

A Related Fixed Point Theorem on Five Metric Spaces

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Abstract: In the present paper a fixed point theorem on five metric space is established. This theorem generalizes and extends the result obtained by Luljeta Kikina and Kristaq Kikina¹ from four metric spaces to five metric spaces. Further, a generalization of the Theorem 1.2 proved by them¹ is obtained as corollary of our theorem.

Keywords: Fixed point, metric space, complete metric space.

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1. Introduction

R. K. Jain, H. K. Sahu and B. Fisher² proved the following fixed point theorem.

Theorem 1.1. *Let (X, d_1) , (Y, d_2) and (Z, d_3) be complete metric spaces. Let $T : X \rightarrow Y$, $S : Y \rightarrow Z$, $R : Z \rightarrow X$ be three mappings satisfying the following inequalities:*

$$d_1(RSy, RSTx) \leq \frac{cf_1(x, y)}{g_1(x, y)},$$

$$d_2(TRz, TRSy) \leq \frac{cf_2(y, z)}{g_2(y, z)},$$

$$d_3(STx, STRz) \leq \frac{cf_3(z, x)}{g_3(z, x)},$$

for all $x \in X$, $y \in Y$ and $z \in Z$ for

which $g_1(x, y) \neq 0$, $g_2(x, y) \neq 0$, $g_3(z, x) \neq 0$, where $0 \leq c < 1$ and

$$f_1(x, y) = \max\{d_1(x, RSTx)d_3(STx); d_1(x, RSTx)d_2(y, TRSy); \\ d_1(x, RSy)d_2(y, Tx)\},$$

$$f_2(y, z) = \max\{d_2(y, TRSy)d_1(Rz, RSy); d_2(y, TRSy)d_3(z, STRz); \\ d_2(y, TRz)\}d_3(z, Sy)\},$$

$$f_3(z, x) = \max\{d_3(z, STRz)d_2(Tx, TRz); d_3(z, STRz)d_1(x, RSTx); \\ d_3(z, STx)\}d_1(x, Rz)\},$$

$$g_1(x, y) = \max\{d_1(x, RSy), d_1(x, RSTx)d_2(Tx, TRSy)\},$$

$$g_2(y, z) = \max\{d_2(y, TRz), d_2(y, TRSy)d_3(Sy, STRz)\},$$

$$g_3(z, x) = \max\{d_3(z, STx), d_3(z, STRz)d_1(Rz, RSTx)\},$$

Then RST has a unique fixed point $\alpha \in X$, TRS has a unique fixed point $\beta \in Y$, and STR has a unique fixed point $\gamma \in Z$. Further $T\alpha = \beta$, $S\beta = \gamma$ and $R\gamma = \alpha$.

The following theorem is proved by Luljeta Kikina and Kristaq Kikina¹:

Theorem 1.2. Let (X, d_1) , (Y, d_2) , (Z, d_3) and (U, d_4) be complete metric spaces. Let $T : X \rightarrow Y$, $S : Y \rightarrow Z$, $R : Z \rightarrow U$ and $Q : U \rightarrow X$ be four mappings satisfying the following inequalities :

$$d_1(QRSy, QRSTx) \leq \frac{cF_1(x, y)}{G_1(x, y)},$$

$$d_2(TQRz, TQRSy) \leq \frac{cF_2(y, z)}{G_2(y, z)},$$

$$d_3(STQu, STQRz) \leq \frac{cF_3(z, u)}{G_3(z, u)},$$

$$d_4(RSTx, RSTQu) \leq \frac{cF_4(u, x)}{G_4(u, x)},$$

for all $x \in X$, $y \in Y$, $z \in Z$ and $u \in U$ for which

$G_1(x, y) \neq 0$, $G_2(y, z) \neq 0$, $G_3(z, u) \neq 0$, $G_4(u, x) \neq 0$, where $0 \leq c < 1$ and

$$f_1(x, y) = \max\{d_1(x, QRSTx) d_4(RSy, RSTx); d_1(x, QRSTx) d_3(Sy, STx); \\ d_1(x, QRSTx) d_2(y, TQRSy); d_1(x, QRSy) d_2(y, Tx)\},$$

$$f_2(y, z) = \max\{d_2(y, TQRSy) d_1(QRz, QRSy); d_2(y, TQRSy) d_4(Rz, RSy); \\ d_2(y, TQRS) d_3(z, STQRz); d_2(y, TQRz) d_3(z, Sy)\},$$

$$f_3(z, u) = \max\{d_3(z, STQRz) d_2(TQu, TQRz); d_3(z, STQRz) d_1(Qu, QRz); \\ d_3(z, STQRz) d_4(u, RSTQu); d_3(z, STQu) d_4(u, Rz)\},$$

$$f_4(u, x) = \max\{d_4(u, RSTQu) d_3(STx, STQu); d_4(u, RSTQu) d_2(Tx, TQu); \\ d_4(u, RSTQu) d_1(x, QRSTx); d_4(u, RSTx) d_1(x, Qu)\},$$

$$G_1(x, y) = \max\{d_1(x, QRSy), d_1(x, QRSTx), d_2(Tx, TQRSy)\},$$

$$G_2(y, z) = \max\{d_2(y, TQRz), d_2(y, TQRSy), d_3(Sy, STQRz)\},$$

$$G_3(z, u) = \max\{d_3(z, STQu), d_3(z, STQRz), d_4(Rz, RSTQu)\},$$

$$G_4(u, x) = \max\{d_4(u, RSTx), d_4(u, RSTQu), d_1(Qu, QRSTx)\},$$

Then $QRST$ has a unique fixed point $\alpha \in X$, $TQRS$ has a unique fixed point $\beta \in Y$, $STQR$ has a unique fixed point $\gamma \in Z$ and $RSTQ$ has a unique fixed point $\delta \in U$. Further $T\alpha = \beta$, $S\beta = \gamma$, $R\gamma = \delta$ and $Q\delta = \alpha$.

In this paper we obtain a new result (cf. Theorem 2.1) concerning fixed points for mappings on five metric spaces. We generalize and extend the result of theorem 1.2 for four metric spaces to five metric spaces. As a corollary of our theorem we obtain a new theorem which generalizes the theorem 1.2.

2. Main Results

Now, we will give and prove our theorem as follows:

Theorem 2.1. Let $(X, d_1), (Y, d_2), (Z, d_3), (U, d_4)$ and (V, d_5) be complete metric spaces. Let $T : X \rightarrow Y$, $S : Y \rightarrow Z$, $R : Z \rightarrow U \rightarrow V$ and $P : V \rightarrow X$ be five mappings satisfying the following inequalities:

$$(1) \quad d_1(PQRSy, PQRSx) \leq \frac{cA_1(x, y)}{B_1(x, y)},$$

$$(2) \quad d_2(TPQRz, TPQRSy) \leq \frac{cA_2(y, z)}{B_2(y, z)},$$

$$(3) \quad d_3(STPQu, STPQRz) \leq \frac{cA_3(z, u)}{B_3(z, u)},$$

$$(4) \quad d_4(RSTPv, RSTPQu) \leq \frac{cA_4(u, v)}{B_4(u, v)},$$

$$(5) \quad d_5(QRSTx, QRSTPv) \leq \frac{cA_5(v, x)}{B_5(v, x)},$$

for all $x \in X, y \in Y, z \in Z, u \in U$ and $v \in V$, which

$B_1(x, y) \neq 0, B_2(y, z) \neq 0, B_3(z, u) \neq 0, B_4(u, v) \neq 0, B_5(v, x) \neq 0$, where
 $0 \leq c < 1$ and

$$A_1(x, y) = \max\{d_1(x, PQRSx)d_5(QRSy, QRSTx); d_1(x, PQRSx)\} d_4(RSy, RSTx); d_1(x, x)d_3(Sy, STx); sd_1(x, PQRSx)d_2(y, TPQRSy); d_1(x, PQRSy)d_2(y, Tx)\},$$

$$A_2(y, z) = \max\{d_2(y, TPQRSy)d_1(PQRz, PQRSy); d_2(y, TPQRSy)\} d_5(QRz, QRSy); d_2(y, TPQRSy)d_4(Rz, RSy); d_2(y, TPQRSy)d_3(z, STPQRz); d_2(y, TPQRz)d_3(z, Sy)\},$$

$$A_3(z, u) = \max\{d_3(z, STPQRz)d_2(TPQu, TPQRz); d_3(z, STPQRz)\} d_1(PQu, PQRz); d_3(z, STPQRz)d_5(Qu, QRz); d_3(z, STPQRz)d_4(u, RSTPQu); d_3(z, STPQu)d_4(u, Rz)\},$$

$$A_4(u, v) = \max\{d_4(u, RSTPQu)d_3(STPv, STPQu); d_4(u, RSTPQu)\} d_2(TPv, TPQu); d_4(u, RSTPQu)d_1(Pv, PQu); d_4(u, RSTPQu)d_5(v, QRSTPv); d_1(u, RSTPv)d_5(v, Qu)\},$$

$$A_5(v, x) = \max\{d_5(v, QRSTPv)d_4(RSTx, RSTPv); d_5(v, QRSTPv)\} d_3(STx, STPv); d_5(v, QRSTPv)d_2(Tx, TPv); d_5(v, QRSTPv)d_1(x, PQRSx); d_5(v, QRSTx)d_1(x, Pv)\}.$$

$$\begin{aligned}
B_1(x, y) &= \max\{d_1(x, PQRSy), d_1(x, PQRSTx), d_2(Tx, TPQRSy)\}, \\
B_2(y, z) &= \max\{d_2(y, TPQRz), d_2(y, TPQRSy), d_3(Sy, STPQRz)\}, \\
B_3(z, u) &= \max\{d_3(z, STPQu), d_3(z, STPQRz), d_4(Rz, RSTPQu)\}, \\
B_4(u, v) &= \max\{d_4(u, RSTPv), d_4(u, RSTPQu), d_5(Qu, QRSTPv)\}, \\
B_5(v, x) &= \max\{d_5(v, QRSTx), d_5(v, QRSTPv), d_1(Pv, PQRSTx)\}.
\end{aligned}$$

Then $PQRST$ has a unique fixed point $\alpha \in X$, $TPQRS$ has a unique fixed point $\beta \in Y$, $STPQR$ has a unique fixed point $\gamma \in Z$, $RSTPQ$ has a unique fixed point $\delta \in U$, and $QRSTP$ has a unique fixed point $\mu \in V$. Further $T\alpha = \beta$, $S\beta = \gamma$, $R\gamma = \delta$, $Q\delta = \mu$, $P\mu = \alpha$.

Proof : Let $x_0 \in X$ be an arbitrary point. We define five sequences (x_n) , (y_n) , (z_n) , (u_n) and (v_n) with X , Y , Z , U and V respectively as follows :

$$X_n = (PQRST)^n X_0; \quad y_n = Tx_{n-1}; \quad Z_n = Sy_n; \quad u_n = RZ_n; \quad v_n = Qu_n \text{ for all } n.$$

We assume that $X_n \neq X_{n+1}$, $y_n \neq y_{n+1}$, $Z_n \neq Z_{n+1}$, $u_n \neq u_{n+1}$ and $V_n \neq V_{n+1}$ for some n , then $y_{n+1} = y_{n+2}$, $z_{n+1} = z_{n+2}$, $u_{n+1} = u_{n+2}$, $v_{n+1} = v_{n+2}$ and we could put $x_n = \alpha$, $y_{n+1} = \beta$, $z_{n+1} = \gamma$, $u_{n+1} = \delta$, $V_{n+1} = \mu$. If $y_n = y_{n+1}$, then $z_n = z_{n+1}$, $u_{n+1} = u_{n+1}$, $v_n = v_{n+1}$ and the iter equalities imply that $x_n = x_{n+1}$.

Taking $z = z_{n-1}$ and $y = y_n$ in (2), we obtain :

$$\begin{aligned}
(6) \quad d_2(y_n, y_{n+1}) &= d_2(TPQRz_{n-1}, TPQRSy_n) \leq \frac{cA_2(y_n, z_{n-1})}{B_2(y_n, z_{n-1})} \\
&= \frac{c \max\{d_2(y_n, y_{n+1})d_1(x_{n-1}, x_n); d_2(y_n, y_{n+1})d_5(v_{n-1}, v_n); \}}{\max\{d_2(y_n, y_n); d_2(y_n, y_{n+1}); d_3(z_n, z_n)\}} \\
&\quad \frac{d_2(y_n, y_{n+1})d_4(u_{n-1}, u_n); d_2(y_n, y_{n+1})d_3(z_{n-1}, z_n);}{d_2(y_n, y_n)d_3(z_{n-1}, z_n)} \\
&\quad \frac{d_2(y_n, y_n)d_3(z_{n-1}, z_n)}{\max\{d_2(y_n, y_n); d_2(y_n, y_{n+1}); d_3(z_n, z_n)\}} \\
&= \frac{c \max\left\{d_2(y_n, y_{n+1}) \begin{bmatrix} d_1(x_{n-1}, x_n), d_5(v_{n-1}, v_n), \\ d_4(u_{n-1}, u_n), d_3(z_{n-1}, z_n) \end{bmatrix} \right\}}{d_2(y_n, y_{n+1})}
\end{aligned}$$

$$c \max \{d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n), d_4(u_{n-1}, u_n), d_5(v_{n-1}, v_n)\}.$$

Taking $u = u_{n-1}$ and $z = z_n$ in (3) we obtain :

$$\begin{aligned} d_3(z_n, z_{n+1}) &= d_3(STPQu_{n-1}, STPQRz_n) \leq \frac{cA_3(z_n, u_{n-1})}{B_3(z_n, u_{n-1})} = \\ &= \frac{c \max \{d_3(z_n, z_{n+1})d_2(y_n, y_{n+1}); d_3(z_n, z_{n+1})d_1(x_{n-1}, x_n)\}}{\max \{d_3(z_n, z_n); d_3(z_n, z_{n+1}); d_4(u_n, u_n)\}} \\ &= \frac{d_3(z_n, z_{n+1})d_5(v_{n-1}, v_n); d_3(z_n, z_{n+1})d_4(u_{n-1}, u_n); d_3(z_n, z_n)d_4(u_{n-1}, u_n)}{\max \{d_3(z_n, z_n); d_3(z_n, z_{n+1}); d_4(u_n, u_n)\}} = \\ &= \frac{c \max \left\{ d_3(z_n, z_{n+1}) \begin{bmatrix} d_2(y_n, y_{n+1}), d_1(x_{n-1}, x_n), \\ d_5(v_{n-1}, v_n), d_4(u_{n-1}, u_n) \end{bmatrix} \right\}}{d_3(z_n, z_{n+1})} \\ &= c \max \{d_2(y_n, y_{n+1}), d_1(x_{n-1}, x_n), d_5(v_{n-1}, v_n), d_4(u_{n-1}, u_n)\} \end{aligned}$$

and using (6) we obtain :

$$\begin{aligned} (7) \quad d_3(z_n, z_{n+1}) &\leq c \max \left\{ \begin{aligned} &cd_1(x_{n-1}, x_n), cd_3(z_{n-1}, z_n), \\ &cd_4(u_{n-1}, u_n), cd_5(v_{n-1}, v_n), \\ &d_1(x_{n-1}, x_n), d_5(v_{n-1}, v_n)d_4(v_{n-1}, v_n) \end{aligned} \right\} \\ &= c \max \{d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n), d_4(u_{n-1}, u_n), d_5(v_{n-1}, v_n)\} \end{aligned}$$

Taking $u = u_n$ and $v = v_{n-1}$ in (4) we obtain :

$$\begin{aligned} d_4(u_n, u_{n+1}) &= d_4(RSTPv_{n-1}, RSTPQu_n) \leq \frac{cA_4(u_n, v_{n-1})}{B_4(u_n, v_{n-1})} \\ &= \frac{c \max \{d_4(u_n, u_{n+1})d_3(z_n, z_{n+1}); d_4(u_n, u_{n+1})d_2(y_n, y_{n+1}); \\ &\quad \max \{d_4(u_n, u_n); d_4(u_n, u_{n+1}); d_5(v_n, v_n)\} \\ &\quad d_4(u_n, u_{n+1})d_1(x_{n-1}, x_n); d_4(u_n, u_{n+1})d_5(v_{n-1}, v_n); \\ &\quad d_4(u_n, u_n)d_5(v_{n-1}, v_n)\}}{\max \{d_4(u_n, u_n); d_4(u_n, u_{n+1}); d_5(v_n, v_n)\}} \end{aligned}$$

$$\begin{aligned}
&= \frac{c \max \left\{ d_4(u_n, u_{n+1}) \left[\begin{array}{l} d_3(z_n, z_{n+1}), d_2(y_n, y_{n+1}), \\ d_1(x_{n-1}, x_n), d_5(v_{n-1}, v_n) \end{array} \right] \right\}}{d_4(u_n, u_{n+1})} \\
&= c \max \{ d_3(z_n, z_{n+1}), d_2(y_n, y_{n+1}), d_1(x_{n-1}, x_n), d_5(v_{n-1}, v_n) \}
\end{aligned}$$

and using (6) and (7) we obtain :

$$(8) \quad d_4(u_n, u_{n+1}) \leq c \max \left\{ \begin{array}{l} cd_1(x_{n-1}, x_n), cd_3(z_{n-1}, z_n), cd_4(u_{n-1}, u_n), \\ cd_5(v_{n-1}, v_n), cd_1(x_{n-1}, x_n), \end{array} \right.$$

$$\begin{aligned}
&cd_3(z_{n-1}, z_n), cd_4(u_{n-1}, u_n), cd_5(v_{n-1}, v_n), d_1(x_{n-1}, x_n), d_5(v_{n-1}, v_n) \} \\
&= c \max \{ d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n), d_4(u_{n-1}, u_n), d_5(v_{n-1}, v_n) \}
\end{aligned}$$

Taking $v = v_n$ and $x = x_{n-1}$ in (5) we obtain :

$$\begin{aligned}
d_5(v_n, v_{n+1}) &= d_5(QRSTx_{n-1}, QRSTPv_n) \leq \frac{cA_5(v_n, x_{n-1})}{B_5(v_n, x_{n-1})} = \\
&= \frac{c \max \{ d_5(v_n, v_{n+1})d_4(u_n, u_{n+1}); d_5(v_n, v_{n+1})d_3(z_n, z_{n+1}); \}}{\max \{ d_5(v_n, v_n); d_5(v_n, v_{n+1}); d_1(v_n, v_n) \}} \\
&= \frac{d_5(v_n, v_{n+1})d_2(y_n, y_{n+1}); d_5(v_n, v_{n+1})d_1(x_{n-1}, x_n);}{\max \{ d_5(v_n, v_n); d_5(v_n, v_{n+1}); d_1(v_n, v_n) \}} = \\
&= \frac{c \max \left\{ d_5(v_n, v_{n+1}) \left[\begin{array}{l} d_4(u_n, u_{n+1}), d_3(z_n, z_{n+1}), \\ d_2(y_n, y_{n+1}), d_1(x_{n-1}, x_n) \end{array} \right] \right\}}{d_5(v_n, v_{n+1})} \\
&= c \max \{ d_4(u_n, u_{n+1}), d_3(z_n, z_{n+1}), d_2(y_n, y_{n+1}), d_1(x_{n-1}, x_n) \}
\end{aligned}$$

and using (6), (7) and (8), we obtain :

$$(9) \quad d_5(v_n, v_{n+1}) \leq c \max \left\{ \begin{array}{l} cd_1(x_{n-1}, x_n), cd_3(z_{n-1}, z_n), cd_4(u_{n-1}, u_n), \\ cd_5(v_{n-1}, v_n), cd_1(x_{n-1}, x_n), cd_3(z_{n-1}, z_n), \\ cd_4(u_{n-1}, u_n), cd_5(v_{n-1}, v_n), cd_1(x_{n-1}, x_n), \\ cd_3(z_{n-1}, z_n), cd_4(u_{n-1}, u_n), cd_5(v_{n-1}, v_n), d_1(x_{n-1}, x_n) \end{array} \right\}$$

$d_5(v_n, v_{n+1}) \leq c \max \{d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n), d_4(u_{n-1}, u_n), d_5(v_{n-1}, v_n)\}$ Taking $x = x_n$ and $y = y_n$ in (1) we obtain :

$$\begin{aligned}
 d_1(x_n, x_{n+1}) &= d_1(PQRSy_n, PQRSTx_n) \leq \frac{cA_1(x_n, y_n)}{B_1(x_n, y_n)} = \\
 &= \frac{c \max \{d_1(x_n, x_{n+1})d_5(v_n, v_{n+1}); d_1(x_n, x_{n+1})d_4(u_n, u_{n+1});\}}{\max \{d_1(x_n, x_n); d_1(x_n, x_{n+1}); d_2(y_{n+1}, y_{n+1})\}} \\
 &= \frac{d_1(x_n, x_{n+1})d_3(z_n, z_{n+1}); d_1(x_n, x_{n+1})d_2(y_n, y_{n+1}); d_1(x_n, x_n)d_2(y_n, y_{n+1})}{\max \{d_1(x_n, x_n); d_1(x_n, x_{n+1}); d_2(y_{n+1}, y_{n+1})\}} \\
 &= \frac{c \max \left\{ d_1(x_n, x_{n+1}) \left[\begin{array}{l} d_5(v_n, v_{n+1}), d_4(u_n, u_{n+1}), \\ d_3(z_n, z_{n+1}), d_2(y_n, y_{n+1}) \end{array} \right] \right\}}{d_1(x_n, x_{n+1})} \\
 &= c \max \{d_5(v_n, v_{n+1}), d_4(u_n, u_{n+1}), d_3(z_n, z_{n+1}), d_2(y_n, y_{n+1})\}.
 \end{aligned}$$

and using (6), (7), (8) and (9), we obtain :

$$(10) \quad d_1(x_n, x_{n+1}) \leq c \max \left\{ \begin{array}{l} d_1(x_{n-1}, x_n), d_3(z_{n-1}, z_n), \\ d_4(u_{n-1}, u_n), d_5(v_{n-1}, v_n) \end{array} \right\},$$

It now follows by induction on using inequalities (6), (7), (8), (9) and (10) that :

$$\begin{aligned}
 d_1(x_n, x_{n+1}) &\leq c^{n-1} \max \{d_1(x_1; x_2); d_3(z_1; z_2); d_4(u_1; u_2); d_5(v_1; v_2)\}. \\
 d_2(y_n, y_{n+1}) &\leq c^{n-1} \max \{d_1(x_1; x_2); d_3(z_1; z_2); d_4(u_1; u_2); d_5(v_1; v_2)\}. \\
 d_3(z_n, z_{n+1}) &\leq c^{n-1} \max \{d_1(x_1; x_2); d_3(z_1; z_2); d_4(u_1; u_2); d_5(v_1; v_2)\}. \\
 d_4(u_n, u_{n+1}) &\leq c^{n-1} \max \{d_1(x_1; x_2); d_3(z_1; z_2); d_4(u_1; u_2); d_5(v_1; v_2)\}. \\
 d_5(v_n, v_{n+1}) &\leq c^{n-1} \max \{d_1(x_1; x_2); d_3(z_1; z_2); d_4(u_1; u_2); d_5(v_1; v_2)\}.
 \end{aligned}$$

Since $0 \leq c < 1$, the sequences (x_n) , (y_n) , (z_n) , (u_n) and (v_n) are Cauchy sequences. Since (X, d_1) , (Y, d_2) , (Z, d_3) , (U, d_4) and (V, d_5) are complete metric spaces, we have:

$$\lim_{n \rightarrow \infty} x_n = \alpha \in X; \lim_{n \rightarrow \infty} y_n = \beta \in Y; \lim_{n \rightarrow \infty} z_n = \gamma \in Z;$$

$$\lim_{n \rightarrow \infty} u_n = \delta \in U; \lim_{n \rightarrow \infty} v_n = \mu \in V;$$

Taking $x = x_n$ and $y = \beta$ in (1) we obtain :

$$\begin{aligned} d_1(PQRS\beta, x_{n+1}) &= d_1(PQRS\beta, PQRSx_n) \leq \frac{cA_1(x_n, \beta)}{B_1(x_n, \beta)} = \\ &= \frac{c \max\{d_1(x_n, x_{n+1})d_5(QRS\beta, v_{n+1}); d_1(x_n, x_{n+1})d_4(RS\beta, u_{n+1});\}}{\max\{d_1(x_n, PQRS\beta); d_1(x_n, x_{n+1}); d_2(y_{n+1}, TPQRS\beta)\}} \\ &\quad \frac{d_1(x_n, x_{n+1})d_3(S\beta, z_{n+1}); d_1(x_n, x_{n+1})d_2(\beta, TPQRS\beta);}{d_1(x_n, PQRS\beta)d_2(\beta, y_{n+1})} \\ &\quad \frac{d_1(x_n, PQRS\beta)d_2(\beta, y_{n+1})}{\max\{d_1(x_n, PQRS\beta); d_1(x_n, x_{n+1}); d_2(y_{n+1}, TPQRS\beta)\}} \end{aligned}$$

Letting n tending in infinity we get $d_1(PQRS\beta, \alpha) \leq 0$ from which it follows $PQRS\beta = \alpha$.

In the same way, using the inequalities (2), (3), (4) and (5) it can be shown that

$$QRST\alpha = \delta, RSTP\delta = \gamma, STPQ\gamma = \mu, TPQR\mu = \beta.$$

Taking $y = y_n$ and $z = s\beta$ in (2) we obtain :

$$\begin{aligned} d_2(TPQRS\beta, y_{n+1}) &= d_2(TPQRS\beta, TPQRSy_n) \leq \frac{cA_2(y_n, S\beta)}{B_1(y_n, S\beta)} = \\ &= \frac{c \max\{d_2(y_n, y_{n+1})d_1(PQRS\beta, x_n); d_2(y_n, y_{n+1})d_5(QRS\beta, v_n); \}}{\max\{d_2(y_n, TPQRS\beta); d_2(y_n, y_{n+1}); d_3(z_n, STPQS\beta)\}} \\ &\quad \frac{d_2(y_n, y_{n+1})d_4(RS\beta, u_n); d_2(y_n, y_{n+1})d_3(S\beta, STPQRS\beta);}{d_2(y_n, TPQRS\beta)d_3(S\beta, Z_n)} \\ &\quad \frac{d_2(y_n, TPQRS\beta)d_3(S\beta, Z_n)}{\max\{d_2(y_n, TPQRS\beta); d_2(y_n, y_{n+1}); d_3(z_n, STPQS\beta)\}} \end{aligned}$$

Letting n tending to infinity and since $PQRS\beta = \alpha$ we get

$$d_2(TPQRS\beta, \beta) \leq \frac{cd_2(\beta, TPRS\beta)d_3(S\beta, \delta)}{\max\{d_2(y_n, TPQRS\beta), d_3(\gamma, ST\alpha)\}}.$$

We distinguish here two cases :

$$\text{If } \max\{d_2(\beta, TPQRS\beta), d_3(\gamma, ST\alpha)\} = d_2(\beta, TPQRS\beta)$$

we have

$$\begin{aligned} d_2(TPQRS\beta, \beta) &= d_2(T\alpha, \beta) \leq \frac{c \max\{d_2(\beta, TPRS\beta)d_3(S\beta, \delta)\}}{\max\{d_2(y_n, TPQRS\beta), d_3(\gamma, ST\alpha)\}} \\ &= cd_3(S\beta, \gamma). \end{aligned}$$

$$\text{If } \max\{d_2(\beta, TPQRS\beta), d_3(\gamma, ST\alpha)\} = d_3(\gamma, ST\alpha)$$

and $d_2(\beta, TPQRS\beta) \neq 0$ we have

$$\begin{aligned} d_2(TPQRS\beta, \beta) &= d_2(T\alpha, \beta) \leq \frac{c \max\{d_2(\beta, TPRS\beta)d_3(S\beta, \gamma)\}}{\{d_3(\gamma, ST\alpha)\}}, \\ &\leq \frac{cd_2(\beta, TPRS\beta)d_3(S\beta, \gamma)}{d_2(\beta, TPQRS\beta)} = cd_3(S\beta, \gamma). \end{aligned} \quad T$$

hus, we always have :

$$(11) \quad d_2(TPQRS\beta, \beta) = d_2(T\alpha, \beta) \leq cd_3(S\beta, \gamma).$$

In the same way, taking $z = z_n$ and $u = R\gamma$ using (3) we obtain :

$$\begin{aligned} d_3(STPQR\gamma, z_{n+1}) &= d_3(STPQR\gamma, STPQRz_n) \leq \frac{cA_3(z_n, R\gamma)}{B_3(z_n, R\gamma)} \\ &= \frac{c \max\{d_3(z_n, z_{n+1})d_2(TPQR\gamma, y_{n+1}); d_3(z_n, z_{n+1})d_1(PQR\gamma, x_n); \}}{\max\{d_3(z_n, STPQR\gamma); d_3(z_n, z_{n+1}); d_4(u_n, RSTPQR\gamma)\}} \\ &\quad \frac{d_3(z, z_{n+1})d_5(QR\gamma, v_n); d_3(z_n, z_{n+1})d_4(R\gamma, RSTPQR\gamma)}{; d_3(z_n, STPQR\gamma)d_4(R\gamma, u_n)} \\ &\quad \max\{d_3(z_n, STPQR\gamma); d_3(z_n, z_{n+1}); d_4(u_n, RSTPQR\gamma)\} \end{aligned}$$

Letting n tending to infinity and since $TPQR\gamma = \beta$ we get

$$\begin{aligned} d_3(STPQR\gamma, \gamma) &= d_3(S\beta, \gamma) \leq \frac{cd_3(\gamma, STPR\gamma)d_4(R\gamma, \gamma)}{\max\{d_3(\gamma, STPQR\gamma), d_4(\delta, RS\beta)\}} \\ &\leq cd_4(R\gamma, \gamma). \end{aligned}$$

Thus,

$$(12) \quad d_3(STPQR\gamma, \gamma) = d_3(S\beta, \gamma) \leq cd_4(R\gamma, \gamma),$$

In the same way, using the inequalities (4), (5), (9) and (10), it can be shown that

$$(13) \quad d_4(RSTPQ\delta, \delta) = d_4(R\gamma, \delta) \leq cd_5(Q\delta, \mu),$$

and

$$(14) \quad d_5(QSTP\mu, \mu) = d_5(Q\delta, \mu) \leq cd_1(P\mu, \alpha),$$

and

$$(15) \quad d_1(PQRST\alpha, \alpha) = d_1(P\mu, \alpha) \leq cd_2(T\alpha, \beta),$$

Using (11), (12), (13), (14) and (15), we obtain :

$$\begin{aligned} d_2(TPQRS\beta, \beta) &= d_2(T\alpha, \beta) \leq cd_3(S\beta, \gamma) \leq c^2 d_4(R\gamma, \gamma), \\ &\leq c^3 d_5(Q\delta, \mu) \leq c^4 d_1(P\mu, \alpha) \leq c^5 d_2(T\alpha, \beta) \\ &= c^5 d_2(TPQRS\beta, \beta), \end{aligned}$$

from which it follows

$$TPQRS\beta = \beta; T\alpha = \beta; S\beta = \gamma; R\gamma = \delta; Q\delta = \mu; P\mu = \alpha.$$

$$0 \leq c < 1.$$

Since It is also obvious that

$$STPQR\gamma = \gamma, RSTPQ\delta = \delta, QRSTP\mu = \mu, PQRST\alpha = \alpha.$$

Therefore, $\alpha, \beta, \gamma, \delta,$ and μ are fixed points of $PQRST, TPQRS, STPQR, RSTPQ$ and $QRSTP$.

Now let we show their unicity. Let assume now that α' is another fixed point of $PQRST$ different from α .

Using the inequality (1) for $y = T\alpha$ and $x = \alpha'$ we get

$$\begin{aligned} d_1(\alpha, \alpha') &= d_1(PQRST\alpha, PQRST\alpha') \leq \\ &c \max \left\{ d_1(\alpha', \alpha') d_5(QRST\alpha, QRST\alpha'); \right. \\ &\left. d_1(\alpha', \alpha') d_4(RST\alpha, RST\alpha'); \right\} \\ &\leq \frac{\left\{ d_1(\alpha', \alpha') d_5(QRST\alpha, QRST\alpha'); \right.}{\max \{ d_1(\alpha', \alpha); d_1(\alpha', \alpha'); d_2(T\alpha', T\alpha) \}}, \end{aligned}$$

$$\begin{aligned}
& \frac{d_1(\alpha', \alpha') d_3(ST\alpha, ST\alpha'); d_1(\alpha', \alpha') d_2(T\alpha, TPQRST\alpha);}{d_1(\alpha', \alpha') d_2(T\alpha, T\alpha')} \\
& \frac{d_1(\alpha', \alpha') d_2(T\alpha, T\alpha')}{\max\{d_1(\alpha', \alpha); d_1(\alpha', \alpha'); d_2(T\alpha', T\alpha)\}} \\
& \leq \frac{cd_1(\alpha', \alpha) d_2(T\alpha', T\alpha)}{\max\{d_1(\alpha, \alpha'); d_2(T\alpha', T\alpha)\}}
\end{aligned}$$

If $\max\{d_1(\alpha, \alpha'); d_2(T\alpha', T\alpha)\} = d_2(T\alpha', T\alpha)$ then we get $d_1(\alpha, \alpha') \leq cd_1(\alpha', \alpha)$ from which it follows $\alpha = \alpha'$.

If $\max\{d_1(\alpha, \alpha'); d_2(T\alpha', T\alpha)\} = d_1(\alpha, \alpha')$, then
 (16) $d_1(\alpha, \alpha') \leq cd_2(T\alpha, T\alpha')$

In the same way, taking $z = ST\alpha$ and $y = T\alpha'$ in (2) we obtain :

$$\begin{aligned}
& d_2(T\alpha, T\alpha') = d_2(TPQRST\alpha, TPQRST\alpha') \leq \\
& \leq \frac{c \max\left\{d_2(T\alpha', TPQRST\alpha') d_1(\alpha, \alpha'); d_2(T\alpha', T\alpha')\right\}}{\max\left\{d_5(QRST\alpha, QRST\alpha'); d_2(T\alpha', T\alpha); d_2(T\alpha', T\alpha'); d_3(ST\alpha', ST\alpha)\right\}} \\
& \frac{d_2(T\alpha', T\alpha') d_4(RST\alpha, RST\alpha'); d_2(T\alpha', T\alpha') d_3(ST\alpha, STPQRST\alpha);}{d_2(T\alpha', T\alpha') d_3(ST\alpha, ST\alpha')} \\
& \frac{cd_2(T\alpha', T\alpha) d_3(ST\alpha, ST\alpha')}{\max\{d_2(T\alpha', T\alpha); d_2(T\alpha', T\alpha'); d_3(ST\alpha', ST\alpha)\}} \\
& \leq cd_3(ST\alpha', ST\alpha).
\end{aligned}$$

Thus,

$$(17) \quad d_2(T\alpha, T\alpha') \leq cd_3(ST\alpha', ST\alpha),$$

By (3), for $u = RST\alpha$ and $z = ST\alpha'$ we get

$$(18) \quad d_3(ST\alpha, ST\alpha') \leq cd_4(RST\alpha', RST\alpha),$$

By (4), for $v = QRST\alpha$ and $u = RST\alpha'$ we get

$$(19) \quad d_4(RST\alpha, RST\alpha') \leq cd_5(QRST\alpha', QRST\alpha),$$

By (5), for $x = PQRST\alpha$ and $v = QST\alpha'$ we get

$$(20) \quad d_5(QRST\alpha, QRST\alpha') \leq cd_1(PQRST\alpha', PQRST\alpha) = cd_1(\alpha', \alpha),$$

By (16), (17), (18), (19) and (20) it follows

$$\begin{aligned} d_1(\alpha, \alpha') &\leq cd_2(T\alpha, T\alpha') \leq c^2 d_3(ST\alpha', ST\alpha), \\ &\leq c^3 d_4(RST\alpha', RST\alpha), \\ &\leq c^4 d_5(QRST\alpha', QRST\alpha) \leq c^5 d_1(\alpha', \alpha), \end{aligned}$$

which is impossible, since $0 \leq c < 1$.

Thus, α is a unique fixed point of $PQRST$.

In the same way it can be proved that β is a unique fixed point of $TPQRS$, γ is a unique fixed point of $STPQR$, δ is a unique fixed point of $RSTPQ$ and μ is a unique fixed point of $QRSTP$.

This completes the proof of the theorem.

Corollary 2.2. If we consider this theorem the metric space (V, d_5) the same with the metric space (X, d_1) , (that is $V = X$, $d_5 = d_1$), and the mapping P as the identity mapping of X ($Px = x \forall x \in X$), then we obtain a fixed point theorem on four metric spaces which is a generalization of the theorem.

Proof : We apply the inequalities (1), (2), (3), (4) and (5) for $d_1 = d$, $V = X$, $v = x$ and $P = I$ as the identity mapping.

The inequality (1) takes the form :

$$(1) \quad d_1(QRSy, QRSTx) \leq \frac{cA_1(x, y)}{B_1(x, y)},$$

where

$$\begin{aligned}
A_1(x, y) = & \max \{d_1(x, QRSTx) d_5(QRSy, QRSTx); d_1(x, QRSTx) \\
& d_4(RSy, RSTx); d_1(x, QRSTx) d_3(Sy, STx); \\
& d_1(x, QRSTx) d_2(y, TQRSy); d_1(x, QRSy) d_2(y, Tx); \} \geq \\
& \max \{d_1(x, QRSTx) d_4(RSy, RSTx); d_1(x, QRSTx) d_3(Sy, STx); \\
& d_1(x, QRSTx) d_2(y, TQRSy); d_1(x, QRSy) d_2(y, Tx); \} \\
& = F_1(x, y).
\end{aligned}$$

and

$$\begin{aligned}
B_1(x, y) = & \max \{d_1(x, QRSy); d_1(x, QRSTx); d_2(Tx, TQRSy)\} \\
& = G_1(x, y).
\end{aligned}$$

The inequality (2) takes the form :

$$(2) \quad d_2(TQRz, TQRSy) \leq \frac{cA_2(y, z)}{B_2(y, z)},$$

where

$$\begin{aligned}
A_2(y, z) = & \max \{d_2(y, TQRSy) d_1(QRz, QRSy); d_2(y, TQRSy) \\
& d_5(QRz, QRSy); d_2(y, TQRSy) d_4(Rz, RSy); \\
& d_2(y, TQRSy) d_3(z, STQRz); d_2(y, TQRz) d_3(z, Sy); \} \geq \\
& \max \{d_2(y, TQRSy) d_1(QRz, QRSy); d_2(y, TQRSy) d_4(Rs, RSy); \\
& d_2(y, TQRSy) d_3(z, STQRz); d_2(y, TQRz) d_3(z, Sy); \} \\
& = F_2(y, z).
\end{aligned}$$

and

$$\begin{aligned}
B_2(y, z) = & \max \{d_2(y, TQRz); d_2(y, TQRSy); d_3(Sy, STQRz)\} \\
& = G_2(y, z).
\end{aligned}$$

The inequality (3) takes the form :

$$(3) \quad d_3(STQu, STQRz) \leq \frac{cA_3(z, u)}{B_3(z, u)},$$

where

$$\begin{aligned}
A_3(z, u) &= \max \{ d_3(z, STQRz) d_2(TQu, TQRz); d_3(z, STQRz) \\
&\quad d_1(Qu, QRz); d_3(z, STQRz) d_5(Qu, QRz); \\
&\quad d_3(z, STQRz) d_4(u, RSTQu); d_3(z, STQu) d_4(u, Rz); \} \geq \\
&\geq \max \{ d_3(z, STQRz) d_2(TQu, TQRz); d_3(z, STQRz) d_1(Qu, QRz); \\
&\quad d_3(z, STQRz) d_4(u, RSTQu); d_3(z, STQu) d_4(u, Rz); \} \\
&= F_3(z, u).
\end{aligned}$$

and

$$\begin{aligned}
B_3(z, u) &= \max \{ d_3(z, STQu); d_3(z, STQRz); d_4(Rz, RSTQu) \} \\
&= G_3(z, u).
\end{aligned}$$

The inequality (4) takes the form :

$$(4) \quad d_4(RSTx, RSTQu) \leq \frac{cA_4(u, x)}{B_4(u, x)},$$

where

and

$$\begin{aligned}
B_4(u, x) &= \max \{ d_4(u, RSTx); d_4(u, RSTQu); d_5(Qu, QRSTx) \} \\
&= G_4(u, x).
\end{aligned}$$

The inequality (5) takes the form :

$$(5) \quad d_5(QRSTx, QRSTx) \leq \frac{cA_5(x, x)}{B_5(x, x)},$$

which holds true for all $x \in X$ since the left side is zero. Thus, in case of four mappings (*special case* $P = I$), the satisfying conditions (1), (2), (3), (4) and (5) of theorem 2.1 is reduced in satisfying of the following conditions :

$$(1) \quad d_1(QRSy, QRSTx) \leq \frac{cA_1(x, y)}{B_1(x, y)},$$

$$(2) \quad d_2(TQRz, TQRSy) \leq \frac{cA_2(y, z)}{B_2(y, z)},$$

$$(3) \quad d_3(STQu, STQRz) \leq \frac{cA_3(z, u)}{B_3(z, u)},$$

$$(4) \quad d_4(RSTx, RSTQu) \leq \frac{cA_4(u, x)}{B_4(u, x)},$$

and from the above it follows whenever the conditions

$$d_1(QRSy, QRSTx) \leq \frac{cF_1(x, y)}{G_1(x, y)},$$

$$d_2(TQRz, TQRSy) \leq \frac{cF_2(y, z)}{G_2(y, z)},$$

$$d_3(STQu, STQRz) \leq \frac{cF_3(z, u)}{G_3(z, u)},$$

$$d_4(RSTx, RSTQu) \leq \frac{cF_4(u, x)}{G_4(u, x)},$$

of theorem 1.2 of LuljetaKikina and KristaqKikina¹ are satisfied, the conditions (1'), (2'), (3') and (4') of the special case of Theorem 2.1 are satisfied. Therefore, we obtain a generalization of Theorem 1.2 proved by LuljetaKikina and Kristaq Kikina¹.

References

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