

On Certain Generalized *BP*-Birecurrent Finsler Space

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Abstract: In this paper, we introduce a Finsler space which Cartan's second curvature tensor P_{jkh}^i satisfies the generalized birecurrence property in sense of Berwald, this space characterize by the following condition

$$B_l B_m P_{jkh}^i = a_{lm} P_{jkh}^i + b_{lm} (\delta_j^i g_{kh} - \delta_k^i g_{jh}) - 2y^t \mu_m B_t (\delta_j^i C_{khl} - \delta_k^i C_{jhl}),$$

$P_{jkh}^i \neq 0$, where $B_l B_m$ is Berwald's covariant differential operator of the second order with respect to x^m and x^l , successively, B_l is Berwald's covariant differential operator of the first order with respect to x^l , C_{jhl} is (h) -torsion tensor, a_{lm} , b_{lm} are non- zero covariant tensor field of second order and μ_m is non- zero covariant vector field, we will call this space a generalized *BP*-birecurrent space. We obtain Berwald's covariant derivative of second order for some tensors, we find the relationship between the covariant derivative in sense of Berwald and the h -covariant derivative in sense of Cartan, we obtain the necessary and sufficient condition for some tensors to be generalized birecurrent in sense of Berwald, we get the relationship between covariant tensors field λ_i , μ_i , a_{lm} and b_{lm} in this space, we show that the behavior of some tensors as birecurrent in this space.

Keywords: Generalized *BP*-birecurrent space, Cartan's second curvature tensor, Berwald covariant derivative, h -covariant derivative.

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1. Introduction

Qasem¹ studied a generalized H – birecurrent space which satisfies the recurrence property, Hadi² discussed a generalized BK – birecurrent space, Qasem and Hadi³⁻⁵ introduced a generalized BR – birecurrent space, a generalized H^h – birecurrent space and a generalized R^h – birecurrent space, Assallal⁶ introduced a generalized P^h – birecurrent space, Saleem⁷ studied the projective curvature tensor W_{jkh}^i which satisfies the generalized birecurrence property and introduced U – generalized birecurrent space. Pandey and Verma⁸ discussed C^h – birecurrent space, Dikshit⁹ introduced R^h – birecurrent space, Ali¹⁰ studied K^h – birecurrent space, Qasem and Hanballa¹¹ introduced K^h – generalized birecurrent space, Hanballa¹² discussed BK – birecurrent space, also Otman⁵ introduced BP – birecurrent space.

Let F_n be an n – dimensional Finsler space equipped with the metric function $F(x, y)$ satisfying the request conditions¹³.

The vector y_i defined by

$$(1.1) \quad y_i = g_{ij}(x, y)y^j.$$

The vector y_i also satisfies the following relations

$$(1.2) \quad g_{ij} = \dot{\partial}_i y_j = \dot{\partial}_j y_i.$$

The two sets of quantities g_{ij} and its associative g^{ij} , which are components of a metric tensor connected by

$$(1.3) \quad g_{ij}g^{ik} = \delta_j^k = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{if } k \neq j \end{cases}.$$

In view of (1.1) and (1.3), we have

$$(1.4) \quad (a) \ \delta_j^i y_i = y_j \quad (b) \ \delta_j^i y^j = y^i \quad (c) \ \delta_j^i g_{jr} = g_{ir} \quad \text{and} \quad (d) \ \dot{\partial}_j y^i = \delta_j^i.$$

Matsumoto¹⁴ introduced $(h)hv$ -torsion tensor which is positively homogeneous of degree -1 in y^i and symmetric in all its indices and defined by

$$(1.5) \quad C_{ijk} = \frac{1}{2} \dot{\partial}_i g_{jk} = \frac{1}{4} \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k F^2.$$

also which satisfy

$$(1.6) \quad \begin{cases} (a) & C_{ijk} y^i = C_{jki} y^i = 0 \\ (b) & \delta_j^i C_{ikl} = C_{jkl} \end{cases}.$$

the unit vector l^i with the direction of y^i given by

$$(1.7) \quad l^i = \frac{y^i}{F}.$$

Cartan h -covariant differentiation (Cartan's second kind covariant differentiation) with respect to x^k given by

$$(1.8) \quad X_{|k}^i := \partial_k X^i - (\dot{\partial}_r X^i) G_k^r + X^r \Gamma_{rk}^{*i},$$

where Γ_{rk}^{*i} is a function called it Cartan's connection parameter.

The h -covariant derivative of the vector y^i vanish identically i.e.

$$(1.9) \quad y_{|k}^i = 0.$$

Let us consider new connection parameter G_{jk}^i of Berwald which are connected with Cartan's connection parameter Γ_{jk}^{*i} by

$$(1.10) \quad G_{jk}^i = \Gamma_{jk}^{*i} + C_{jk|h}^i y^h.$$

Berwald covariant derivative $B_k T_j^i$ of an arbitrary tensor field T_j^i with respect to x^k given by

$$(1.11) \quad B_k T_j^i := \partial_k T_j^i - (\partial_r T_j^i) G_k^r + T_j^r G_{rk}^i - T_r^i G_{jk}^r.$$

The processes of Berwald covariant differentiation with respect to x^h and the partial differentiation with respect to y^k commute according to

$$(1.12) \quad (\partial_k B_h - B_h \partial_k) T_j^i = T_j^r G_{khr}^i - T_r^i G_{khj}^r,$$

for an arbitrary tensor field T_j^i . Berwald covariant derivative of the vector y^i vanish identically, i.e.

$$(1.13) \quad B_k y^i = 0.$$

But, in general, Berwald covariant derivative of the metric tensor g_{ij} does not vanish and given by

$$(1.14) \quad B_k g_{ij} = -2C_{ijk|h} y^h = -2y^h B_h C_{ijk}.$$

The tensor R_{jkh}^i called h -curvature tensor (Cartan's third curvature tensor) is positively homogeneous of degree zero in y^i and defined by¹³

$$R_{jkh}^i := \partial_h \Gamma_{jk}^{*i} + (\partial_l \Gamma_{jh}^{*i}) G_k^l + G_{jm}^i (\partial_h G_k^m - G_{hl}^m G_k^l) + \Gamma_{mh}^{*i} \Gamma_{jk}^{*m} - h/k,$$

which satisfies the relations

$$(1.15) \quad \begin{cases} (a) R_{jkh}^i = K_{jkh}^i + C_{js}^i H_{kh}^s, \\ (b) R_{jkh}^i y^j = H_{kh}^i. \end{cases}$$

R -Ricci tensor R_{jk} given by

$$(1.16) \quad \begin{cases} (a) R_{jk} = K_{jk} + C_{js}^r H_{kr}^s, \\ (b) R_{jk} y^j = R_k. \end{cases}$$

Also, the h -curvature tensor R_{jkh}^i satisfies the following identity known as Bianchi identity¹³

$$(1.17) \quad R_{ijk|h}^r + R_{ihj|k}^r + R_{ikh|j}^r = -y^m \left(R_{mkh}^s P_{ijs}^r + R_{mjk}^s P_{ih s}^r + R_{mhj}^s P_{iks}^r \right).$$

The tensor P_{jkh}^i called $h\nu$ -curvature tensor (Cartan's second curvature tensor) is positively homogeneous of degree -1 in y^i and defined by¹³

$$P_{jkh}^i := \hat{\partial}_h \Gamma_{jk}^{*i} + C_{jr}^i P_{kh}^r - C_{jh|k}^i.$$

Which satisfies the relations

$$(1.18) \quad P_{jkh}^i y^j = \Gamma_{jkh}^{*i} y^j = P_{kh}^i = C_{kh|r}^i y^r,$$

where P_{jkh}^i called $\nu(h\nu)$ -torsion tensor satisfies

$$(1.19) \quad P_{kh}^i y^k = 0.$$

Using (1.18) in (1.10), we get

$$(1.20) \quad P_{kh}^i = G_{kh}^i - \Gamma_{kh}^{*i}.$$

P -Ricci tensor P_{jk} , curvature vector P_k and curvature scalar P of Cartan's second curvature tensor given by

$$(1.21) \quad \begin{cases} (a) P_{jk} = P_{jki}^i \\ (b) P_k = P_{ki}^i \\ (c) P = P_k y^k, \end{cases}$$

respectively.

The $h\nu$ -curvature tensor P_{jkh}^i satisfies the following

$$(1.22) \quad P_{jkh}^i - P_{jkh}^i = -S_{jkh|r}^i y^r$$

and

$$(1.23) \quad P_{jkh}^i - P_{kjh}^i = C_{kh|j}^i + C_{sj}^i P_{kh}^s - j/k.$$

2. Some Tensors in Generalize BP – Birecurrent Space

Let us consider a Finsler space F_n which Cartan's second curvature tensor P_{jkh}^i satisfies the condition

$$(2.1) \quad B_m P_{jkh}^i = \lambda_m P_{jkh}^i + \mu_m (\delta_j^i g_{kh} - \delta_k^i g_{jh}), \quad P_{jkh}^i \neq 0,$$

where B_m is Berwald's covariant differential operator with respect to x^m , λ_m and μ_m are non-zero covariant vectors field, this space introduced by Alaa et al.^{15, 16}, they called it a generalized BP – recurrent space.

Now, Taking B –covariant derivative for the condition (2.1) with respect to x^l , we get

$$\begin{aligned} B_l B_m P_{jkh}^i &= B_l (\lambda_m) P_{jkh}^i + \lambda_m B_l (P_{jkh}^i) + B_l (\mu_m) (\delta_j^i g_{kh} - \delta_k^i g_{jh}) \\ &\quad + \mu_m B_l (\delta_j^i g_{kh} - \delta_k^i g_{jh}). \end{aligned}$$

In view of the condition (2.1) and using (1.14), we can written above equation as

$$(2.2) \quad B_l B_m P_{jkh}^i = a_{lm} P_{jkh}^i + b_{lm} (\delta_j^i g_{kh} - \delta_k^i g_{jh}) - 2y^t \mu_m B_t (\delta_j^i C_{khl} - \delta_k^i C_{jhl}),$$

where $a_{lm} = B_l \lambda_m + \lambda_m \lambda_l$ and $b_{lm} = \lambda_m \mu_l + B_l \mu_m$ are non-zero covariant tensors field of second order known as recurrence tensors .

Definition 2.1: A Finsler space F_n which Cartan's second curvature tensor P_{jkh}^i satisfies the condition (2.2) will be called a generalized BP –birecurrent space and the tensor will be called a generalized B –birecurrent and we will be denoted them briefly as $G(BP)$ – BRF_n and GB – BR , respectively.

Remark 2.1: The condition⁵ $B_l B_m P_{jkh}^i = a_{lm} P_{jkh}^i$, $P_{jkh}^i \neq 0$, looks as a birecurrent. Also consider as a particular case of the condition (2.1), but it doesn't do.

Let us consider a $G(BP)$ – BRF_n which is characterized by the condition (2.2).

Transvecting the condition (2.2) by y^j , using (1.13), (1.18), (1.4b), (1.1) and (1.6a), we get

$$(2.3) \quad B_l B_m P_{kh}^i = a_{lm} P_{kh}^i + b_{lm} (y^i g_{kh} - \delta_k^i y_h) - 2y^t \mu_m B_t (y^i C_{khl}).$$

Thus, we conclude

Theorem 2.1: *Berwald's covariant derivative of the second order for the (v)hv-torsion tensor P_{kh}^i given by eq. (2.3) in $G(BP) - BRF_n$.*

Contracting the indices i and h in the condition (2.2), using (1.21a), (1.4c) and (1.6b), we get

$$(2.4) \quad B_l B_m P_{jk} = a_{lm} P_{jk}.$$

Contracting the indices i and h in eq. (2.3) and using (1.21b), (1.1), (1.4a) and (1.6a), we get

$$(2.5) \quad B_l B_m P_k = a_{lm} P_k.$$

Transvecting eq. (2.5) by y^k , using (1.18c) and put $(y_k y^k = 1)$, we get

$$(2.6) \quad B_l B_m P = a_{lm} P.$$

Thus, we conclude

Theorem 2.2. *The P -Ricci tensor P_{jk} , the curvature vector P_k and the curvature scalar P behave as birecurrent in $G(BP) - BRF_n$.*

Taking B -covariant derivative for (1.22) twice with respect to x^m and x^l , respectively, using the condition (2.2), we get

$$\begin{aligned} B_l B_m (-S_{jkh|}^i y^r) &= a_{lm} (P_{jkh}^i - P_{jhk}^i) + b_{lm} (\delta_h^i g_{jk} - \delta_k^i g_{jh}) \\ &\quad - 2y^t \mu_m B_t (\delta_h^i C_{jkl} - \delta_k^i C_{jhl}). \end{aligned}$$

Using (1.22) in above equation, we get

$$(2.7) \quad B_l B_m (-S_{jkh|}^i y^r) = a_{lm} (-S_{jkh|}^i y^r) + b_{lm} (\delta_h^i g_{jk} - \delta_k^i g_{jh})$$

$$-2y^t \mu_m B_t \left(\delta_h^i C_{jkl} - \delta_k^i C_{jhl} \right).$$

Thus, we conclude

Theorem 2.3. *Berwald's covariant derivative of the second order for the tensor $(-S_{jkh|r}^i y^r)$ given by eq. (2.7) in $G(BP) - BRF_n$.*

Taking B -covariant derivative for (1.23) twice with respect to x^m and x^l , respectively, and using the condition (2.2), we get

$$\begin{aligned} B_l B_m \left(C_{kh|j}^i + C_{sj}^i P_{kh}^s - j/k \right) &= a_{lm} \left(P_{jkh}^i - P_{kjh}^i \right) + \gamma_{lm} \left(\delta_j^i g_{kh} - \delta_k^i g_{jh} \right) \\ &\quad - 2y^t \alpha_m B_t \left(\delta_j^i C_{khl} - \delta_k^i C_{jhl} \right), \end{aligned}$$

where $y_{lm} = 2b_{lm}$, $\alpha_m = 2\mu_m$.

Using (1.23) in above equation, we get

$$\begin{aligned} B_l B_m \left(C_{kh|j}^i + C_{sj}^i P_{kh}^s - j/k \right) &= a_{lm} \left(C_{kh|j}^i + C_{sj}^i P_{kh}^s - j/k \right) \\ &\quad + \gamma_{lm} \left(\delta_j^i g_{kh} - \delta_k^i g_{jh} \right) - 2y^t \alpha_m B_t \left(\delta_j^i C_{khl} - \delta_k^i C_{jhl} \right). \end{aligned}$$

Thus, we conclude

Theorem 2.4. *The tensor $(C_{kh|j}^i + C_{sj}^i P_{kh}^s - j/k)$ is a generalized B -birecurrent in $G(BP) - BRF_n$.*

3. Necessary and Sufficient Condition for Some Tensors to be Generalized B -Birecurrent

In this section, we will shed light about the necessary and sufficient condition for some tensors to be generalized birecurrent in a $G(BP) - BRF_n$ which is characterized by the condition (2.2). For a Riemannian space V_4 , the projective curvature tensor P_{jkh}^i (Cartan's second curvature tensor) defined as follows¹⁷

$$(3.1) \quad P_{jkh}^i = R_{jkh}^i - \frac{1}{3} \left(\delta_h^i R_{jk} - \delta_k^i R_{jh} \right)$$

Taking B -covariant derivative for eq. (3.1) twice with respect to x^m and x^l , respectively, using the condition (2.2) and eq. (3.1) in the resulting equation, we get

$$(3.2) \quad B_l B_m R_{jkh}^i = a_{lm} R_{jkh}^i + b_{lm} (\delta_j^i g_{kh} - \delta_k^i g_{jh}) - 2y^t \mu_m B_t (\delta_j^i C_{khl} - \delta_k^i C_{jhl}) \\ - \frac{1}{3} a_{lm} (\delta_h^i R_{jk} - \delta_k^i R_{jh}) + \frac{1}{3} B_l B_m (\delta_h^i R_{jk} - \delta_k^i R_{jh}).$$

This shows that

$$B_l B_m R_{jkh}^i = a_{lm} R_{jkh}^i + b_{lm} (\delta_j^i g_{kh} - \delta_k^i g_{jh}) - 2y^t \mu_m B_t (\delta_j^i C_{khl} - \delta_k^i C_{jhl}),$$

if and only if

$$(3.3) \quad B_l B_m (\delta_h^i R_{jk} - \delta_k^i R_{jh}) = a_{lm} (\delta_h^i R_{jk} - \delta_k^i R_{jh}).$$

Thus, we conclude

Theorem 3.1: In $G(BP) - BRF_n$, for $n=4$, Cartan's third curvature tensor R_{jkh}^i is a generalized B -birecurrent if and only if the tensor $(\delta_h^i R_{jk} - \delta_k^i R_{jh})$ is birecurrent.

Transvecting the condition (3.1) by y^j , using (1.18), (1.15b) and (1.16b), we get

$$(3.4) \quad P_{kh}^i = H_{kh}^i - \frac{1}{3} (\delta_h^i R_k - \delta_k^i R_h).$$

Taking B -covariant derivative for eq. (3.4) twice with respect to x^m and x^l , respectively, using eq. (2.3) and eq. (3.4) in the resulting equation, we get

$$(3.5) \quad B_l B_m H_{kh}^i = a_{lm} H_{kh}^i + b_{lm} (y^i g_{kh} - \delta_k^i y_h) - 2y^t \mu_m B_t (y^i C_{khl}) \\ - \frac{1}{3} a_{lm} (\delta_h^i R_k - \delta_k^i R_h) + \frac{1}{3} B_l B_m (\delta_h^i R_k - \delta_k^i R_h).$$

This shows that

$$(3.6) \quad B_l B_m H_{kh}^i = a_{lm} H_{kh}^i + b_{lm} (y^i g_{kh} - \delta_k^i y_h) - 2y^t \mu_m B_t (y^i C_{khl}),$$

if and only if

$$(3.7) \quad B_l B_m (\delta_h^i R_k - \delta_k^i R_h) = a_{lm} (\delta_h^i R_k - \delta_k^i R_h).$$

Thus, we conclude

Theorem 3.2: In $G(BP) - BRF_n$, for $n = 4$, Berwald's covariant derivative of the second order for the torsion tensor H_{kh}^i of Berwald curvature tensor given by eq. (3.6) if and only if the tensor $(\delta_h^i R_k - \delta_k^i R_h)$ is birecurrent.

Using (1.15a) and (1.16a) in (3.1), we get

$$(3.8) \quad P_{jkh}^i = K_{jkh}^i + C_{js}^i H_{kh}^s - \frac{1}{3} \{ (K_{jk} + C_{js}^r H_{kr}^s) \delta_h^i - (K_{jh} + C_{js}^r H_{hr}^s) \delta_k^i \}.$$

Taking B -covariant derivative for eq. (3.8) twice with respect to x^m and x^l , respectively, using the condition (2.2) and eq. (3.8) in the resulting equation, we get

$$\begin{aligned} B_l B_m K_{jkh}^i &= a_{lm} K_{jkh}^i + b_{lm} (\delta_j^i g_{kh} - \delta_k^i g_{jh}) - 2y^t \mu_m B_t (\delta_j^i C_{khl} - \delta_k^i C_{jhl}) \\ &+ a_{lm} (C_{js}^i H_{kh}^s) - \frac{1}{3} a_{lm} \{ (K_{jk} + C_{js}^r H_{kr}^s) \delta_h^i - (K_{jh} + C_{js}^r H_{hr}^s) \delta_k^i \} \\ &- B_l B_m (C_{js}^i H_{kh}^s) + \frac{1}{3} B_l B_m \{ (K_{jk} + C_{js}^r H_{kr}^s) \delta_h^i - (K_{jh} + C_{js}^r H_{hr}^s) \delta_k^i \}. \end{aligned}$$

This shows that

$$B_l B_m K_{jkh}^i = a_{lm} K_{jkh}^i + b_{lm} (\delta_j^i g_{kh} - \delta_k^i g_{jh}) - 2y^t \mu_m B_t (\delta_j^i C_{khl} - \delta_k^i C_{jhl}),$$

if and only if

$$\begin{aligned} (3.9) \quad & \frac{1}{3} B_l B_m \{ (K_{jk} + C_{js}^r H_{kr}^s) \delta_h^i - (K_{jh} + C_{js}^r H_{hr}^s) \delta_k^i \} - B_l B_m (C_{js}^i H_{kh}^s) \\ &= \frac{1}{3} a_{lm} \{ (K_{jk} + C_{js}^r H_{kr}^s) \delta_h^i - (K_{jh} + C_{js}^r H_{hr}^s) \delta_k^i \} - a_{lm} (C_{js}^i H_{kh}^s). \end{aligned}$$

Thus, we conclude

Theorem 3.3: In $G(BP) - BRF_n$, for $n=4$, Cartan's fourth curvature tensor K_{jkh}^i is a generalized B -birecurrent if and only if eq. (3.9) holds.

Using (1.15b) in Bianchi identity (1.17), we get

$$(3.10) \quad R_{ijk|h}^r + R_{ihj|k}^r + R_{ikh|j}^r = - \left(H_{kh}^s P_{ijs}^r + H_{jk}^s P_{ihs}^r + H_{hj}^s P_{iks}^r \right).$$

Taking B -covariant derivative for eq. (3.10) twice with respect to x^m and x^l , respectively, using the conditions (2.1) and (2.2) in the resulting equation, after that using eq. (3.10), we get

$$\begin{aligned} B_l B_m \left(R_{ijk|h}^r + R_{ihj|k}^r + R_{ikh|j}^r \right) &= a_{lm} \left(R_{ijk|h}^r + R_{ihj|k}^r + R_{ikh|j}^r \right) \\ &- b_{lm} \left(\delta_i^r g_{js} H_{kh}^s - \delta_j^r g_{is} H_{kh}^s + \delta_i^r g_{hs} H_{jk}^s - \delta_h^r g_{is} H_{jk}^s + \delta_i^r g_{ks} H_{hj}^s - \delta_k^r g_{is} H_{hj}^s \right) \\ &- 2y^t \mu_m B_t \left(\delta_i^r C_{jst} H_{kh}^s - \delta_j^r C_{ist} H_{kh}^s + \delta_i^r C_{hst} H_{jk}^s - \delta_h^r C_{ist} H_{jk}^s + \delta_i^r C_{kst} H_{hj}^s \right. \\ &- \left. \delta_k^r C_{ist} H_{hj}^s \right) - \lambda_l \left(P_{ijs}^r B_m H_{kh}^s + P_{ihs}^r B_m H_{jk}^s + P_{iks}^r B_m H_{hj}^s \right) - \lambda_m \left(P_{ijs}^r B_l H_{kh}^s \right. \\ &+ \left. P_{ihs}^r B_l H_{jk}^s + P_{iks}^r B_l H_{hj}^s \right) - \mu_l \left\{ \left(\delta_i^r g_{js} - \delta_j^r g_{is} \right) B_m H_{kh}^s + \left(\delta_i^r g_{hs} - \delta_h^r g_{ih} \right) B_m H_{jk}^s \right. \\ &+ \left. \left(\delta_i^r g_{ks} - \delta_k^r g_{is} \right) B_m H_{hj}^s \right\} - \mu_m \left\{ \left(\delta_i^r g_{js} - \delta_j^r g_{is} \right) B_l H_{kh}^s + \left(\delta_i^r g_{hs} - \delta_h^r g_{ih} \right) B_l H_{jk}^s \right. \\ &+ \left. \left(\delta_i^r g_{ks} - \delta_k^r g_{is} \right) B_l H_{hj}^s \right\} - \left\{ \left(B_l B_m H_{jk}^s \right) P_{ihs}^r + \left(B_l B_m H_{hj}^s \right) P_{iks}^r + \left(B_l B_m H_{kh}^s \right) P_{ijs}^r \right\}. \end{aligned}$$

This shows that

$$(3.11) \quad B_l B_m \left(R_{ijk|h}^r + R_{ihj|k}^r + R_{ikh|j}^r \right) = a_{lm} \left(R_{ijk|h}^r + R_{ihj|k}^r + R_{ikh|j}^r \right),$$

if and only if

$$\begin{aligned} (3.12) \quad &- b_{lm} \left(\delta_i^r g_{js} H_{kh}^s - \delta_j^r g_{is} H_{kh}^s + \delta_i^r g_{hs} H_{jk}^s - \delta_h^r g_{is} H_{jk}^s + \delta_i^r g_{ks} H_{hj}^s - \delta_k^r g_{is} H_{hj}^s \right) \\ &- 2y^t \mu_m B_t \left(\delta_i^r C_{jst} H_{kh}^s - \delta_j^r C_{ist} H_{kh}^s + \delta_i^r C_{hst} H_{jk}^s - \delta_h^r C_{ist} H_{jk}^s + \delta_i^r C_{kst} H_{hj}^s \right. \\ &- \left. \delta_k^r C_{ist} H_{hj}^s \right) - \lambda_l \left(P_{ijs}^r B_m H_{kh}^s + P_{ihs}^r B_m H_{jk}^s + P_{iks}^r B_m H_{hj}^s \right) - \lambda_m \left(P_{ijs}^r B_l H_{kh}^s \right. \end{aligned}$$

$$\begin{aligned}
& +P_{ihs}^r B_l H_{jk}^s + P_{iks}^r B_m H_{hj}^s) - \mu_l \left\{ (\delta_i^r g_{js} - \delta_j^r g_{is}) B_m H_{kh}^s + (\delta_i^r g_{hs} - \delta_h^r g_{ih}) \right. \\
& B_m H_{jk}^s + (\delta_i^r g_{ks} - \delta_k^r g_{is}) B_m H_{hj}^s \left. \right\} - \mu_m \left\{ (\delta_i^r g_{js} - \delta_j^r g_{is}) B_l H_{ks}^s \right. \\
& + (\delta_i^r g_{hs} - \delta_h^r g_{ih}) B_l H_{jk}^s + (\delta_i^r g_{ks} - \delta_k^r g_{is}) B_l H_{hj}^s \left. \right\} - \left\{ (B_l B_m H_{jk}^s) P_{ihs}^r \right. \\
& \left. + (B_l B_m H_{hj}^s) P_{iks}^r + (B_l B_m H_{kh}^s) P_{ijs}^r \right\} = 0.
\end{aligned}$$

Transvecting (3.10) by y^i , using (1.15b), (1.9) and (1.18), we get

$$(3.13) \quad H_{jk|h}^r + H_{hj|k}^r + H_{kh|j}^r = - \left(H_{kh}^s P_{js}^r + H_{jk}^s P_{hs}^r + H_{hj}^s P_{ks}^r \right).$$

Transvecting the condition (2.1) by y^i , using (1.18), (1.13), (1.4b) and (1.1), we get

$$(3.14) \quad B_m P_{kh}^i = \lambda_m P_{kh}^i + \mu_m (y^i g_{kh} - \delta_k^i y_h).$$

Taking B -covariant derivative for eq. (3.13) twice with respect to x^l and x^m , respectively, using eqs. (3.14) and (2.3) in the resulting equation, after that using eq. (3.13), we get

$$\begin{aligned}
& B_l \left(H_{jk|h}^r + H_{hj|k}^r + H_{kh|j}^r \right) = a_{lm} \left(H_{jk|h}^r + H_{hj|k}^r + H_{kh|j}^r \right) \\
& - b_{lm} \left\{ (y^r g_{js} - \delta_j^r y_s) H_{kh}^s + (y^i g_{hs} - \delta_h^r y_s) H_{jk}^s + (y^i g_{ks} - \delta_k^r y_s) H_{hj}^s \right\} \\
& - 2y^t \mu_m B_t \left(y^r C_{jst} H_{kh}^s + y^r C_{hst} H_{jk}^s + \delta_i^r C_{kst} H_{hj}^s \right) \\
& - \lambda_l \left(P_{js}^r B_m H_{kh}^s + P_{hs}^r B_m H_{jk}^s + P_{ks}^r B_m H_{hj}^s \right) - \lambda_m \left(P_{js}^r B_l H_{kh}^s \right. \\
& + P_{hs}^r B_l H_{jk}^s + P_{ks}^r B_l H_{hj}^s) - \mu_l \left\{ (y^r g_{js} - \delta_j^r y_s) B_m H_{kh}^s + (y^r g_{hs} - \delta_h^r y_s) \right. \\
& B_m H_{jk}^s + (y^r g_{ks} - \delta_k^r y_s) B_m H_{hj}^s \left. \right\} - \mu_m \left\{ (y^r g_{js} - \delta_j^r y_s) B_l H_{ks}^s \right. \\
& + (y^r g_{hs} - \delta_h^r y_s) B_l H_{jk}^s + (y^r g_{ks} - \delta_k^r y_s) B_l H_{hj}^s \left. \right\} - \left\{ (B_l B_m H_{jk}^s) P_{hs}^r \right. \\
& \left. + (B_l B_m H_{hj}^s) P_{ks}^r + (B_l B_m H_{kh}^s) P_{js}^r \right\}.
\end{aligned}$$

This shows that

$$(3.15) \quad B_l B_m \left(H_{jk|h}^r + H_{hj|k}^r + H_{kh|j}^r \right) = a_{lm} \left(H_{jk|h}^r + H_{hj|k}^r + H_{kh|j}^r \right),$$

if and only if

$$(3.16) \quad \begin{aligned} & b_{lm} \left\{ \left(y^r g_{js} - \delta_j^r y_s \right) H_{kh}^s + \left(y^i g_{hs} - \delta_h^r y_s \right) H_{jk}^s + \left(y^i g_{ks} - \delta_k^r y_s \right) H_{hj}^s \right\} \\ & + 2y^t \mu_m B_t \left(y^r C_{jst} H_{kh}^s + y^r C_{hst} H_{jk}^s + \delta_i^r C_{kst} H_{hj}^s \right) \\ & + \lambda_l \left(P_{js}^r B_m H_{kh}^s + P_{hs}^r B_m H_{jk}^s + P_{ks}^r B_m H_{hj}^s \right) + \lambda_m \left(P_{js}^r B_l H_{kh}^s \right. \\ & + P_{hs}^r B_l H_{jk}^s + P_{ks}^r B_l H_{hj}^s \left. \right) + \mu_l \left\{ \left(y^r g_{js} - \delta_j^r y_s \right) B_m H_{kh}^s + \left(y^r g_{hs} - \delta_h^r y_s \right) \right. \\ & B_m H_{jk}^s + \left. \left(y^r g_{ks} - \delta_k^r y_s \right) B_m H_{hj}^s \right\} + \mu_m \left\{ \left(y^r g_{js} - \delta_j^r y_s \right) B_l H_{ks}^s \right. \\ & + \left. \left(y^r g_{hs} - \delta_h^r y_s \right) B_l H_{jk}^s + \left(y^r g_{ks} - \delta_k^r y_s \right) B_l H_{hj}^s \right\} + \left\{ \left(B_l B_m H_{jk}^s \right) P_{hs}^r \right. \\ & + \left. \left(B_l B_m H_{hj}^s \right) P_{ks}^r + \left(B_l B_m H_{kh}^s \right) P_{js}^r \right\} = 0. \end{aligned}$$

The equations (3.11) and (3.15) show that the tensors $\left(R_{ijk|h}^r + R_{ihj|k}^r + R_{ikh|j}^r \right)$ and $\left(H_{jk|h}^r + H_{hj|k}^r + H_{kh|j}^r \right)$ behave as birecurrent if and only if eqs. (3.12) and (3.16) respectively, holds. Thus, we conclude

Theorem 3.4: *The behavior of the tensors and $\left(H_{jk|h}^r + H_{hj|k}^r + H_{kh|j}^r \right)$ as birecurrent if and only if eqs. (3.12) and (3.16), respectively hold in $G(BP) - BRF_n$.*

Differentiating eq. (2.3) partially with respect to y^j , using (1.4d), (1.5) and (1.2), we get

$$\begin{aligned} \dot{\partial}_j \left(B_l B_m P_{kh}^i \right) &= \left(\dot{\partial}_j a_{lm} \right) P_{kh}^i + a_{lm} \left(\dot{\partial}_j P_{kh}^i \right) + \left(\dot{\partial}_j b_{lm} \right) \left(y^i g_{kh} - \delta_k^i y_h \right) \\ &+ b_{lm} \left(\delta_j^i g_{kh} + 2y^i C_{jkh} - \delta_k^i g_{jh} \right) - 2y^t \mu_m B_t \left(\delta_j^i C_{khl} \right). \end{aligned}$$

Using the commutation formula shown by (1.12) twice for the tensor $\left(B_m P_{kh}^i \right)$ and then use it for the tensor P_{kh}^i in above equation, we get

$$B_l B_m \left(\dot{\partial}_j P_{kh}^i \right) + B_l \left(P_{kh}^r G_{mjr}^i \right) - B_l \left(P_{rh}^i G_{mjk}^r \right) - B_l \left(P_{kr}^i G_{mjh}^r \right)$$

$$\begin{aligned}
& -\left(B_r P_{kh}^i\right) G_{jlm}^r + \left(B_m P_{kh}^r\right) G_{ljr}^i - \left(B_m P_{rh}^i\right) G_{ljk}^r - \left(B_m P_{kr}^i\right) G_{ijh}^r \\
& = \left(\dot{\partial}_j a_{lm}\right) P_{kh}^i + a_{lm} \left(\dot{\partial}_j P_{kh}^i\right) + \left(\dot{\partial}_j b_{lm}\right) \left(y^i g_{kh} - \delta_k^i y_h\right) \\
& + b_{lm} \left(\delta_j^i g_{kh} + 2y^i C_{jkh} - \delta_k^i g_{jh}\right) - 2y^t \mu_m B_t \left(\delta_j^i C_{khl}\right).
\end{aligned}$$

This shows that

$$(3.17) \quad B_l B_m \left(\dot{\partial}_j P_{kh}^i\right) = a_{lm} \left(\dot{\partial}_j P_{kh}^i\right),$$

if and only if

$$\begin{aligned}
(3.18) \quad & B_l \left(P_{kh}^r G_{mjr}^i\right) - B_l \left(P_{rh}^i G_{mjk}^r\right) - B_l \left(P_{kr}^i G_{mjh}^r\right) \\
& - \left(B_r P_{kh}^i\right) G_{jlm}^r + \left(B_m P_{kh}^r\right) G_{ljr}^i - \left(B_m P_{rh}^i\right) G_{ljk}^r - \left(B_m P_{kr}^i\right) G_{ijh}^r \\
& = \left(\dot{\partial}_j a_{lm}\right) P_{kh}^i + \left(\dot{\partial}_j b_{lm}\right) \left(y^i g_{kh} - \delta_k^i y_h\right) \\
& + b_{lm} \left(\delta_j^i g_{kh} + 2y^i C_{jkh} - \delta_k^i g_{jh}\right) - 2y^t \mu_m B_t \left(\delta_j^i C_{khl}\right).
\end{aligned}$$

Differentiating eq. (2.4) partially with respect to y^h using (1.2), we get

$$\dot{\partial}_h \left(B_l B_m P_{jk}\right) = \left(\dot{\partial}_h a_{lm}\right) P_{jk} + a_{lm} \left(\dot{\partial}_h P_{jk}\right).$$

Using the commutation formula shown by (1.12) twice for the tensor $(B_m P_{jk})$ and then use it for the tensor P_{jk} in above equation, we get

$$\begin{aligned}
& B_l B_m \left(\dot{\partial}_h P_{jk}\right) - B_l \left(P_{rk} G_{mhj}^r\right) - B_l \left(P_{jr} G_{mhk}^r\right) - \left(B_r P_{jk}\right) G_{lhm}^r \\
& - \left(B_m P_{rk}\right) G_{lhj}^r - \left(B_m P_{jr}\right) G_{lhk}^r = \left(\dot{\partial}_h a_{lm}\right) P_{jk} + a_{lm} \left(\dot{\partial}_h P_{jk}\right).
\end{aligned}$$

This shows that

$$(3.19) \quad B_l B_m \left(\dot{\partial}_h P_{jk}\right) = a_{lm} \left(\dot{\partial}_h P_{jk}\right),$$

if and only if

$$(3.20) \quad B_l(P_{rk}G_{mhj}^r) + B_l(P_{jr}G_{mhk}^r) + (B_rP_{jk})G_{lhm}^r + (B_mP_{rk})G_{lhj}^r \\ + (B_mP_{jr})G_{lhk}^r + (\dot{\partial}_h a_{lm})P_{jk} = 0.$$

Differentiating eq. (2.5) partially with respect to y^j , we get

$$\dot{\partial}_j(B_lB_mP_k) = (\dot{\partial}_ja_{lm})P_k + a_{lm}(\dot{\partial}_jP_k).$$

Using the commutation formula shown by (1.12) for the tensor (B_mP_k) and then use it for the tensor P_k in above equation, we get

$$B_lB_m(\dot{\partial}_jP_k) - B_l(P_rG_{mjk}^r) - (B_rP_k)G_{ljm}^r + (B_mP_r)G_{ljk}^r \\ = (\dot{\partial}_ja_{lm})P_k + a_{lm}(\dot{\partial}_jP_k).$$

This shows that

$$(3.21) \quad B_lB_m(\dot{\partial}_jP_k) = a_{lm}(\dot{\partial}_jP_k),$$

if and only if

$$(3.22) \quad B_l(P_rG_{mjk}^r) + (B_rP_k)G_{ljm}^r + (B_mP_r)G_{ljk}^r + (\dot{\partial}_ja_{lm})P_k = 0.$$

The equations (3.17), (3.19) and (3.21) show that the tensors $(\dot{\partial}_jP_{kh}^i)$, $(\dot{\partial}_hP_{jk})$ and $(\dot{\partial}_jP_k)$ be have as birecurrent if and only if eqs. (3.18), (3.20) and (3.22), respectively holds. Thus, we conclude

Theorem 3.5. *The behavior of the tensors $(\dot{\partial}_jP_{kh}^i)$, $(\dot{\partial}_hP_{jk})$ and $(\dot{\partial}_jP_k)$ as birecurrent if and only if eqs. (3.18), (3.20) and (3.22), respectively hold in $G(BP) - BRF_n$.*

Using the definition of the covariant derivative in sense of Berwald for the $h\nu$ -curvature tensor P_{jkh}^i shown by (1.11) with respect to x^m , we get

$$B_mP_{jkh}^i = \partial_mP_{jkh}^i + P_{jkh}^rG_{rm}^i - P_{rkh}^iG_{jm}^r - P_{jrh}^iG_{km}^r - P_{jkr}^iG_{hm}^r - (\partial_rP_{jkh}^i)G_m^r.$$

Using (1.20) in above equation, we get

$$\begin{aligned} B_m P_{jkh}^i &= \partial_m P_{jkh}^i + P_{jkh}^r \Gamma_{rm}^{*i} - P_{rkh}^i \Gamma_{jm}^{*r} - P_{jrh}^i \Gamma_{km}^{*r} - P_{jkr}^i \Gamma_{hm}^{*r} - (\dot{\partial}_r P_{jkh}^i) G_m^r \\ &= P_{jkh}^r P_{rm}^i - P_{rkh}^i P_{jm}^r - P_{jrh}^i P_{km}^r - P_{jkr}^i P_{hm}^r. \end{aligned}$$

Using the definition of the h -covariant derivative in sense of Cartan shown by (1.8) with respect to x^m in above equation, we get

$$(3.23) \quad B_m P_{jkh}^i = P_{jkh|m}^i + P_{jkh}^r P_{rm}^i - P_{rkh}^i P_{jm}^r - P_{jrh}^i P_{km}^r - P_{jkr}^i P_{hm}^r.$$

Taking B -covariant derivative for eq. (3.23) with respect to x^l and using the condition (2.2), we get

$$\begin{aligned} B_l P_{jkh|m}^i &= a_{lm} P_{jkh}^i + b_{lm} (\delta_j^i g_{kh} - \delta_k^i g_{jh}) - 2y^t \mu_m B_t (\delta_j^i C_{khl} - \delta_k^i C_{jhl}) \\ &\quad - B_l (P_{jkh}^r P_{rm}^i - P_{rkh}^i P_{jm}^r - P_{jrh}^i P_{km}^r - P_{jkr}^i P_{hm}^r). \end{aligned}$$

This shows that

$$(3.24) \quad B_l P_{jkh|m}^i = a_{lm} P_{jkh}^i + b_{lm} (\delta_j^i g_{kh} - \delta_k^i g_{jh}) - 2y^t \mu_m B_t (\delta_j^i C_{khl} - \delta_k^i C_{jhl}),$$

if and only if

$$(3.25) \quad B_l (P_{jkh}^r P_{rm}^i - P_{rkh}^i P_{jm}^r - P_{jrh}^i P_{km}^r - P_{jkr}^i P_{hm}^r) = 0.$$

Thus, we conclude

Theorem 3.6. *The covariant derivative in sense of Berwald with respect to x^l of the h -covariant derivative in sense of Cartan with respect to x^m for Cartan's second curvature tensor P_{jkh}^i which given by eq. (3.24) is a generalized birecurrent if and only if eq. (3.25) holds.*

Remark 3.1: we used a generalized birecurrence property for both, sense of Berwald and Cartan's covariant derivative for some tensors.

Remark 3.2: In $G(BP) - BRF_n$, for the $h\nu$ -curvature tensor P_{jkh}^i , the covariant derivative in sense of Berwald coincides with the h -covariant derivative in sense of Cartan if and only if eq. (3.25) holds.

These (Remark 3.1 and Remark 3.2) will be true for another tensors in the following discussions and theorems. Transvecting (3.23) by y^j , using (1.18), (1.13), (1.9) and (1.19), we get

$$(3.26) \quad B_m P_{kh}^i = P_{kh|m}^i + P_{kh}^r P_{rm}^i - P_{rh}^i P_{km}^r - P_{kr}^i P_{hm}^r .$$

Taking B -covariant derivative for eq. (3.26) with respect to x^l and using eq. (2.3), we get

$$\begin{aligned} B_l P_{kh|m}^i &= a_{lm} P_{kh}^i + b_{lm} (y^i g_{kh} - \delta_k^i y_h) - 2y^t \mu_m B_t (y^i C_{khl}) \\ &\quad - B_l (P_{kh}^r P_{rm}^i - P_{rh}^i P_{km}^r - P_{kr}^i P_{hm}^r) . \end{aligned}$$

This shows that

$$(3.27) \quad B_l P_{kh|m}^i = a_{lm} P_{kh}^i + b_{lm} (y^i g_{kh} - \delta_k^i y_h) - 2y^t \mu_m B_t (y^i C_{khl}) ,$$

if and only if

$$(3.28) \quad B_l (P_{kh}^r P_{rm}^i - P_{rh}^i P_{km}^r - P_{kr}^i P_{hm}^r) = 0 .$$

Thus, we conclude

Theorem 3.7: *The covariant derivative in sense of Berwald with respect to x^l of the h -covariant derivative in sense of Cartan with respect to x^m for the (v)hv-torsion tensor P_{kh}^i given by eq. (3.27) if and only if eq. (3.28) holds.*

Contracting the indices i and h in (3.23), using (1.21a), we get

$$B_m P_{jk} = P_{jk|m} + P_{jki}^r P_{rm}^i - P_{rk} P_{jm}^r - P_{jr} P_{km}^r - P_{jkr}^i P_{im}^r ,$$

or

$$(3.29) \quad B_m P_{jk} = P_{jk|m} - P_{rk} P_{jm}^r - P_{jr} P_{km}^r .$$

Taking B -covariant derivative for eq. (3.29) with respect to x^l and using eq. (2.4), we get

$$B_l P_{jk|m} = a_{lm} P_{jk} - B_l (P_{rk} P_{jm}^r + P_{jr} P_{km}^r) .$$

This shows that

$$(3.30) \quad B_l P_{jk|m} = a_{lm} P_{jk} ,$$

if and only if

$$(3.31) \quad B_l (P_{rk} P_{jm}^r + P_{jr} P_{km}^r) = 0 .$$

Contracting the indices i and h in (3.26), using (1.21b), we get

$$B_m P_k = P_{k|m} + P_{ki}^r P_{rm}^i - P_r P_{km}^r - P_{kr}^i P_{im}^r ,$$

or

$$(3.32) \quad B_m P_k = P_{k|m} - P_r P_{km}^r .$$

Taking B -covariant derivative for eq. (3.32) with respect to x^l and using eq. (2.5), we get

$$B_l P_{k|m} = a_{lm} P_k - B_l (P_r P_{km}^r) .$$

This shows that

$$(3.33) \quad B_l P_{k|m} = a_{lm} P_k ,$$

if and only if

$$(3.34) \quad B_l (P_r P_{km}^r) = 0 .$$

The equations (3.30) and (3.33) show that when the covariant derivative in sense of Berwald coincide sense of Cartan, the behavior of the P -Ricci tensor P_{jk} and the curvature vector P_k as birecurrent if and only if eqs. (3.31) and (3.34), respectively hold . Thus, we conclude

Theorem 3.8: *The covariant derivative in sense of Berwald with respect to x^l of the h -covariant derivative in sense of Cartan with respect to x^m for the P -Ricci tensor P_{jk} and the curvature vector P_k behave as birecurrent if and only if eqs. (3.32) and (3.35), respectively hold .*

4. Conclusion

We introduced the generalized BP -birecurrent Finsler space, we proved that some tensors satisfy the generalized birecurrence property, we got Berwald's covariant derivative of second order for some tensors, we obtained some tensors are non-vanishing, we proved that the covariant derivative in sense of Berwald coincides with the h -covariant derivative in sense of Cartan, also we found the necessary and sufficient condition for some tensors to be generalized birecurrent in sense of Berwald, we obtained that the behavior of some tensors as birecurrent in this space.

References

1. F. Y. Qasem, On Generalized H -birecurrent Finsler Space, *International Journal of Mathematics and its Applications*, **4(2-B)** (2016), 51-57.
2. W. H. Hadi, *Study of Certain Types of Generalized Birecurrent in Finsler Spaces*, Ph.D. Thesis, Faculty of Education-Aden, University of Aden, (Aden) (Yemen), 2016.
3. F. Y. Qasem and W. H. Hadi, On a Generalized BR -Birecurrent Finsler Space, *American Scientific Research Journal for Engineering, Technology and Science*, (2016), 9-18.
4. F. Y. Qasem and W. H. Hadi, On a Generalized H^h -Birecurrent Finsler Space, *International Journal of Sciences Basic and Applied Research*, **25(2)** (2016), 207-217.
5. F. Y. Qasem and W. H. Hadi, On a Generalized R^h -Birecurrent Finsler Space, *International Journal of Mathematics and physical Sciences Research*, **3(2)** (2016), 93-99.
6. F. A. Assallal, *On Certain Generalized h -Birecurrent of Curvature Tensor*, M. Sc. Thesis, University of Aden, (Aden) (Yemen), 2018.
7. A. A. Saleem, *On Certain Generalized Birecurrent and Trirecurrent Finsler Spaces*, M. Sc. Thesis, University of Aden, (Aden) (Yemen), 2011.
8. P. N. Pandey and R. Verma, On C^h -Birecurrent Finsler Space, *Journal of International Academy of Physical Science*, **2** (1998), 43-50.
9. S. Dikshit, *Certain Types of Recurrences in Finsler Spaces*, D. phil. Thesis, University of Allahabad, (Allahabad) (India), 1992.
10. M. A. Ali, *On K^h -Birecurrent Finsler Space*, M. sc. Thesis, University of Aden, (Aden) (Yemen), 2014.
11. F. Y. Qasem and A. M. Hanballa, On Study K^h -Generalized Birecurrent Finsler Space, *International Journal of Sciences Basic and Applied Research*, **25(3)** (2016), 208-216.

12. A. M. Hanballa, *On Covariant Differentiation of Curvature Tensors in Finsler Spaces*, Ph. D. Thesis, Faculty of Education-Aden, University of Aden, (Aden) (Yemen), 2016.
13. H. Rund, *The Differential Geometry of Finsler Spaces*, Springer-Verlag, Berlin Göttingen, 1959; 2nd Edit. (in Russian), Nauka, (Moscow), 1981.
14. M. Matsumoto, On h – Isotropic and C^h – Recurrent Finsler, *J. Math. Kyoto Univ.*, **11** (1971), 1-9.
15. Alaa A. Abdallah, A. A. Navlekar and Kirtiwant P. Ghadle, On Study Generalized BP – Recurrent Finsler Space, *International Journal of Mathematics Trends and Technology*, **65(4)** (2019), 74-79 .
16. Alaa A. Abdallah, A. A. Navlekar and Kirtiwant P. Ghadle, The Necessary and Sufficient Condition for Some Tensors Which Satisfy a Generalized BP – Recurrent Finsler Space, *International Journal of Scientific and Engineering Research*, **10(11)** (2019), 135-140.
17. A. Zafar and A. Musavvir, On Some Properties of W – Curvature Tensor, *Palestine Journal of Mathematics*, **3(1)** (2014), 61-69.
18. A. M. Otman, *On Covariant Differentiation for Curvature Tensor of Third Order in the Sense of Berwald*, M. sc. Thesis, University of Aden, (Aden) (Yemen), 2018.