

Some Curvature Properties on Sasakian Manifolds

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Abstract: The object of present paper is to study projective curvature tensor in a sasakian manifolds. In which semi-symmetric and projectively semi-symmetric Sasakian manifolds are considered. And we have shown that semi-symmetric, projectively semi-symmetric Sasakian manifold becomes Einstein manifold. Also we prove that projective curvature tensor is identical to curvature tensor and

$$P(\bar{X}, Y, Z, W) + P(X, \bar{Y}, Z, W) + P(X, Y, \bar{Z}, W) + P(X, Y, Z, \bar{W}) = 0.$$

Key words : Sasakian manifolds, projective curvature tensor, semi-symmetric, projective semi-symmetric.

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1. Introduction

Sasakian manifolds were introduced in 1960 by the Japanese geometer Shigeo Sasaki as an odd - dimensional analogue of Kähler manifolds. Sasakian manifolds is a branch of contact manifolds. Contact manifolds is an emerging branch of modern mathematics. The notation of contact geometry has evolved from the mathematical formalism of classical mechanics¹. Two important classes of contact manifolds are K-contact manifolds and Sasakian manifolds²⁻³. K – contact and Sasakian manifolds have been studied by several authors⁴⁻⁷ and many others. It is well known that every Sasakian manifold is K-contact, but the converse is not true, in general. However a three dimensional K-contact manifold is Sasakian⁸. The nature of a manifold mostly depends on its curvature tensor. Using the tools of conformal transformation geometers have deduced conformal curvature tensor. In the similar way with the help of projective transformation the

notation of projective curvature has been defined. Apart from conformal curvature tensor, the projective curvature tensor is another important tensor from the differential geometric point of view.

The object of present paper is to study the curvature properties of projective curvature tensor. Section 2 is preliminaries which will be used in later sections. In section 3 we have considered semi-symmetric Riemannian manifolds studied by Szabo⁹, Sekigawa and Tanno¹⁰, Sekigawa and Takagi¹¹, Sekigawa¹² and prove that a semi-symmetric Sasakian manifold is an Einstein manifold. In section 4 we consider projective semi-symmetric Sasakian manifold and shown that a projective semi-symmetric Sasakian manifold is an Einstein manifold. Finally in section 5 we study the properties of projective curvature tensor in K- contact and Sasakian manifolds. Also shown that in a projective semi-symmetric Sasakian manifold, the projective curvature tensor is identical to curvature tensor.

2. Preliminaries

A $(2n+1)$ - dimensional Riemannian manifold (M, g) is called an almost contact manifold if following relation holds^{2,13}

$$(2.1) \quad \phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0,$$

$$(2.2) \quad \eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \quad \eta(\phi X) = 0,$$

$$(2.3) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(2.4) \quad g(\phi X, Y) = -g(X, \phi Y), \quad g(\phi X, X) = 0,$$

$$(2.5) \quad (\nabla_X \eta)(Y) = g(\nabla_X \xi, Y),$$

where ϕ is a $(1,1)$ tensor, η is a 1- form, ξ is the corresponding vector field to the 1-form η and g is Riemannian metric. An almost contact metric manifold is called contact metric manifold if $d\eta(X, Y) = \Phi(X, Y) = g(X, \phi Y)$, Φ is called the fundamental 2-form of the manifold. If, in addition, ξ is a killing vector, the manifold is called a K- contact manifold^{2,3&13}. It is well known that a contact metric manifold is K- contact if and only if $\nabla_X \xi = -\phi X$, for any vector field X on M . Also a contact metric manifold

is K-contact if and only if the (1,1) type tensor field h defined by $h = \frac{1}{2} L_{\xi} \phi$ is equal to zero, where L denotes Lie differentiation. It is known that an almost contact structure is normal² if and only if

$$[\phi, \phi] + 2d\eta \otimes \xi = 0$$

A normal contact metric manifold is known as a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

for any vector fields X, Y on M . Every Sasakian manifold is K-contact but the converse is not true, in general. However a three – dimensional K-contact manifold is Sasakian¹¹. It is well known that (M, g) is Sasakian if and only if

$$(2.6) \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y$$

for all vector fields X and Y on M .

A complete regular contact metric manifold M^{2n+1} carries a K- contact structure (ϕ, ξ, η, g) , defined in terms of the almost Kähler structure (J, G) of the base manifold M^{2n} . Here the K-contact structure (ϕ, ξ, η, g) is Sasakian if and only if the base manifold (M^{2n}, J, G) is Kählerian. If (M^{2n}, J, G) is only almost Kähler, then (ϕ, ξ, η, g) is only K-contact². In a Sasakian manifold the Ricci operator Q commutes with ϕ , that is, $\phi Q = Q\phi$. Recently in¹⁴ it has been shown that there exists K-contact manifold with $\phi Q = Q\phi$ which are not Sasakian. It is to be noted that a K-contact manifold being intermediate between a contact metric manifold and a Sasakian manifold.

Let M be a $(2n+1)$ – dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in the Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well known projective curvature tensor P defined by

$$(2.7) \quad P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}(S(Y, Z)X - S(X, Z)Y),$$

vanishes on the manifold for $X, Y, Z \in TM$, where R is the curvature tensor of type (1,3) and S is the Ricci tensor of type (0,2). In fact, M is projectively flat if and only if it is of constant curvature. For a $(2n+1)$ -dimensional K-contact manifold^{2,13}, we always have

$$(2.8) \quad R(\bar{X}, Y, Z, W) + R(X, \bar{Y}, Z, W) + R(X, Y, \bar{Z}, W) \\ + R(X, Y, Z, \bar{W}) + (\nabla_{\xi} R)(X, Y, Z, W) = 0,$$

$$(2.9) \quad S(\bar{X}, Y) + S(X, \bar{Y}) + (\nabla_{\xi} S)(X, Y) = 0,$$

$$(2.10) \quad g(\bar{X}, Y) + g(X, \bar{Y}) = 0,$$

Also for a $(2n+1)$ -dimensional Sasakian manifold, we have

$$(2.11) \quad \nabla_X \xi = -\phi X,$$

$$(2.12) \quad S(X, \xi) = 2n\eta(X),$$

$$(2.13) \quad R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \quad R(\xi, X)\xi = -X + \eta(X)\xi,$$

$$(2.14) \quad S(\bar{X}, Y) + S(X, \bar{Y}) = 0,$$

$$(2.15) \quad (\nabla_{\xi} R)(X, Y, Z, W) = 0, \quad (\nabla_{\xi} S)(X, Y) = 0,$$

$$(2.16) \quad 'P(X, Y, Z, \xi) = \eta(X) \left\{ g(Y, Z) - \frac{1}{2n} S(Y, Z) \right\} \\ - \eta(Y) \left\{ g(X, Z) - \frac{1}{2n} S(X, Z) \right\},$$

$$(2.17) \quad 'P(\xi, Y, Z, W) = \eta(W) \left\{ g(Y, Z) - \frac{1}{2n} S(Y, Z) \right\},$$

$$(2.18) \quad 'P(\xi, Y, Z, \xi) = g(Y, Z) - \frac{1}{2n} S(Y, Z),$$

$$(2.19) \quad P(\xi, Y, Z) = \left\{ g(Y, Z) - \frac{1}{2n} S(Y, Z) \right\} \xi,$$

3. Semi - Symmetric Sasakian Manifold

A Sasakian manifold (M^{2n+1}, g) is said to be semi-symmetric¹⁵ if it satisfies the relation

$$(3.1) \quad (R(X, Y)R)(U, V)W = 0,$$

where $R(X, Y)$ is the curvature operator. From (3.1), we have

$$R(X, Y)R(U, V)W - R(R(X, Y)U, V)W - R(U, R(X, Y)V)W \\ - R(U, V)R(X, Y)W = 0.$$

Substituting $X = W = \xi$ in above equation, we have

$$R(\xi, Y)R(U, V)\xi - R(R(\xi, Y)U, V)\xi - R(U, R(\xi, Y)V)\xi \\ - R(U, V)R(\xi, Y)\xi = 0.$$

Now, with the help of (2.6) and (2.13) in above equation, we obtain

$$(3.2) \quad \eta(V)R(\xi, Y)U - \eta(U)R(\xi, Y)V - g(Y, U)R(\xi, V)\xi + \eta(U)R(Y, V)\xi \\ - g(Y, V)R(U, \xi)\xi + \eta(V)R(U, Y)\xi + R(U, V)Y - \eta(Y)R(U, V)\xi = 0.$$

Again using (2.6) and (2.13) in (3.2), we have

$$(3.3) \quad R(U, V)Y = g(Y, V)U - g(Y, U)V.$$

Contracting with respect to U, we have

$$S(Y, V) = 2ng(Y, V).$$

Hence we have the following theorem

Theorem 3.1: *A semi- symmetric Sasakian manifold is an Einstein manifold.*

4. Projectively Semi - Symmetric Sasakian Manifolds

In view of (2.7) we have the projective curvature tensor of a $(2n+1)$ - dimensional Sasakian manifold as

$$(4.1) \quad P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}(S(Y, Z)X - S(X, Z)Y)$$

Now from the above equation and with the help of (2.2) & (2.13), we get

$$(4.2) \quad P(\xi, V)\xi = P(V, \xi)\xi = 0,$$

for any vector field V . We suppose that a Sasakian manifold is projectively semi-symmetric¹⁶, that is, $(R(X, Y)P)(U, V)W = 0$. Hence

$$(4.3) \quad \begin{aligned} &R(X, Y)P(U, V)W - P(R(X, Y)U, V)W - P(U, R(X, Y)V)W \\ &- P(U, V)R(X, Y)W = 0 \end{aligned}$$

Substituting $U = W = \xi$ in (4.3) it follows that

$$\begin{aligned} &R(X, Y)P(\xi, V)\xi - P(R(X, Y)\xi, V)\xi - P(\xi, R(X, Y)V)\xi \\ &- P(\xi, V)R(X, Y)\xi = 0. \end{aligned}$$

Using (4.2) we obtain from above equation

$$(4.4) \quad P(R(X, Y)\xi, V)\xi + P(\xi, V)R(X, Y)\xi = 0.$$

Using (2.6) in (4.4), we have

$$(4.5) \quad \eta(Y)[P(X, V)\xi + P(\xi, V)X] - \eta(X)[P(Y, V)\xi + P(\xi, V)Y] = 0.$$

Now using (2.7) in (4.5), we obtain

$$(4.6) \quad \left\{ \begin{aligned} &\eta(Y)[R(X, V)\xi - \frac{1}{2n}\{S(V, \xi)X - S(X, \xi)V\} \\ &\quad + R(\xi, V)X - \frac{1}{2n}\{S(V, X)\xi - S(\xi, X)V\}] \\ &- \eta(X)[R(Y, V)\xi - \frac{1}{2n}\{S(V, \xi)Y - S(Y, \xi)V\} \\ &\quad + R(\xi, V)Y - \frac{1}{2n}\{S(V, Y)\xi - S(\xi, Y)V\}] = 0. \end{aligned} \right.$$

Using (2.12) in (4.6), we obtain

$$(4.7) \quad \left\{ \begin{array}{l} \eta(Y)[R(X, V)\xi - \eta(V)X + R(\xi, V)X - \frac{1}{2n}S(V, X)\xi] - \eta(X)[R(Y, V)\xi \\ - \eta(V)Y + R(\xi, V)Y - \frac{1}{2n}S(V, Y)\xi] = 0. \end{array} \right.$$

Using (2.6) and (2.13) in (4.7), we have

$$(4.8) \quad \frac{1}{2n}[\eta(X)S(V, Y)\xi - \eta(Y)S(V, X)\xi] = \eta(X)g(V, Y)\xi \\ - \eta(Y)g(V, X)\xi.$$

Since $\xi \neq 0$, we have

$$\eta(X)S(V, Y) = 2n\eta(X)g(V, Y),$$

is a solution of above equation.

Since $\eta(X) \neq 0$, $S(V, Y) = 2ng(V, Y)$.

Hence we obtain the following theorem

Theorem 4.1: *A projectively semi-symmetric Sasakian manifold is an Einstein manifold.*

5. Curvature Properties of Projective Curvature Tensor

Let us consider the projective curvature tensor $P(X, Y, Z, W)$ on a Riemannian manifold

$$(5.1) \quad P(X, Y, Z, W) = R(X, Y, Z, W) - \frac{1}{2n}[g(X, W)S(Y, Z) \\ - g(Y, W)S(X, Z)],$$

From (5.1), we obtain

$$(5.2) \quad \left\{ \begin{aligned} & 'P(\bar{X}, Y, Z, W) + 'P(X, \bar{Y}, Z, W) + 'P(X, Y, \bar{Z}, W) + 'P(X, Y, Z, \bar{W}) \\ & = R(\bar{X}, Y, Z, W) + R(X, \bar{Y}, Z, W) + R(X, Y, \bar{Z}, W) + R(X, Y, Z, \bar{W}) \\ & - \frac{1}{2n} [g(\bar{X}, W)S(Y, Z) + g(X, W)S(\bar{Y}, Z) + g(X, W)S(Y, \bar{Z}) \\ & + g(X, \bar{W})S(Y, Z) - g(Y, W)S(\bar{X}, Z) - g(\bar{Y}, W)S(X, Z) \\ & - g(Y, W)S(X, \bar{Z}) - g(Y, \bar{W})S(X, Z)]. \end{aligned} \right.$$

If the manifold is K-contact then from (2.8), (2.9), (2.10) and (5.2), we have

$$\begin{aligned} & 'P(\bar{X}, Y, Z, W) + 'P(X, \bar{Y}, Z, W) + 'P(X, Y, \bar{Z}, W) + 'P(X, Y, Z, \bar{W}) \\ & = -(\nabla_{\xi} R)(X, Y, Z, W) - \frac{1}{2n} [(\nabla_{\xi} S)(X, Z)g(Y, W) \\ & - (\nabla_{\xi} S)(Y, Z)g(X, W)]. \end{aligned}$$

Hence we obtain following theorem

Theorem 5.1: *In a $(2n+1)$ -dimensional K-contact manifold M , we have*

$$\begin{aligned} & 'P(\bar{X}, Y, Z, W) + 'P(X, \bar{Y}, Z, W) + 'P(X, Y, \bar{Z}, W) + 'P(X, Y, Z, \bar{W}) \\ & = -(\nabla_{\xi} R)(X, Y, Z, W) - \frac{1}{2n} [(\nabla_{\xi} S)(X, Z)g(Y, W) \\ & - (\nabla_{\xi} S)(Y, Z)g(X, W)]. \end{aligned}$$

If the manifold is Sasakian, that is, in view of

$(\nabla_{\xi} R)(X, Y, Z, W) = 0$, $(\nabla_{\xi} S)(X, Y) = 0$, we have the following theorem:

Theorem 5.2: *In a $(2n+1)$ -dimensional Sasakian manifold M , the relation $'P(\bar{X}, Y, Z, W) + 'P(X, \bar{Y}, Z, W) + 'P(X, Y, \bar{Z}, W) + 'P(X, Y, Z, \bar{W}) = 0$ hold.*

Now by virtue of (5.1) and Theorem 4.1 we have

$$(5.3) \quad 'P(X, Y, Z, W) = R(X, Y, Z, W).$$

Hence we obtain the following theorem

Theorem 5.3: *In a projective semi-symmetric Sasakian manifold projective curvature tensor is identical to curvature tensor.*

Now with the help of (2.16), (2.17), (2.18), (2.19) and Theorem 4.1 we have following results

Corollary 5.1: *In a projective semi-symmetric Sasakian manifold M , we have*

- (1) $'P(X, Y, Z, \xi) = 0,$
- (2) $'P(\xi, Y, Z, W) = 0,$
- (3) $'P(\xi, Y, Z, \xi) = 0,$
- (4) $'P(\xi, Y, Z) = 0.$

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