

On Non-Invariant Hypersurface of Lorentzian Para-Sasakian Manifold with Generalized Symmetric Metric Connection

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Abstract: The present paper deals with the study of non-invariant hypersurface $M(f, g, u, v, \lambda)$ of a LP-Sasakian manifold $\bar{M}$ with generalized symmetric metric connection. Some properties of this structure have been discussed. A necessary and sufficient condition for a totally umbilical non-invariant hyper-surface $M(f, g, u, v, \lambda)$ of a LP-Sasakian manifold $\bar{M}$ with generalized symmetric metric connection to be totally geodesic is proved.

Keywords: LP-Sasakian manifold, generalized symmetric metric connection, totally geodesic.

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1. Introduction

A linear connection $\bar{\nabla}$ is said to be generalized symmetric connection if its torsion tensor T is of the form

$$(1.1) \quad T(X, Y) = \alpha \{ \eta(Y)X - \eta(X)Y \} + \beta \{ \eta(Y)\phi X - \eta(X)\phi Y \},$$

for any vector fields X, Y on a manifold, where α and β are smooth functions, ϕ is a tensor of type $(1,1)$ and η is a 1-form. Moreover, the connection $\bar{\nabla}$ is said to be a generalized symmetric metric connection if there is a Riemannian metric g in $\bar{M}$ such that $\bar{\nabla}_g = 0$, otherwise it is non-metric. In the equation (1.1), if $\alpha = 0$ ($\beta = 0$), then the generalized symmetric connection is called β -quarter-symmetric connection (α -semi-symmetric connection). Moreover, if we choose $(\alpha, \beta) = (1, 0)$ and $(\alpha, \beta) = (0, 1)$, then the generalized symmetric connection is reduced to a semi-symmetric connection and quarter-symmetric connection, respectively. Hence a generalized symmetric connection can be viewed as a generalization of semi-symmetric connection

and quarter-symmetric connection. These two connections are important for the study of geometry and to understand application in physics. In 1932, Hayden¹ introduced a metric connection with non-zero torsion on a Riemannian manifold. The properties of Riemannian manifolds with semi-symmetric (symmetric) and non-metric connection have been studied by many authors²⁻⁶. The idea of quarter-symmetric linear connections in a differential manifold was introduced by Golab⁴. A linear connection is said to be a quarter-symmetric connection, if its torsion tensor T is of the form

$$(1.2) \quad T(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$

for any vector fields X, Y on a manifold, where u is a 1-form and ϕ is a tensor of type $(1,1)$. If $\phi = I$, then the quarter-symmetric connection is reduced to a semi-symmetric connection. Hence quarter-symmetric connection can be viewed as a generalization of semi-symmetric connection.

On the other hand, in 1989, Matsumoto⁷ introduced the notion of Lorentzian para-Sasakian manifolds. Mihai and Rosca⁸ studied the same manifolds independently and they obtained several results on such manifolds. Lorentzian para-Sasakian manifolds have also been studied by Matsumoto and Mihai⁹, Mihai, Shaikh and De¹⁰. Srivastava and Kushwaha¹¹ studied Lorentzian para-Sasakian manifolds admitting a special semi symmetric recurrent metric connection. Perktas, Kilic and Tripathi¹² investigated curvature tensors with respect to semi-symmetric connection in a Lorentzian para-Sasakian manifold. In 2016, Bahadir¹³⁻¹⁴ got some results on Lorentzian para-Sasakian manifold with quarter-symmetric non-metric connection and in 2018, studied Lorentzian para-Sasakian manifold with generalised-symmetric metric connection.

Blair and Ludden¹⁵ showed that a non-invariant hypersurface of an almost contact manifold admits an almost complex structure and a distinguished 1-form induced by the contact form of the manifold. They also investigated non-invariant hypersurfaces of an almost contact metric manifold. The notion of (f, g, u, v, λ) -structure was given by Goldberg and Yano¹⁶. It is well known that a hypersurface of an almost contact metric manifold always admits a (f, g, u, v, λ) -structure. Yanno and Okumura¹⁸ proved that there always exists a (f, g, u, v, λ) structure on a non-invariant hypersurface of an almost contact metric manifold. They also proved that there does not exist invariant hypersurface of a contact manifold. Prasad¹⁹ studied the non-invariant hypersurfaces of a trans-Sasakian manifolds. In this paper, we study non-invariant hypersurface of a LP-Sasakian manifold with generalized symmetric metric connection.

2. Preliminaries

Let $\bar{M}$ be a $(2n + 1)$ -dimensional almost contact metric manifold with almost contact metric structure (ϕ, ξ, η, g) . Then we have

$$(2.1) \quad \phi^2 = X + \eta(X) \quad \eta(\xi) = -1,$$

$$(2.2) \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

$$(2.3) \quad g(X, \xi) = \eta(X),$$

$$(2.4) \quad g(\phi X, Y) = g(X, \phi Y) = \varphi(X, Y),$$

for all $X, Y \in \bar{M}$. Such a structure (ϕ, ξ, η, g) is termed as Lorentzian para-contact structure. Also in a Lorentzian para-contact structure the following relations hold:

$$(2.5) \quad \eta(\phi) = 0, \quad \phi(\xi) = 0.$$

A Lorentzian para-contact manifold $\bar{M}$ is called Lorentzian para-Sasakian manifold (LP-Sasakian manifold) if

$$(2.6) \quad (\bar{\nabla}_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,$$

$$(2.7) \quad \bar{\nabla}_X \xi = \phi X,$$

for all $X, Y \in \bar{M}$, where $\bar{\nabla}$ is called operator of covariant differentiation with respect to Lorentzian metric g . We define a generalized symmetric metric connection $\bar{\nabla}$ in LP-Sasakian manifold $\bar{M}$ by

$$(2.8) \quad \bar{\nabla}_X Y = \nabla_X Y + \alpha \{ \eta(Y)X - g(X, Y)\xi \} + \beta \{ \eta(Y)\phi X - g(\phi X, Y)\xi \},$$

$$(2.9) \quad (\bar{\nabla}_X \phi)Y = [(1 - \beta)g(X, Y) + (2 - 2\beta)\eta(X)\eta(Y) - \alpha g(\phi X, Y)\xi] + (1 - \beta)\eta(Y)X - \alpha\eta(Y)\phi X,$$

$$(2.10) \quad \bar{\nabla}_X \xi = (1 - \beta)\phi X - \alpha X - \alpha\eta(X)\xi,$$

$$(2.11) \quad (\bar{\nabla}_X \eta)Y = (1 - \beta)g(\phi X, Y) - \alpha g(\phi X, \phi Y),$$

for all $X, Y \in (T \bar{M})$.

A hypersurface of an almost contact metric manifold $\bar{M}$ is called a non-invariant hypersurface, if the transform of a tangent vector of the hypersurface under the action of tensor ϕ defining the contact structure is never tangent to the hypersurface. Let X be a tangent vector on non-invariant hypersurface of an almost contact metric manifold $\bar{M}$ then ϕX is never tangent to the hypersurface.

Let M be a non invariant hypersurface of an almost contact metric manifold, then we have

$$(2.12) \quad \phi X = fX + u(X)\hat{N},$$

$$(2.13) \quad \phi \hat{N} = -U,$$

$$(2.14) \quad \xi = V + \lambda \hat{N}, \lambda = \eta(\hat{N})$$

$$(2.15) \quad \eta(X) = v(X),$$

where f is a (1,1) tensor field, u, v are 1-form, $\hat{N}$ is unit normal to the hypersurface $X \in TM$ and $u(X) \neq 0$, then we get an induced (f, g, u, v, λ) -structure on M satisfying the conditions

$$(2.16) \quad f^2 = -I + u \otimes U + v \otimes V,$$

$$(2.17) \quad fU = -\lambda V, fV = \lambda U,$$

$$(2.18) \quad uof = \lambda v, vof = -\lambda u,$$

$$(2.19) \quad u(U) = 1 - \lambda^2, u(V) = v(U), v(V) = 1 - \lambda^2,$$

$$(2.20) \quad g(fX, fY) = g(X, Y) - u(X)u(Y) - v(X)v(Y),$$

$$(2.21) \quad g(X, fY) = -g(fX, Y) - g(X, U) = u(X),$$

$$(2.22) \quad g(X, V) = v(X),$$

for all $X, Y \in TM$ and $\lambda = \eta(\hat{N})$

We denote by g the metric tensor of $\bar{M}$ as well as on M . Let $\bar{\nabla}$ be the generalized symmetric connection on $\bar{M}$ and ∇ be the induced connection on M with respect to unit normal vector $\hat{N}$. Gauss and Weingarten formulas are given by

$$(2.23) \quad \bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y)\hat{N},$$

$$(2.24) \quad \bar{\nabla}_X \hat{N} = -A_{\hat{N}}X,$$

where σ is the second fundamental form of M related to $A_{\hat{N}}$ by

$$(2.25) \quad \sigma(X, Y) = g(A_{\hat{N}}X, Y)$$

for all $X, Y \in TM, \hat{N} \in T^\perp M$.

3. Non-invariant hypersurfaces of LP-Sasakian manifold with generalized symmetric metric connection

Lemma 3.1: Let $M(f, g, u, v, \lambda)$ be a non-invariant hypersurface of a LP-Sasakian manifold with generalized symmetric metric connection. Then

$$(3.1) \quad (\bar{\nabla}_X \phi)Y = (\nabla_X f)Y - u(Y)A_{\hat{N}}X + \sigma(X, Y)U + ((\nabla_Y u)X + \sigma(X, fY))\hat{N}$$

$$(3.2) \quad (\bar{\nabla}_X \eta)Y = (\nabla_X v)Y - \lambda \sigma(X, Y),$$

$$(3.3) \quad \bar{\nabla}_X \xi = \nabla_X V - \lambda A_{\hat{N}}X + (\sigma(X, V) + X\lambda)\hat{N}$$

Proof: Consider

$$(\bar{\nabla}_X \varphi)Y = (\bar{\nabla}_X \varphi)Y - \varphi(\bar{\nabla}_X Y),$$

Using equation (2.12) and (2.23), we get

$$(\bar{\nabla}_X \varphi)Y = \nabla_X(fY) + \sigma(X, fY)\hat{N} + u(Y)(\bar{\nabla}_X \hat{N}) + (\bar{\nabla}_X u(Y))\hat{N} - f(\nabla_X Y) - u(\nabla_X Y)\hat{N} + \sigma(X, Y)U,$$

or

$$(\bar{\nabla}_X \varphi)Y = (\nabla_X f)Y - u(Y)A_{\hat{N}}X + \sigma(X, Y)U + ((\nabla_Y u)X + \sigma(X, fY))\hat{N}.$$

Also, we have

$$(\bar{\nabla}_{X\eta})Y = \bar{\nabla}_{X\eta}(Y) - \eta(\bar{\nabla}_X Y).$$

Using equation (2.15) and (2.23), we obtain

$$(\bar{\nabla}_{X\eta})Y = \nabla_X(v(Y)) - v(\bar{\nabla}_X Y) - \lambda\sigma(X, Y),$$

or

$$(\bar{\nabla}_X \eta)Y = (\nabla_X v)Y - \lambda\sigma(X, Y).$$

Further, consider

$$\bar{\nabla}_X \xi = \nabla_X \xi + \sigma(X, \xi)\hat{N},$$

using equation (2.14), we get

$$\bar{\nabla}_X \xi = \nabla_X V + \lambda \nabla_X \hat{N} + (X\lambda)\hat{N} + \sigma(X, V)\hat{N},$$

again using equation (2.24), we have

$$\bar{\nabla}_X \xi = \nabla_X V - \lambda A_{\hat{N}}X + (\sigma(X, V) + X\lambda)\hat{N}.$$

Theorem 3.2: Let $M(f, g, u, v, \lambda)$ be a non-invariant hypersurface of Lorentzian para-Sasakian manifold $\bar{M}$ with generalized symmetric metric connection. Then

$$(3.4) \quad \sigma(X, \xi)U = (\beta - 1)f^2X + (1 - \beta)u(X)U + \alpha f(X),$$

$$(3.5) \quad u(\nabla_X \xi) = (1 - \beta)u(fX) - \alpha u(X).$$

Proof: Consider

$$(\bar{\nabla}_X \varphi)\xi = \bar{\nabla}_X \varphi\xi - \varphi(\bar{\nabla}_X \xi),$$

using equation (2.5) and (2.10), we get

$$= -\varphi((1 - \beta)\varphi X - \alpha X - \alpha\eta(X)\xi).$$

Again using equation (2.12), we obtain

$$(3.6) \quad (\bar{\nabla}_X \varphi)\xi = -(1 - \beta)f^2X - (1 - \beta)u(fX)\hat{N} + (1 - \beta)u(X)U + \alpha f(X) + \alpha u(X)\hat{N}.$$

We also know that

$$(3.7) \quad (\bar{\nabla}_X \varphi)\xi = -f(\nabla_X \xi) - u(\nabla_X \xi)\hat{N} + \sigma(X, \xi)U.$$

From equation (3.6) and (3.7), we have

$$\begin{aligned} -f(\nabla_X \xi) - u(\nabla_X \xi)\hat{N} + \sigma(X, \xi)U &= -(1 - \beta)f^2X \\ &\quad - (1 - \beta)u(fX)\hat{N} + (1 - \beta)u(X)U \\ &\quad + \alpha f(X) + \alpha u(X)\hat{N}. \end{aligned}$$

Equating tangential and normal component on both side of the above equation, we get (3.4) and (3.5).

Theorem 3.3: Let $M(f, g, u, v, \lambda)$ be a non-invariant hypersurface of Lorentzian para-Sasakian manifold $\bar{M}$ with generalized symmetric metric connection. Then

$$(3.8) \quad (\nabla_X f)Y = u(Y)A_{\bar{N}}X - \sigma(X, Y)U + (1 - \beta)g(X, Y)V + (2 - 2\beta)v(X)v(Y)V - \alpha g(fX, Y)V + (1 - \beta)v(Y)X - \alpha v(Y)f(X),$$

$$(3.9) \quad (\nabla_X u)Y = (1 - \beta)\lambda g(X, Y) + (2 - 2\beta)\lambda v(X)v(Y) - \alpha \lambda g(fX, Y) - \alpha u(X)v(Y) - \sigma(X, fY),$$

$$(3.10) \quad \nabla_X V = \lambda A_{\bar{N}}X + (1 - \beta)fX - \alpha X - \alpha v(X)V,$$

$$(3.11) \quad \sigma(X, V) = (1 - \beta)v(X) - \alpha \lambda v(X) - X\lambda,$$

$$(3.12) \quad (\nabla_X v)Y = \lambda \sigma(X, Y) + (1 - \beta)g(fX, Y) - \alpha[g(X, Y) - v(X)v(Y)].$$

Proof: Using (2.12), (2.14) in equation (2.9) and (3.1), we obtain

$$\begin{aligned} (\nabla_X f)Y - u(Y)A_{\bar{N}}X + \sigma(X, Y)U + ((\nabla_X u)Y + \sigma(X, fY))\hat{N} \\ = [(1 - \beta)g(X, Y) + (2 - 2\beta)v(X)v(Y) - \alpha g(fX \\ + u(X)\hat{N}, Y)]\xi + (1 - \beta)v(Y)X - \alpha v(Y)(fX + u(X)\hat{N}), \\ = (1 - \beta)g(X, Y)V + (1 - \beta)\lambda g(X, Y)\hat{N} + (2 - \\ 2\beta)v(X)v(Y)V + (2 - 2\beta)\lambda v(X)v(Y)\hat{N} - \alpha g(fX, Y)V - \\ \alpha \lambda g(fX, Y)\hat{N} + (1 - \beta)v(Y)X - \alpha v(Y)fX - \alpha u(X)v(Y)\hat{N}. \end{aligned}$$

Equating tangential and normal component of the above equation, we get (3.8) and (3.9).

Also using equations (2.10), (2.12) and (2.15), we get

$$(3.13) \quad \bar{\nabla}_X \xi = (1 - \beta)fX + (1 - \beta)u(X)\hat{N} - \alpha X - \alpha v(X)V - \alpha \lambda v(X)\hat{N},$$

and from equation (3.3), we have

$$(3.14) \quad \bar{\nabla}_X \xi = \nabla_X V - \lambda A_{\bar{N}}X + (\sigma(X, V) + X\lambda)\hat{N}.$$

Equating tangential and normal part of equations (3.13) and (3.14), we get equations (3.10) and (3.11). Also using equations (2.11), (2.12) and (3.2) we get (3.12).

Theorem 3.4: Let $M(f, g, u, v, \lambda)$ be a non-invariant hypersurface of Lorentzian para-Sasakian manifold $\bar{M}$ with generalized symmetric metric connection. Then

$$(3.15) \quad \begin{aligned} (\bar{\nabla}_X \varphi)Y &= (1 - \beta)(g(X, Y)V + v(Y)X) \\ &- \alpha(g(fX, Y)V + v(Y)fX) + (2 - 2\beta)v(X)v(Y)V \\ &+ ((1 - \beta)\lambda g(X, Y) - \alpha \lambda g(fX, Y) \\ &- \alpha u(X)v(Y) + (2 - 2\beta)\lambda v(X)u(Y))\hat{N}. \end{aligned}$$

Proof: From equation (3.1), we have

$$(3.16) \quad (\bar{\nabla}_X \varphi)Y = (\nabla_X f)Y - u(Y)A_{\bar{N}}X + \sigma(X, Y)U + ((\nabla_Y u)X + \sigma(X, fY))\hat{N}.$$

And also from (3.8) and (3.9), we get

$$(3.17) \quad (\nabla_X f)Y = u(Y)A_{\bar{N}}X - \sigma(X, Y)U + (1 - \beta)g(X, Y)V \\ + (2 - 2\beta)v(X)v(Y)V - \alpha g(fX, Y)V \\ + (1 - \beta)v(Y)X - \alpha v(Y)f(X),$$

and

$$(3.18) \quad (\nabla_X u)Y = (1 - \beta)\lambda g(X, Y) + (2 - 2\beta)\lambda v(X)v(Y) \\ - \alpha \lambda g(fX, Y) - \alpha u(X)v(Y) - \sigma(X, fY).$$

Using equations (3.17), (3.18) in equation (3.16), we get (3.15).

Theorem 3.5: Let $M(f, g, u, v, \lambda)$ be a totally umbilical non-invariant hypersurface of Lorentzian para-Sasakian manifold $\bar{M}$ with generalised symmetric metric connection. Then it is totally geodesic if and only if

$$(3.19) \quad (1 - \beta)u(X) - \alpha \lambda v(X) - X\lambda = 0.$$

Proof: From equation (2.10), we get

$$\bar{\nabla}_X \xi = (1 - \beta)\varphi X - \alpha X - \alpha \eta(X)\xi,$$

using equations (2.12), (2.14) and (2.15) in above equation we obtain

$$\bar{\nabla}_X \xi = (1 - \beta)fX + (1 - \beta)u(X)\hat{N} - \alpha X - \alpha v(X)V - \alpha \lambda v(X)\hat{N}.$$

Equating the normal part of the above equation and equation (3.3), we get

$$(3.20) \quad \sigma(X, Y) = (1 - \beta)u(X) - \alpha \lambda v(X) - X\lambda.$$

If M is totally umbilical, then $A_{\bar{N}} = \zeta I$. From equation (2.25), we have

$$\sigma(X, Y) = g(A_{\bar{N}}X, Y) = g(\zeta X, Y) = \zeta g(X, Y),$$

$$(3.21) \quad \sigma(X, V) = \zeta g(X, V) = v(X).$$

From equations (3.20) and (3.21), we get

$$(3.22) \quad (1 - \beta)u(X) - \alpha \lambda v(X) - X\lambda - \zeta v(X) = 0.$$

If M is totally geodesic then $\zeta = 0$, if and only if (3.19) is true.

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