Circular Waves in Thermoelastic Plates Sandwiched between Liquid Layers

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Abstract: This paper examines in detail the characteristics of the propagation of thermoelastic waves in a homogeneous, transversely isotropic, thermally conducting elastic plate bordered with layers (or half-spaces) of conducting viscous fluid on both sides. Complex secular equations for symmetric and antisymmetric wave motion of the circular plate, in completely separate terms, are derived. The waves of short wavelength and Leaky Lamb waves have also been discussed. The results have been deduced and compared with the existing one in relevant publications available in the literature at various stages of this work. Finally, in order to illustrate the analytical results, the numerical solution is carried out for cobalt plate bordered with water by using the functional iteration method. The analytical and numerical results are found to be in close agreement.

Keywords: Crested waves, viscous, conducting, thermal relaxation, Biot’s constant, transversely isotropic.

1. Introduction

Lamb waves are a form of elastic perturbation that can propagate in a solid plate with free boundaries; there are two groups of waves, symmetric and anti-symmetric. The temperature changes due to the elastic deformation cannot be ignored; therefore it is required to determine the thermal and mechanical fields in the body concurrently. The theory to include the effect of temperature change, known as the theory of thermoelasticity, has also been well established. According to the theory, the temperature field is coupled with the elastic strain field. In thermoelasticity, classical heat transfer, Fourier’s conduction equation is extensively used in many engineering applications. Ultrasonic nondestructive inspection plays important role in multilayered structures testing, especially in aeroindustry, where composites are replacing metallic parts. Detecting damage in composite materials, several
techniques have been developed; however Lamb wave methods have recently emerged as a consistent way to situate defects in such materials. Every technique implemented in the literature offers their own unique advantages in detecting certain types of defects. This study will enhance the knowledge base of the NDE community as well as the seismologists and geophysicists where anisotropy effects are significant.

Viktorov studied the characteristics of Rayleigh and Lamb waves in elastokinetics. The propagation of plane harmonic waves in a homogeneous an anisotropic material has also been studied in coupled theory of Thermoelasticity by Chadwick. Dhaliwal and Sherief extended the generalized Thermoelasticity theories to anisotropic elastic bodies. Watkins et al. calculated the attenuation of Lamb waves in the presence of an inviscid liquid using an acoustic impedance method. Massalas et al. studied the influence of a constant flux on the static and dynamic response of a simply supported and clamped circular plate with edge immovable and constrained. Grafi, Zhu and Wu brought out detailed analysis of the characteristics of Lamb waves in elastokinetics. The investigations on the propagation of Circular crested waves in anisotropic plates bordered with inviscid liquid maintained at uniform temperature have been carried out by Sharma and Pathania. Sharma and Sharma investigated the propagation of Lamb wave in thermoviscoelastic plate loaded with viscous fluid layer of varying temperature. Fabien Josse et al. presented an analytical solution for the resonance condition of the piezoelectric quartz resonator with one surface in contact with a viscous conductive liquid. Yang and Shue presented theoretical and experimental study on a leaky Lamb wave propagating in a piezoelectric plate loaded by a dielectric/conductive fluid. Lee and Kuo examined in detail the characteristics of a leaky Lamb wave propagating in a piezoelectric plate immersed in a dielectric or conductive fluid.

In this work we consider the problem of circular crested wave propagation in an infinite homogeneous, transversely isotropic, thermoelastic plate bordered with a viscous liquid layers or half spaces on both sides in the context of generalized (Lord-Shulman (LS) and Green-Lindsay (GL)) theories of thermoelasticity. It is noticed that the motion for circular crested waves is also governed by Rayleigh-Lamb type secular equations as in the case of a rectangular plate. More general dispersion equations of Lamb type waves are derived and discussed. The secular equations for leaky Lamb waves have also been deduced. The analytical results so obtained have been verified numerically and are illustrated graphically in case of cobalt material and water.
2. Formulation of the Problem

We consider an infinite homogeneous, transversely isotropic, thermally conducting elastic plate of thickness $2d$, initially at temperature $T_0$, having a conducting viscous liquid layer of thickness $h$ on both sides. We take origin of the co-ordinate system $(r, \theta, z)$ on the middle surface of the plate. The $z$-axis is taken along the thickness of the plate. The plate is axi-symmetric with $z$-axis as axis of symmetry. We take $r-z$ plane as the plane of incidence. The basic governing equations in non-dimensional form for generalized thermoelasticity in the absence of body forces and heat sources and for viscous fluid (homogeneous liquid) medium, are given by Sharma and Pathania\textsuperscript{8}, Sharma and Sharma\textsuperscript{9} respectively.

\begin{align*}
(1) \quad u_{rr} + \frac{1}{r} u_r - \frac{1}{r^2} u + c_2 u_{zz} + c_3 w_{rz} - \left(T + t_0 \delta_{zz} T^* \right)_{,r} &= \ddot{u} , \\
(2) \quad c_3 \left(u_{rr} + \frac{1}{r} u_r \right) + c_2 w_{rr} + \frac{1}{w_{rr}} c_1 w_{zz} - \beta \left(T + t_0 \delta_{zz} T^* \right)_{,z} &= \ddot{w} , \\
(3) \quad T_{rr} + \frac{1}{r} T_r + \delta T_{,r} - \left(\dot{T} + t_0 \dot{T}^* \right) &= \epsilon \left(\frac{\partial}{\partial t} + t_0 \delta_{zz} \frac{\partial^2}{\partial t^*} \right) \left(u_{rr} + \frac{1}{r} u + \beta w_{zz} \right) , \\
(4) \quad \nu \frac{\partial}{\partial t} \nabla^2 \hat{u}_{Lij} + \left(\delta^2 + \frac{1}{3} \nu \frac{\partial}{\partial t} \right) \nabla \cdot \hat{u}_{Lij} - \frac{\beta L}{\rho} \nabla T_{Lij} &= \ddot{\hat{u}}_{Lij} , \\
\quad a^* \nabla^2 T_{Lij} - \dot{T}_{Lij} &= \frac{\epsilon \nu}{\beta L} \frac{\delta^2}{\beta^2 L} \nabla \cdot \hat{u}_{Lij} , \quad (i, j = 1, 2) ,
\end{align*}

where $\beta_1 = (c_{11} + c_{12}) \alpha_1 + c_{13} \alpha_3$, $\beta_3 = 2c_{13} \alpha_1 + c_{33} \alpha_3$, $\beta = \frac{\beta_3}{\beta_1}$, $K = \frac{K_3}{K_1}$,

\begin{align*}
\epsilon &= \frac{\beta L^2 T_0}{\rho C_e c_{11}} , \quad \beta_L = \frac{\beta L}{\beta_1} , \quad \delta^2 = \frac{c_L^2}{v_L^2} , \quad c_L^2 = \frac{\lambda_L}{\rho L} , \quad \beta_L^* = 3 \lambda_L \alpha^* , \\
\epsilon_L &= \frac{\beta L^2 T_0^*}{\rho L C_L^* \lambda_L} , \quad v_L^* = \frac{\mu L \omega^*}{\rho L v_L^2} , \quad \nu^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} , \quad a^* = \frac{\rho C_L^* K}{\rho L C_L^*} , \quad k^* = \frac{k_L}{k_s}.
\end{align*}
Here $t_0$ and $t_1$ are the thermal relaxation times. $\delta_{ik}$ is the Kronecker’s with $k = 1$ for Lord-Shulman (LS) theory and $k = 2$ for Green-Lindsay (GL) theory of generalized thermoelasticity. $\varepsilon$ is the thermomechanical coupling constant and $v_1$ is the longitudinal wave velocity. $\mathbf{u}(r, z, t) = (u, 0, w)$ is the displacement vector, $T(r, z, t)$ is the temperature change, $c_j$ is the velocity of sound in the liquid, $\alpha^*$ is the coefficient of volume thermal expansion. $\varepsilon_L$ is the thermomechanical coupling in the liquid. Along the $z$ direction for the top liquid layer ($j = 1$) and for the bottom liquid layer ($j = 2$). $c_L$, $\rho_L$ are velocity of sound, the density of the liquid respectively, $\lambda_L$ is the bulk modulus, $C_v^*$ is the specific heat at constant volume of fluid. $T_{L_i}$ is the temperature deviation of liquid medium from ambient temperature $T_0^*$. $\mathbf{u}_{L_j}(r, z, t) = (u_{L_j}, 0, w_{L_j})$ ($j = 1, 2$) is the velocity vector. $u_{L_j}$ and $w_{L_j}$ are respectively. Along the $z$ direction for the top liquid layer ($j = 1$) and for the bottom liquid layer ($j = 2$). Moreover liquid is assumed to be thermally conducting hypothetical solid. Here the dot notation is used for time differentiation and comma denotes spatial derivatives.

3. Boundary Conditions

The boundary conditions at the solid-liquid interfaces $z = \pm d$ to be satisfied are:

**Mechanical conditions**

\begin{align}
\sigma_{zz} &= (\sigma_{zz})_L, \quad \sigma_{xz} = (\sigma_{xz})_L, \quad u = u_L, \quad w = w_L.
\end{align}

**Thermal conditions**

The conduction-convection boundary condition is given by

\begin{align}
T &= T_{L_{ij}}, T_{,zz} + \overline{k} T_{L_{ij},zz} = 0,
\end{align}

where $H$ is Biot’s heat transfer coefficient.

4. Solution of the Problem

We assume solutions of the form
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\[ (7) \quad \{ u, w, T \} = \left\{ J_1(\xi r), VJ_0(\xi r), WJ_0(\xi r) \right\} S e^{i \xi (x \sin \theta + mz - ct)}, \]

\[ (8) \quad \{ \varphi_{L_j}, \psi_{L_j} \} = \left\{ \Phi_{L_0}, J_0 (\xi r), \Psi_{L_0}, J_0 (\xi r) \right\} S e^{i \xi (mz - ct)}, \]

where \( c = \frac{\omega}{\xi} \) is the non-dimensional phase velocity, \( \omega \) is the frequency and \( \xi \) is the wave number. \( J_0, J_1 \) are respectively, the Bessel functions of order zero and one. \( \theta \) is the angle of inclination of wave normal with axis of symmetry. \( m \) is still unknown parameter. \( V, W \) are respectively, the amplitude ratios of displacement \( w \) temperature \( T \) and \( \Phi_{L_0}, \Psi_{L_0} \) are the amplitude ratios of \( \varphi_{L_0}, \psi_{L_0} \) respectively.

On seeking the solution of (1)-(3) in the form (7), displacements, temperature change can be expressed as

\[ u = \sum_{q=1}^{5} \left( A_q \cos \xi m_q z + B_q \sin \xi m_q z \right) J_1 (\xi r) e^{-i \omega t}, \]

\[ w = \sum_{q=1}^{5} V_q \left( -A_q \sin \xi m_q z + B_q \cos \xi m_q z \right) J_0 (\xi r) e^{-i \omega t}, \]

\[ T = \sum_{q=1}^{5} W_q \left( A_q \cos \xi m_q z + B_q \sin \xi m_q z \right) J_0 (\xi r) e^{-i \omega t}. \]

where \( A_q, B_q \) are amplitudes. \( V_q, W_q \) are amplitude ratios and \( m_q \) are given as

\[ \sum m_1^2 = \frac{P - Je^2}{c_1 c_2} + \frac{1 - c^2 \tau_0}{K} + \frac{i \varepsilon \xi c^3 \beta^2 \tau_1}{c_1 K}, \]

\[ \sum m_2 m_2' = \frac{(c_2 - c^2)(1 - c^2)}{c_1 c_2} + \frac{(1 - c^2 \tau_0) (P - Je^2)}{c_1 c_2 K} + \frac{i \varepsilon \xi c^3 \tau_0 \tau_1}{c_1 c_2 K} \left[ c_1 - 2c_2 \beta + (1 - c^2) \beta^2 \right], \]

\[ \sum m_i m_i m_i = \frac{c_2 - c^2}{c_1 c_2 K} \left[ (1 - c^2 \tau_0) (1 - c^3) + i \varepsilon \xi c^3 \tau_0 \tau_1 \right], \]
\[ P = c_1 + c_2^2 - c_3^2, J = c_1 + c_2, \tau_0 = t_0 + i\omega^{-1}, \tau'_0 = t_0\delta_{t_k} + i\omega^{-1}, \tau_1 = t_0\delta_{t_k} + i\omega^{-1}, \]

\[
V_q = \begin{cases} 
-i m_q a_q, & q = 1, 3 \\
ib_q/m_q, & q = 5 
\end{cases}
\]

\[
W_q = \begin{cases} 
\frac{i(c_2 + c_3a_q)m_q^2 + 1-c^2}{c\tau_1}, & q=1, 3 \\
\frac{[c_1c_2m_q^4 + (PJe^2)m_q^2 + (c_2 - c^2)(1-c^2)]a_q}{c\beta\tau_1(c_1m_q^2 + 1-c_1/\beta - c^2)}, & q=5 
\end{cases}
\]

\[
a_q = \frac{c_2m_q^2 + (1-c_3/\beta) - c^2}{(c_1-c_3\beta)m_q^2 + c_2 - c^2}, \quad b_q = \left( c_2m_q^2 + 1-c^2 + ic\tau_1 W_q \right) \frac{1}{c_3}.
\]

On using stress strain relation

(11) \[ \sigma_{zz} = (c_3-c_2)\left( u_{zr} + \frac{1}{r} \right) + c_1w_{zr} + i\omega\tau_1\beta T, \sigma_{rz} = u_{rz} + w_{rz} \]

axial stress and shear stress in \( rz \) plane and temperature change are obtained as

\[
\sigma_{zz} = -\zeta \sum_{q=1}^{5} P_q \left( A_q \cos \xi m_q z + B_q \sin \xi m_q z \right) J_0 \left( \xi r \right) e^{-i\omega t},
\]

\[
\sigma_{rz} = i\xi c_2 \sum_{q=1}^{5} F_q \left( -A_q \sin \xi m_q z + B_q \cos \xi m_q z \right) J_1 \left( \xi r \right) e^{-i\omega t},
\]

(12) \[ T_{rz} = \sum_{q=1}^{5} N_q \left( A_q \cos \xi m_q z + B_q \sin \xi m_q z \right) J_0 \left( \xi r \right) e^{-i\omega t}, \]

where
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\[ P_q = \begin{cases} 
  c_2(1+a_q) - c^2 a_q, & q = 1, 3 \\
  c_2 m_q^2 - (c_2 - c^2) b_q m_q^2 \frac{c_2}{m_q^2}, & q = 5 \\
  f_q = m_q^2 - b_q m_q, & q = 5 
\end{cases} \]

\[ N_q = i \xi W_q m_q, q = 1, 3, 5 \]

In the liquid layers, in addition to introducing non-dimensional quantities, we take

\[ u_{ij} = \frac{\partial \phi_{Lq}}{\partial r} + \frac{\partial \psi_{Lj}}{\partial z}, \quad w_{ij} = -\frac{\partial \phi_{Lq}}{\partial z} - \frac{\partial \psi_{Lj}}{\partial r}, r, \]

where \( \phi_{Lq} \) and \( \psi_{Lj} \), \( j = 1, 2 \) are respectively, the scalar velocity potential and vector velocity potential. We obtain

\[ \nabla^2 \phi_{Lq} - \frac{1}{\delta_L (1 + \epsilon_L) + \frac{4 \nu_L}{3} \frac{\partial}{\partial t}} \frac{\partial \phi_{Lq}}{\partial t} = 0, \]

\[ \nabla^2 \psi_{Lj} - \frac{1}{\nu_L} \frac{\partial \psi_{Lj}}{\partial t} = 0 \]

\[ a^2 \nabla^2 T_{ij} - \frac{\dot{T}_{ij}}{\beta_L \rho} = \frac{\epsilon_L \rho_L}{\beta_L \rho} \nabla^2 \phi_{Lq}, \quad (i, j = 1, 2) \]

On seeking solution of form (8) for \( d < z < d + h \) and \( -(d + h) < z < -d \), such that the acoustical pressure is zero at \( z = \pm (d + h) \), obtained as,

\( \Phi_{L1}, T_{k1} = (1, S_L \xi^2) A_1 \sin \xi m_{11} \left[ z - (d + h) \right] e^{-i\omega t} \),

\( \Phi_{L2}, T_{k2} = (1, S_L \xi^2) A_2 \sin \xi m_{11} \left[ z + (d + h) \right] e^{-i\omega t} \),
\[ \Psi_{L_1} = A \omega \sin \xi m_0 \left[ z - (d + h) \right] e^{-i \omega t} \]
\[ \Psi_{L_2} = -A \omega \sin \xi m_0 \left[ z + (d + h) \right] e^{-i \omega t} \]

(17)

\[ S_{L_n} = -\frac{i \omega \varepsilon_{L_n} \rho \epsilon_{L_n} \delta \epsilon^2 \left( a^2 \epsilon^2 + i \omega \right)}{\bar{\epsilon}_{L_n} \rho \left( a^2 \epsilon^2 + i \omega \right) \left( \delta \epsilon^2 - \frac{4}{3} \nu_L \frac{\partial}{\partial t} \right) \left( a^2 \epsilon^2 + i \omega \right) + i \omega \varepsilon_{L_n} \delta \epsilon^2} \]

\[ m_{ij,8i} = \left( b_i^2 c^2 - 1 \right), i = 1, 2, \]
\[ m_{9,110} = \pm \sqrt{\frac{i c^2 \epsilon^2}{\nu_L \omega} - 1} \]

5. Derivation of the Secular Equations

By invoking the interface conditions (5) to (6) at plate surfaces \( z = \pm d \), we obtain a system of ten simultaneous linear equations in amplitudes. System of equations have a non-trivial solution if the determinant of the coefficients of amplitudes vanishes, which leads to a characteristic equation for the propagation of modified guided thermoelastic waves in the plate. The characteristic equation for the thermoelastic plate waves in this case, after applying lengthy algebraic reductions of the determinant along with conditions \( \gamma \neq 0 \) and \( \gamma \neq \left( 2n - 1 \right) \pi / 2, n = 1, 2, 3, ... \) leads to the following secular equations

(18)

\[ \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \end{bmatrix} = \begin{bmatrix} P_{39} G_{1} & T_1 \\ P_{19} G_{1} & T_2 \\ P_{19} G_{1} & T_3 \\ P_{19} G_{1} & T_4 \\ P_{19} G_{1} & T_5 \end{bmatrix} + \begin{bmatrix} P_{79} G_{71} & T_7 \\ P_{19} G_{1} & T_8 \\ P_{19} G_{1} & T_9 \end{bmatrix} = \begin{bmatrix} P_{9} G_{5} \\ P_{9} G_{1} \end{bmatrix}, \]

where

\[ P_{71} = -\bar{\rho} \nu L^{-1} i \xi \left[ i \xi c^2 - 4 \nu L \xi^2 + c \xi^2 m_{71} \right], \]
\[ P_{72} = -\bar{\rho} \nu L^{-1} i \xi \left[ i \xi c^2 - 4 \nu L \xi^2 + c \xi^2 m_{72} \right], \]
\[ P_{9} = -2 \bar{\rho} i \xi^3 c m_9, \]
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\[ f_{71} = -2i\bar{\rho}\xi^3 cm_{71}, \quad f_{72} = -2i\bar{\rho}\xi^3 cm_{72}, \quad f_9 = -\bar{\rho}i\xi^3 c\left[m_9^2 - 1\right], \quad V_{71} = \xi m_{71}, \]

\[ V_{72} = \xi m_{72}, \quad V_9 = -\xi, \quad U_9 = -\xi = U_{72}, \quad U_9 = \xi m_9, \quad W_{71} = -S_{L1}\xi^2, \quad W_{72} = -S_{L2}\xi^2 \]

\[ N_{71} = K S_{L1}\xi^2 \xi m_{71}, \quad N_{72} = K S_{L2}\xi^2 \xi m_{72}, \quad F_9 = f_{10} G'_{1} - f_{30} G'_{3} + f_{59} G'_{5} + f_{79} G'_{7}, \]

(18) (a) \[ P_{q9} = \left(P_q U_{q9} - U_q P_9\right), \quad f_{q9} = \left(f_q V_{q9} - V_q f_9\right), \quad q = 1, 3, 5, 71, 72 \]

\[
G_1 = \begin{bmatrix}
  f_3 & f_3 & -f_{71} & -f_{72} \\
  V_3 & V_3 & -V_{71} & -V_{72} \\
  W_{33}^{+1} & W_{35}^{+1} & -W_{71} T_{71} & -W_{72} T_{72} \\
  N_3 & N_5 & -2N_{71} & -2N_{72}
\end{bmatrix}
\]

(18) (b) \[ G'_1 = \begin{bmatrix}
  P_{33}^{+1} & P_{35}^{+1} & -P_{71} T_{71} & -P_{72} T_{72} \\
  U_{33}^{+1} & U_{35}^{+1} & -U_{71} T_{71} & -U_{72} T_{72} \\
  W_{33}^{+1} & W_{35}^{+1} & -W_{71} T_{71} & -W_{72} T_{72} \\
  N_3 & N_5 & -2N_{71} & -2N_{72}
\end{bmatrix} \]

Here \( T_n = \tan \gamma m_n, \quad n = 1, 3, 5, \quad T_n = \tan \gamma' m_n, \quad n = 71, 72, 9 \gamma = \xi d, \quad G_3, G_5, G_{71}, G_{72} \) and \( G'_3, G'_5, G'_{71}, G'_{72} \) can be written from \( G_1 \) and \( G'_1 \) by replacing \((3, 5, 71, 72)\) in column index elements in \( G_1 \) and \( G'_1 \) with \((1, 5, 71, 72), (1, 3, 71, 72), (1, 3, 5, 72), (1, 3, 5, 71)\) elements respectively. The superscript \(+1\) corresponds to skew-symmetric and \(-1\) refers to symmetric modes.

The Rayleigh-Lamb type equation also governs circular crested thermoelastic waves in a plate bordered with layers of viscous liquid or half space viscous liquid on both sides. Although the frequency wave number relationship holds whether the waves are straight or circularly crested, the displacement and stress vary according to Bessel functions rather than trigonometric functions as far as the radial coordinate is concerned. For large values of \( r \), we have
\[
J_0 (\xi r) \rightarrow \frac{\sin \xi r + \cos \xi r}{\sqrt{\pi \xi r}}, \quad J_1 (\xi r) \rightarrow \frac{\sin \xi r - \cos \xi r}{\sqrt{\pi \xi r}}
\]

Thus, far from the origin the motion becomes periodic in \( r \). Actually, “far” occurs rather rapidly, within four to five zeros of the Bessel function. As \( r \) becomes very large, the straight crested behavior is the limit of circular crested waves.

6. Waves of Short Wavelength

The waves of short wavelength correspond to the case when the transverse wavelength of the plate is quite small as compared to the thickness of the plate.

**Case1:** The dispersion equation for thermoelastic Rayleigh waves of an infinite half space solid bordered with an infinite half space homogeneous liquid can be obtained with conditions i.e Some information on the asymptotic behavior is obtainable by letting \( R \rightarrow \infty \). If we take \( R > \omega / \sqrt{c_2} \), it follows that \( V < \sqrt{c_2} \) and the roots of characteristic equation lies in region I in this case. Then we replace \( m_1 \), \( m_3 \) and \( m_5 \) in the secular equation by \( im'_1 \), \( im'_3 \) and \( im'_5 \).

Herefor \( R \rightarrow \infty \),
\[
\begin{align*}
\frac{\tanh(\gamma m'_1)}{\tanh(\gamma m'_3)} & \rightarrow 1, \\
\frac{\tanh(\gamma m'_5)}{\tanh(\gamma m'_7)} & \rightarrow 1
\end{align*}
\]

\[
\begin{align*}
\frac{\tan(\gamma m_{12})}{\tanh(\gamma m'_3)} & \rightarrow i, \\
\frac{\tan(\gamma m_{15})}{\tanh(\gamma m'_5)} & \rightarrow i, \\
\frac{\tan(\gamma m_{17})}{\tanh(\gamma m'_7)} & \rightarrow i, \\
\end{align*}
\]

so that secular equation (18) reduces to

\[
\begin{align*}
P'_{19} G_1'' - P'_{39} G_3'' + P'_{59} G_5'' = & \mp P'_{719} G_7'' \pm P'_{729} G_7'' \\
& \mp \left[ f'_{19} G_1'' - f'_{39} G_3'' + f'_{59} G_5'' + f'_{719} G_7'' - f'_{729} G_7'' \right].
\end{align*}
\]

**Case2:** The dispersion equation for thermoelastic Rayleigh waves of an infinite half space solid bordered with a homogeneous liquid layer of thickness \( h \) is given as

\[
\begin{align*}
P'_{19} G_1'' - P'_{39} G_3'' + P'_{59} G_5'' = & \mp i P'_{719} G_7'' \tan \gamma m_{71} \pm i P'_{729} G_7'' \tan \gamma r_{72} \\
& \mp i \left[ f'_{19} G_1'' - f'_{39} G_3'' + f'_{59} G_5'' + f'_{719} G_7'' - f'_{729} G_7'' \right] \tan \gamma m_9,
\end{align*}
\]
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where $P_{q9}, f_{q9}, q = 1, 3, 5, 7, 19, 72$, are given by equation (18) (a) $G''_j$ and $G''_j, (j = 1, 3, 5, 7, 19, 72)$ are given by equations (18) (b) by replacing $m_1, m_3, m_5$ in the secular equation by $im'_1, im'_3, im'_5$.

7. Leaky Lamb Waves

When a plate of finite thickness is bordered with a half space homogeneous liquid on both sides, part of Lamb wave energy in the plate is coupled into the liquid as radiation; most of the energy is still in the solid. This type of disturbance is called the leaky Lamb wave.

Case 1: The dispersion equations for leaky Lamb waves i.e. Lamb waves in a transversely isotropic plate bordered with an infinite half space homogeneous liquid $(h \to \infty)$ at both sides are as follows.

$$
\begin{align*}
\left[ \frac{T_1}{T_5} \right]^{\pm 1} - \frac{P_{19} G_3}{P_{19} G_1} \left[ \frac{T_3}{T_5} \right]^{\pm 1} - \frac{iP_{19} G_{21}}{P_{19} G_1 \left[ T_5 \right]^{11}} + \frac{iP_{29} G_{72}}{P_{19} G_1 \left[ T_5 \right]^{11}} - \frac{i\bar{F}_9}{P_{19} G_1 \left[ T_5 \right]^{11}} \\
= - \frac{P_{q9} G_5}{P_{19} G_1}
\end{align*}
$$

where $P_{q9}, f_{q9}, q = 1, 3, 5, 7$ and $\bar{F}_9$ is given by equation (18.1). $G_1, G_3, G_5, G_7$ and $G'_1, G'_3, G'_5, G'_7$ are given by equations (18.2).

Case 2: If we multiply a factor of $i \tan \gamma m_1, i \tan \gamma m_2$ and $i \tan \gamma m_3$ to the terms $(-P_{39})$ and $(-P_{59})$ respectively in equation (21), the dispersion equation (18) of Lamb waves in a transversely isotropic thermoelastic plate bordered with a homogeneous layer of thickness $h$ on both sides can be obtained.

8. Numerical Results and Discussion

In order to illustrate the theoretical results obtained in the previous section, some numerical results are presented. The materials chosen for this purpose is cobalt and liquid chosen for the purpose of numerical calculations is water. The physical data is given by Sharma and Pathania\(^8\), Sharma and Sharma\(^9\).

$$
\rho = 8.836 \times 10^3 \text{ Kg m}^{-3}, \quad c_{11} = 3.071 \times 10^{11} \text{ N m}^{-2},
$$
\[ c_{12} = 1.650 \times 10^{11} \text{Nm}^{-2} \quad c_{13} = 1.027 \times 10^{11} \text{Nm}^{-2} \]

\[ c_{33} = 3.581 \times 10^{11} \text{Nm}^{-2}, \quad c_{44} = 1.510 \times 10^{11} \text{Nm}^{-2} \]

\[ T_0 = 298 K \quad \beta_1 = 7.04 \times 10^6 \text{Nm}^{-2} \text{deg}^{-1} \]

\[ \beta_3 = 6.90 \times 10^6 \text{Nm}^{-2} \text{deg}^{-1}, \quad C_e = 4.27 \times 10^2 \text{JKg}^{-1} \text{deg} K^{-1} \quad K_i = 0.690 \times 10^3 \text{Wm}^{-1} \text{deg}^{-1} \]

\[ K_j = 0.690 \times 10^3 \text{Wm}^{-1} \text{deg}^{-1}, \quad \omega^* = 1.88 \times 10^{11} \text{s}^{-1}, \quad \varepsilon = 1.29 \times 10^{-2} \]

\[ d = 1, \quad h = 0.25 \quad H = 0.1 \quad c_L = 1.5 \times 10^3 \text{m/s}, \quad \rho_L = 1000 \text{Kg m}^{-3} \]

The secular equation (18) is solved by using functional iteration method and using \( c^{-1} = V^{-1} + i \omega^{-1} Q \), phase speed \( V \) and attenuation coefficient \( Q \) have been obtained. The phase velocity and attenuation coefficient of first few symmetric and skew symmetric modes has been computed and corresponding dispersion curves for Rayleigh-Lamb type modes are represented graphically. In Figure 1 and 2, the computed simulated results corresponding to three situations of liquid loading namely, inviscid (ideal) liquid \( (\rho_L \neq 0.0, \mu_L = 0.0) \), low viscous liquid \( (\rho_L \neq 0.0, \mu_L = 0.1) \) and high viscous liquid \( (\rho_L \neq 0.0, \mu_L = 1.0) \). From Figure 1, the phase velocity of acoustic asymmetric mode is observed to increase from zero value at vanishing wave number and become closer to Rayleigh wave velocity. The velocity of the acoustic symmetric mode becomes dispersion less. The phase velocity of optical modes of wave propagation, symmetric and asymmetric; attain quite large values at vanishing wave number to become asymptotically closer to the shear wave speed. These behaviors of optical modes attributed to the fact that the viscosity gives rise to dissipative forces at supersonic speed and hence phase velocity of these modes decreases. In Figure 2, it is observed that the magnitude of attenuation coefficient of low viscous liquid for \( (n=1) \) mode become high and then slashes down sharply as compare to ideal and high viscous liquid. From Figure 3, it is observed that the phase velocity function in case of symmetric and asymmetric modes show constant and stabilized behavior with thermal conductivity ratio. The velocity of the acoustic symmetric mode remains non-dispersive. The magnitude of phase velocity of symmetric and asymmetric \( (n=2) \) mode is observed to be greater than that of the other modes. From Figure 4, it is noticed that the profiles of attenuation
coefficient in other modes have almost negligible variation as compare to the symmetric and asymmetric (n=1) mode. For symmetric and asymmetric (n=1) mode, the attenuation coefficient is observed to increase monotonically and then slashes down. The magnitude of the attenuation coefficient of symmetric modes is greater than the asymmetric modes.

Fig. 1. Variation of phase velocity with wave number for angle of inclination \( \theta = 45^\circ \)

Fig. 2. Variation of attenuation coefficient with wave number for angle of inclination \( \theta = 45^\circ \)

Figure 3: Variation of phase velocity with thermal conductivity ratio for angle of inclination \( \theta = 45^\circ \)
9. Conclusions

At higher wave numbers the phase velocities of various higher (optical) symmetric and skew-symmetric modes attains quite high values and significantly decreases to become asymptotically closer to the shear wave velocity in contrast to that of lowest (acoustic)skew-symmetric mode \( (A_0) \) which tends to the Rayleigh wave speed. The velocity of the acoustic symmetric mode \( (S_0) \) becomes dispersionless. The acoustic skew symmetric \( (A_0) \) mode is observed to be most affected and sensitive. The profiles of attenuation coefficient of acoustic modes are noticed to be highly dispersive. It is observed that the phase velocity profile in case of ideal liquid lies between those of high viscous and low viscous liquid at all wave numbers. Significant effect of thermal conductivity ratio has been observed on dispersion curves, attenuation coefficient profiles in the considered material plate.

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**References**


