(2n+1)(2n)}\left( \frac{a}{2n} + b \right)\eta(Y)\eta(Z) + (2n-1)Z\beta + (\phiZ)\alpha\eta(Y).
\end{equation}
From equations (3.3) and (4.4), we have

\[(4.5) \quad r = \frac{-2n(2n+1)a}{(1-a)2nb-a^2}(\alpha^2 - \beta^2 - \xi \beta)\]

and

\[(4.6) \quad (2n-1)(\beta - \xi \beta \eta) + d\alpha \phi = 0.\]

**Theorem 4.1:** A trans-Sasakian manifold cannot be pseudo projectively flat unless \((2n-1)(\beta - \xi \beta \eta) + d\alpha \phi = 0.\)

From equations (4.5) and (4.6), we get

**Theorem 4.2:** The scalar curvature \(r\) of a pseudo projectively flat trans-Sasakian manifold satisfies

\[(4.7) \quad r = \frac{-2n(2n+1)a}{(1-a)2nb-a^2}(\alpha^2 - \beta^2 - \xi \beta),\]

where \(\alpha\) and \(\beta\) are related by \((2n-1)(\beta - \xi \beta \eta) + d\alpha \phi = 0.\)

**5. \(M\)-Projectively Flat Trans-Sasakian Manifolds**

Let a trans-Sasakian manifold be \(M\)-projectively flat. From equation (1.2), we have

\[(5.1) \quad R(X,Y)Z = \frac{1}{4n} \left[ S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \right].\]

Contracting equation (5.1) with \(U\), we get

\[(5.2) \quad g(R(X,Y)Z,U) = \frac{1}{4n} \left[ S(Y,Z)g(X,U) - S(X,Z)g(Y,U) + g(Y,Z)S(X,U) - g(X,Z)S(Y,U) \right].\]

Putting \(U = \xi\) in equation (5.2) and using equations (2.3), we have
(5.3) \[ g(R(X,Y)Z, \xi) = \frac{1}{4n} [S(Y,Z)\eta(X) - S(X,Z)\eta(Y) \]

\[ + g(Y,Z)S(X,\xi) - g(X,Z)S(Y,\xi) ] \]

Again putting \( X = \xi \) in equation (5.3) and using equations (2.3), (2.10) and (2.12), we have

(5.4) \[ 4n(\alpha^2 - \beta^2 - \xi\beta)[g(Y,Z) - \eta(Y)\eta(Z)] = S(Y,Z) + \{2n(\alpha^2 - \beta^2) \]

\[ - \xi\beta \} \eta(Y)\eta(Z) - (2n-1)Z\beta\eta(Y) - (\phi Z)\alpha\eta(Y) \]

\[ + 2n(\alpha^2 - \beta^2 - \xi\beta)g(Y,Z) - \{2n(\alpha^2 - \beta^2) \]

\[ - \xi\beta \} \eta(Y)\eta(Z) + (2n-1)Y\beta\eta(Z) + (\phi Y)\alpha\eta(Z) . \]

(5.5) \[ S(Y,Z) = 2n(\alpha^2 - \beta^2 - \xi\beta)g(Y,Z) - 4n(\alpha^2 - \beta^2 - \xi\beta)\eta(Y)\eta(Z) \]

\[ + (2n-1)Z\beta\eta(Y) + (\phi Z)\alpha\eta(Y) - (2n-1)Y\beta\eta(Z) \]

\[ - (\phi Y)\alpha\eta(Z) . \]

From equations (3.6) and (5.5), we get

(5.6) \[ r = 2n(2n+1)(\alpha^2 - \beta^2 - \xi\beta) \]

and

(5.7) \[ (2n-1)\text{grad } \beta - \phi(\text{grad } \alpha) = (2n-1)(\xi\beta)\xi. \]

**Theorem 5.1:** A trans-Sasakian manifold cannot be M-projectively flat unless \((2n-1)\text{grad } \beta - \phi(\text{grad } \alpha) = (2n-1)(\xi\beta)\xi.\)

From equations (5.6) and (5.7), we have

**Theorem 5.2:** If a trans-Sasakian manifold is M-projectively flat then scalar curvature \( r = 2n(2n+1)(\alpha^2 - \beta^2 - \xi\beta) \), where \( \alpha \) and \( \beta \) are related by \((2n-1)\text{grad } \beta - \phi(\text{grad } \alpha) = (2n-1)(\xi\beta)\xi.\)
The first author is thankful to UGC, Government of India for providing the UCG- Dr D. S. Kothari Post Doctoral Fellowship.

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