Stability Analysis of Fractional-Order Hindmarsh-Rose Neuron Model with Time-Delays

G. Velmurugan and R. Rakkiyappan

Department of Mathematics
Bharathiar University, Coimbatore - 641 046, Tamil Nadu, India.
E-mail: gvmuruga@gmail.com, rakkigru@gmail.com.

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Abstract: In this paper, the authors considered a class of fractional-order Hindmarsh-Rose (HR) neuron model with time-delays and widely investigated the dynamical behaviors of considered fractional-order HR model with time-delays. Some new sufficient conditions are derived by using the proposed stability theory of linear fractional-order systems to guarantee the stability results of fractional-order HR model with time-delays. Numerical simulations are also presented as evidence for the effectiveness of our derived theoretical results.

Keywords: Hindmarsh-Rose neuron (HR) model - Stability - Fractional-order - Time delays.

1. Introduction

Recently, many mathematicians and researchers have focused their interest and attention to analysis of the dynamical properties of fractional differential equations due to wide range of applications in various fields of science and engineering such as dielectric, electromagnetic waves, viscoelastic systems, physics, biology, chemistry, medicine etc\(^1\)\(^-\)\(^6\). In general, fractional calculus has proven to be a very suitable tool for the description of memory and hereditary properties of various materials and processes\(^6\). Due to this merit, many of the real-world applications are mathematically modeled by using the fractional-order differential equations, which give more accurate results and have more complicated properties than the classical integer-order differential equations. Thus, the study of fractional-order dynamical systems and fractional-order differential equations are great deal of attention and significance in both theory and applications. Moreover, several fractional-order dynamical systems are

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proposed and extensively analysis the dynamical characteristics of considered systems in the existing literature, such as fractional-order Lorenz system, fractional-order Chen system, fractional-order Liu system\textsuperscript{7-9}. Nowadays, the analysis of dynamical spiking/bursting properties of neuron models or neuronal ensembles have become a hotspot in the area of research. As we know that bursts could provide a more reliable approach of information transfer\textsuperscript{10-12}. There are some neuronal models were presented in the literature such as Hodgkin-Huxley, FitzHugh-Nagumo, Morris-Lecar, Hindmarsh-Rose (HR)\textsuperscript{13-16}. Many of the authors studied the dynamical properties of neuronal models in the form of integer-order differential equations such as stability, synchronization, bifurcation and Hopf bifurcation\textsuperscript{13-16}. Moreover, the main advantages of fractional-order systems are that have more degrees of freedom and infinite memory. The neuronal models are incorporated in to the infinite memory by introducing fractional-order. On the other hand, time delay systems have been received much attention in the field of interdisciplinary subjects. The neuron model with timedelay may cause certain dynamical behaviors such as oscillation, divergences, chaos, instability, or other poor performance. Thus, the fractional-order neuron model with time delays have more complicated and interesting one. Many of researchers\textsuperscript{17-19} have focused their interest to analysis of the properties of fractional-order neuron model with time delays and provided some excellent results. HR model is a slow-fast system and its detailed description is given by X. Shi and Z. Wang\textsuperscript{17}. The authors studied the problem of adaptive synchronization of time delay Hindmarsh-Rose neuron system via self-feedback and provided some sufficient conditions to ensure the synchronization based on adaptive control theory and Lyapunov stability theory. Analysis the dynamical behaviors such as stability, Hopf bifurcation and periodically firing of fractional-order Hindmarsh-Rose neuronal model were extensively studied by D. Jun, Z. Guang-jun, X. Yong, Y. Hong and W. Jue\textsuperscript{18}. X. Yong, K. Yan Mei, L. Yong and W. Ying\textsuperscript{19} widely investigated the firing properties and synchronization rate of considered fractional-order Hindmarsh-Rose model neurons. However, the studies on fractional-order Hindmarsh-Rose neuron model are without consideration of time-delays. To the best of our knowledge, the problem of stability and Hopf bifurcation analysis of fractional-order HR neuron model with time delay has not been investigated in the literature.

Motivated by the above discussion, in this paper we consider the fractional-order HR neuron model with time delay and widely studied the stability and Hopf bifurcation properties of considered model. Some new sufficient conditions are derived to ensure the stability of HR neuron model.
with time delays by using stability theorem for linear fractional order systems with multiple time delays. Finally, numerical simulations are proposed to show the effectiveness of our results.

2. Preliminaries

In this section, the description of fractional-order delayed Hindmarsh-Rose neuron (HR) model is given as well as some fundamental definitions, lemmas and Theorems of fractional calculus have been provided, which are used to demonstrate our main results. The Caputo fractional-order derivative is employed throughout this paper.

**Definition 2.1:** The Caputo fractional-order derivative of order \( \mu \) for a function \( h(t) \in C^{n+1}\left([t_0, +\infty), R\right) \) is defined as

\[
t_0 D_t^\mu h(t) = \frac{1}{\Gamma(n-\mu)} \int_{t_0}^t \frac{h^{(n)}(\tau)}{(t-\tau)^{\mu+1-n}} d\tau,
\]

where \( \mu > 0 \) and \( n \) is a positive integer such that \( n - 1 < \mu \leq n \) and \( \Gamma(\cdot) \) is a Gamma function.

**Definition 2.2:** The Laplace transform of the Caputo fractional-order derivative is

\[
L\{t_0 D_t^\mu h(t); s\} = s^\mu H(s) - \sum_{k=0}^{n-1} s^{\mu-k-1} h^{(k)}(t_0), \quad n - 1 < \mu \leq n.
\]

When \( h^{(k)}(t_0) = 0, k = 1, 2, \cdots, n - 1 \), then

\[
L\{t_0 D_t^\mu h(t); s\} = s^\mu H(s),
\]

where \( L\{\cdot\} \) denotes the Laplace transform and \( s \) is the variable of Laplace domain.

**Theorem 2.3:** The following linear autonomous system

\[
D_t^\mu x = Ax \quad \text{and} \quad x(0) = x_0,
\]
where \(0 < \mu \leq 1\), \(x \in \mathbb{R}^n\) and \(A \in \mathbb{R}^{n \times n}\) is an \(n \times n\) matrix, is asymptotically stable if and only if \(|\arg(\lambda)| > \frac{\mu\pi}{2}\) is satisfied for all eigenvalues \((\lambda)\) of the matrix \(A\). In this case, each component of the states decays towards 0 like \(t^{-\mu}\). Also, this system is stable if and only if \(|\arg(\lambda)| \geq \frac{\mu\pi}{2}\) is satisfied for all eigenvalues \((\lambda)\) of the matrix \(A\) with those critical eigenvaluessatisfying \(|\arg(\lambda)| = \frac{\mu\pi}{2}\) having geometric multiplicity of one.

**Theorem 2.4:** For the nonlinear fractional-order system\(^{21}\)

\[
D_\mu^\mu x = g(x),
\]

where \(0 < \mu \leq 1\), \(x \in \mathbb{R}^n\) and \(g\) is a smooth nonlinear function, the equilibrium \(x_0\) is locally asymptotically stable if \(|\arg(\lambda)| > \frac{\mu\pi}{2}\) is satisfied for all eigenvalues \((\lambda)\) of the Jacobian matrix \(J = \frac{\partial g}{\partial x}\) evaluated at the equilibrium \(x_0\), where \(x_0\) satisfies \(g(x_0) = 0\).

In this paper, we consider the fractional-order Hindmarsh-Rose(HR) neuron model with time delay and the description as follows

\[
\begin{align*}
D_\mu^\mu u &= au^2 - bu^3 + v - w(t - \tau) + I, \\
D_\mu^\mu v &= c - du^2 - v, \\
D_\mu^\mu w &= r\beta(u + \delta) - w,
\end{align*}
\]

where \(\mu\) is the fractional-order \((0 < \mu < 1)\), \(u\) represents the membrane potential, \(v\) is associated with the fast current of \(Na^+\) or \(K^+\) ions and \(w\) is associated with the slow current of, for example \(Ca^+\) ions. The parameters \(a, b, c, d, r, \beta, \delta, I\) are real constants. \(\tau > 0\) is time delay and \(I\) represents the external current in biological neurons.
Let \((u^*, v^*, w^*)\) be the equilibrium point of the system (2.6). From (2.6), we can obtain the linearizing equations as follows

\[
\begin{align*}
D^\alpha \ddot{u} &= 2au^* \dot{u} - 3bu^* \dot{u}^2 + \ddot{v} - \ddot{w}(t - \tau), \\
D^\alpha \ddot{v} &= -2du^* \dot{u} - \ddot{v}, \\
D^\alpha \ddot{w} &= r(\beta \dot{u} - \ddot{w}),
\end{align*}
\]

(2.7)

with the following linear transform, \(\ddot{u} = u - u^*, \ddot{v} = v - v^*\) and \(\ddot{w} = w - w^*\). Equation (2.7) can be rewritten as in the following matrix form

\[
D^\alpha \dot{U} = JU(t) + HU(t - \tau),
\]

(2.8)

where \(U = (\ddot{u}, \ddot{v}, \ddot{w})^T\) and \(J\) is the Jacobian matrix of system (2.6) at the equilibrium point \((u^*, v^*, w^*)\), such as

\[
J = \begin{bmatrix}
2au^* - 3bu^*^2 & 1 & 0 \\
-2du^* & -1 & 0 \\
r\beta & 0 & -r
\end{bmatrix}
\quad \text{and} \quad H = \begin{bmatrix}
0 & 0 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Now, taking Laplace transform on both sides of equation (2.8) one can obtain

\[
\begin{align*}
s^\alpha U_1(s) - s^{\alpha-1}\delta_1(0) &= 2au^* U_1(s) - 3bu^*^2 U_1(s) + U_2(s) \\
 &\quad - e^{-s\tau}\left(U_3(s) + \int_0^\tau e^{-st}\delta_3(t)dt\right), \\
s^\alpha U_2(s) - s^{\alpha-1}\delta_2(0) &= U_2(s) - 2du^* U_1(s), \\
s^\alpha U_3(s) - s^{\alpha-1}\delta_3(0) &= r\beta U_1(s) - rU_3(s),
\end{align*}
\]

(2.9)

where \((U_1(s), U_2(s), U_3(s))\) and \((\delta_1(0), \delta_2(0), \delta_3(0))\) are represents the Laplace transform and initial conditions of \((\ddot{u}(t), \ddot{v}(t), \ddot{w}(t))\). Equation (2.9) can be rewritten as in the vector form as follows...
\[(2.10) \quad \Delta(s) \begin{pmatrix} U_1(s) \\ U_2(s) \\ U_3(s) \end{pmatrix} = \begin{pmatrix} k_1(s) \\ k_2(s) \\ k_3(s) \end{pmatrix}, \]

where
\[(2.11) \quad \Delta(s) = \begin{pmatrix} s^{\mu} - 2au^* + 3bu^*^2 & -1 & e^{-st} \\ 2du^* & s^{\mu} + 1 & 0 \\ -r\beta & 0 & s^{\mu} + r \end{pmatrix}, \]

and
\[(2.12) \quad \begin{cases} k_1(s) = s^{\mu-1}\delta_1(0) - e^{-st}\int_t^0 e^{^{-s\tau}}\delta_3(t)dt, \\ k_2(s) = s^{\mu-1}\delta_2(0), \\ k_3(s) = s^{\mu-1}\delta_3(0). \end{cases} \]

Thus, \(\Delta(s)\) is the characteristic matrix of system (2.6) for simplicity and \(k_1(s), k_2(s), k_3(s)\) is the remaining nonlinear part of system (2.6). Also, \(\det(\Delta(s))\) is a characteristic polynomial of (2.6) and the distribution of \(\det(\Delta(s))\)’s eigenvalues are entirely used to studied the stability of system (2.6)

### 3. Results

In this section, we establish some stability theorems to ensure the stability and hopf bifurcation of considered HR neuron model with time delays.

**Theorem 3.1:** If all the roots of the characteristic equation \(\det(\Delta(s)) = 0\) have negative real parts, then the zero solution of system (2.6) is Lyapunov globally asymptotically stable.

**Proof:** Multiplying by \(s\) on both sides of the equation (2.10), one can obtain
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\begin{equation}
\Delta(s) \begin{pmatrix}
    sU_1(s) \\
    sU_2(s) \\
    sU_3(s)
\end{pmatrix} = \begin{pmatrix}
    sk_1(s) \\
    sk_2(s) \\
    sk_3(s)
\end{pmatrix}.
\end{equation}

If all the roots of the transcendental equation \( \det(\Delta(s)) = 0 \) lie in open left half complex plane, i.e., \( \text{Re}(s) < 0 \), then we consider (3.1) in \( \text{Re}(s) \geq 0 \). In this restricted area, (3.1) has a unique solution \((sU_1(s), sU_2(s), sU_3(s))\). Thus, we have

\[ \lim_{s \to 0, \text{Re}(s) \geq 0} sU_i(s) = 0, \quad i = 1, 2, 3. \]

From the assumption of all roots of the characteristic equation \( \det(\Delta(s)) = 0 \) and the final-valued theorem of Laplace transform\(^2\), we get

\[ \lim_{t \to \infty} \tilde{u}(t) = \lim_{s \to 0, \text{Re}(s) \geq 0} sU_1(s) = 0, \]

\[ \lim_{t \to \infty} \tilde{v}(t) = \lim_{s \to 0, \text{Re}(s) \geq 0} sU_2(s) = 0, \]

\[ \lim_{t \to \infty} \tilde{w}(t) = \lim_{s \to 0, \text{Re}(s) \geq 0} sU_3(s) = 0. \]

Hence, we conclude that the considered fractional-order HR neuron model with time delay system (2.6) is Lyapunov globally asymptotically stable.

**Theorem 3.2:** If all the roots of the characteristic equation \( \Delta(s) = \det\left(s^\mu I - J - He^{-st}\right) = 0 \) have negative real parts, then the zero solution of system (2.6) is Lyapunov asymptotically stable.

**Proof:** From (2.9) we can rewrite as follows

\begin{equation}
\Delta(s)U(s) = k(s),
\end{equation}

where \( U(s) = (U_1(s), U_2(s), U_3(s))^T \) represents the Laplace transform of \( u(t) = (\tilde{u}(t), \tilde{v}(t), \tilde{w}(t))^T \) with \( U(s) = L(u(t)) \). Then \( \Delta(s) \) and \( k(s) \) are same as in (2.11) and (2.12).

Now, we can obtain the characteristic equation of (2.6) as follows
\[ (3.3) \quad \Delta(s) = \det \left( s^\mu I - J - He^{-st} \right) = 0. \]

Followed by the paper \textsuperscript{22}, we conclude that all the roots of equation (3.3) have negative real parts then the system (2.6) is Lyapunov asymptotically stable.

4. Numerical Simulation

In this section, we provide Numerical simulations, which are used to verify our theoretical results. The fractional-order HR neuron model with time delays (2.6) have chosen the following parameter values as \( a = 3.0, b = 1.0, c = 1.0, d = 5.0, r = 0.006, \beta = 4.0, \delta = 1.6 \) and time delay \( \tau = 1.0 \). In this paper, we consider the parameters \( I \) and \( \mu \) are variable. Given different values of the parameters \( I \) and \( \mu \) that shows different behavior of considered HR system (2.6). In Figure 1-3 shows that the time trajectory and phase portrait between \( u(t) \) and \( w(t) \) in which the parameter \( I = 1.3 \) and the fractional order \( \mu = 0.9, 0.7, 0.5 \) respectively. For the different values of fractional-order \( \mu = 0.9, 0.7, 0.5 \), it exhibits the spiking/bursting behaviors for HR neuron model (2.6). The time response and State trajectory between \( u(t) \) and \( v(t) \) with the parameter \( I = 3.0, 4.5 \) and the fractional order \( \mu = 0.95 \) are depicts in Figure 4-5. The increasing the values of external direct current \( I = 3.0, 4.5 \), the periodic bursting behavior and regular bursting behavior of HR neuron model (2.6) are shown in Figure 4-5.

![Figure 1: Time response and state trajectory of fractional-order HR neuron model when\( I = 1.3 \) and \( \mu = 0.9 \).](image-url)
Figure 2: Time response and state trajectory of fractional-order HR neuron model with the parameters when $I = 1.3$ and $\mu = 0.7$.

Figure 3: Time response and state trajectory of fractional-order HR neuron model with the parameters when $I = 1.3$ and $\mu = 0.5$.

Figure 4: Time response and state trajectory of fractional-order HR neuron model with the parameters when $I = 3.0$ and $\mu = 0.95$. 
Figure 5: Time response and state trajectory of fractional-order HR neuron model with the parameters when $I = 4.5$ and $\mu = 0.95$.

5. Conclusion

In this paper, we dealt with the problem of stability analysis of fractional-order Hindmarsh-Rose (HR) neuron model with time-delays. First, we presented the fractional-order HR neuron model with time delays. By using the stability theory of linear fractional-order systems, some new sufficient conditions to ensure the stability results of fractional-order HR model with time-delays were derived. Finally, numerical simulations have been given to show the effectiveness of our main results.

References


