

Complex Finsler Spaces with Generalized $(\gamma, |\beta|)$ - Metric

Sweta Kumari and P. N. Pandey

Department of Mathematics
University of Allahabad, Allahabad - 211002, India

Email: jhasweta735@gmail.com, pnpiaps@gmail.com

Anjali Goswami

College of Science and Theoretical Studies
Saudi Electronic University

Riyadh, KSA

Email: a.goswami@seu.edu.sa

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Abstract: In the present paper, the class of complex Finsler spaces with generalized $(\gamma, |\beta|)$ -metric where γ is an m^{th} - root metric and β is a differential $(0, 1)$ - form, is introduced. The fundamental metric tensor, its inverse, angular metric tensor, its inverse and Euler-Lagrange equations of such spaces are determined.

Keywords: Complex Finsler space, generalized $(\gamma, |\beta|)$ -metric, fundamental metric tensor, Euler - Lagrange equations.

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1. Introduction

N. Aldea and G. Munteanu¹ introduced the notion of complex Finsler spaces with (α, β) - metric, where α is a purely Hermitian form and β is a differential $(1, 0)$ - form, in 2009. Further computational and geometrical details of such spaces are given by Aldea and Munteanu^{2, 3} and Sweta Kumari and P. N. Pandey⁴. G. Munteanu^{5, 6} introduced and developed the class of complex Lagrange spaces. Several geometers including R. Miron⁷

and B. Nicolaescu^{8, 9} studied Lagrange spaces with (α, β) -metric. In 2013, S. K. Shukla and P. N. Pandey¹⁰ developed the theory of Lagrange spaces with (γ, β) -metric, where γ is a cubic-root metric and β is a 1-form. Based on this, recently Sweta Kumari and P. N. Pandey¹¹ introduced the notion of $(\gamma, |\beta|)$ - metric in complex Finsler spaces. The aim of the present paper is to generalize the concept of $(\gamma, |\beta|)$ - metric on complex Finsler spaces by taking γ as an m^{th} -root metric and β as a differential $(1, 0)$ -form. Such metric is called generalized $(\gamma, |\beta|)$ - metric and complex Finsler spaces endowed with such metrics are called complex Finsler spaces with generalized $(\gamma, |\beta|)$ -metric.

2. Preliminaries

Let M be a complex manifold of dimension n and $(z^k)_{k=1,2,\dots,n}$ complex coordinates in a local chart. Its complexified tangent bundle $T_{\mathbb{C}} M$ splits into holomorphic tangent bundle $T' M$ and antiholomorphic tangent bundle $T'' M$, i.e. $T_{\mathbb{C}} M = T' M \oplus T'' M$. The holomorphic tangent bundle $T' M$ is itself a complex manifold with local coordinates (z^k, η^k) in a chart, which changes by the following rules

$$(2.1) \quad z'^k = z'^k(z), \quad \eta'^k = \frac{\partial z'^k}{\partial z^j} \eta^j.$$

Further, $T_{\mathbb{C}}(T' M)$ decomposes into holomorphic and antiholomorphic tangent bundles $T'(T' M)$ and $T''(T' M)$ respectively. A natural local frame $\{\partial/\partial z^k, \partial/\partial \eta^k\}$ for $T'(T' M)$ changes according to the rules obtained from Jacobi matrix of (2.1). Since the changing rule of $\partial/\partial z^k$ contains the second order partial derivatives, the concept of complex non-linear connection (c. n. c.) was introduced.

Let $V(T' M) \subset T'(T' M)$ be the vertical bundle spanned by $\{\partial/\partial \eta^k\}$. The complex non-linear connection (c. n. c.) determines a supplementary complex sub-bundle to $V(T' M)$ in $T'(T' M)$, i.e.

$$T'(T'M) = H(T'M) \oplus V(T'M),$$

called the horizontal bundle. It determines an adapted frame $\left\{ \frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j} \right\}$, where $N_k^j(z, \eta)$ are the coefficients of the complex nonlinear connection^{12, 13}.

A complex Finsler metric F on complex manifold M is a continuous function $F: T'M \rightarrow \mathbb{R}^+$ satisfying following conditions¹²

- (i) $L = F^2$ is smooth on $T'M \setminus \{0\}$;
- (ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;
- (iii) $F(z, \lambda\eta) = |\lambda| F(z, \eta)$, for $\forall \lambda \in \mathbb{C}$;
- (iv) the Hermitian matrix $(g_{i\bar{j}}(z, \eta))$ where $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$, is positive definite on $T'M \setminus \{0\}$.

Let us write $L = F^2$. Then, the pair (M, F) is called a complex Finsler space. The variational problem for a complex Finsler space with Lagrangian, L with its fundamental metric tensor lead to a complex geodesic curve. Consider a smooth curve $c: t \in [0, 1] \rightarrow z^i(t) \in M$ and $(z^k(t), \eta^k(t))$ be its extension on $T'M$. The Euler-Lagrange equations for the complex Finsler space with Lagrangian, L are

$$(2.2) \quad E_i(L) \equiv \frac{\partial L}{\partial z^i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \eta^i} \right) = 0,$$

where L is taken along the curve in $T'M$.

A complex Finsler space is said to be with generalized $(\gamma, |\beta|)$ - metric if it is a function of $\gamma(z, \eta)$ and $|\beta|(z, \eta)$ i.e.

$$(2.3) \quad F(z, \eta) = \gamma(z, \eta) + |\beta|(z, \eta)$$

with

$$(2.4) \quad \gamma^m = a_{i_1 \bar{i}_2 \bar{i}_3 \dots \bar{i}_m} \eta^{i_1} \bar{\eta}^{i_2} \bar{\eta}^{i_3} \dots \bar{\eta}^{i_m}$$

and

$$(2.5) \quad |\beta(z, \eta)|^2 = \beta(z, \eta) \bar{\beta}(z, \eta), \quad \beta(z, \eta) = b_{\bar{i}}(z) \eta^{\bar{i}}.$$

$F^n = (M, F(z, \eta))$ determined by (2.3) is called a complex Finsler space with generalized $(\gamma, |\beta|)$ -metric.

3. Fundamental Metric Tensor and Angular Metric Tensor

From the homogeneity property, we have

$$(3.1) \quad \frac{\partial \gamma}{\partial \eta^i} \eta^i = \gamma \quad \text{and} \quad \frac{\partial |\beta|}{\partial \eta^i} \eta^i = |\beta|.$$

Differentiating (2.4) and (2.5) partially with respect to η^i and $\bar{\eta}^j$, we obtain

$$(3.2) \quad \begin{cases} \frac{\partial \gamma}{\partial \eta^i} = \frac{1}{m \gamma^{m-1}} a_i, \quad \frac{\partial |\beta|}{\partial \eta^i} = \frac{\bar{\beta} b_i}{2|\beta|}, \quad \frac{\partial \gamma}{\partial \bar{\eta}^j} = \frac{m-1}{m \gamma^{m-1}} a_{\bar{j}}, \quad \frac{\partial |\beta|}{\partial \bar{\eta}^j} = \frac{\beta b_{\bar{j}}}{2|\beta|} \\ \frac{\partial^2 \gamma}{\partial \eta^i \partial \bar{\eta}^j} = \frac{m-1}{m \gamma^{m-1}} \left\{ a_{i\bar{j}} - \frac{m-1}{m \gamma^m} a_i a_{\bar{j}} \right\}, \quad \frac{\partial^2 |\beta|}{\partial \eta^i \partial \bar{\eta}^j} = \frac{b_i b_{\bar{j}}}{2|\beta|}. \end{cases}$$

where $a_i = a_{i i_2 i_3 i_4 \dots i_m} \bar{\eta}^{i_2} \bar{\eta}^{i_3} \dots \bar{\eta}^{i_m}$ and $a_{\bar{i}} = a_{\bar{i} \bar{j} \bar{i}_3 \bar{i}_4 \dots \bar{i}_m} \bar{\eta}^{i_3} \bar{\eta}^{i_4} \dots \bar{\eta}^{i_m}$.

$$(3.3) \quad L(z, \eta) = F^2 = (\gamma + |\beta|)^2.$$

Using (3.3), we have

$$(3.4) \quad L_\gamma = 2F = L_{|\beta|}, \quad L_{\gamma\gamma} = 2 = L_{|\beta||\beta|} = L_{\gamma|\beta|}.$$

Since, $\eta_i = \frac{\partial L}{\partial \eta^i} = L_\gamma \frac{\partial \gamma}{\partial \eta^i} + L_{|\beta|} \frac{\partial |\beta|}{\partial \eta^i}$, using (3.2) and (3.4), we get

$$(3.5) \quad \left\{ \begin{array}{l} (a) \quad \eta_i = \frac{2F}{m \gamma^{m-1}} a_i + \frac{F \bar{\beta}}{|\beta|} b_i, \quad (b) \quad \bar{\eta}_j = \frac{2F(m-1)}{m \gamma^{m-1}} a_{\bar{j}} + \frac{F \beta}{|\beta|} b_{\bar{j}}. \end{array} \right.$$

Assuming $\rho_0 = \frac{L_\gamma}{m\gamma^{m-1}} = \frac{2F}{m\gamma^{m-1}}$ and $\mu_0 = \frac{L_{|\beta|}}{2|\beta|} = \frac{F}{|\beta|}$, (3. 5) gives

$$(3. 6) \quad \left\{ \begin{array}{ll} (a) & \eta_i = \rho_0 a_i + \mu_0 b_i \bar{\beta}, \quad (b) \quad \bar{\eta}_j = (m-1)\rho_0 a_{\bar{j}} + \mu_0 \beta b_{\bar{j}}. \end{array} \right.$$

Fundamental Metric Tensor

The fundamental metric tensor $g_{i\bar{j}}$ of the complex Finsler space (M, F) with generalized $(\gamma, |\beta|)$ - metric is given by

$$(3.7) \quad \left\{ \begin{array}{l} g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j} \\ = L_{\gamma\gamma} \frac{\partial \gamma}{\partial \eta^i} \frac{\partial \gamma}{\partial \bar{\eta}^j} + L_{\gamma|\beta|} \left(\frac{\partial \gamma}{\partial \eta^i} \frac{\partial |\beta|}{\partial \bar{\eta}^j} + \frac{\partial |\beta|}{\partial \eta^i} \frac{\partial \gamma}{\partial \bar{\eta}^j} \right) \\ + L_{|\beta||\beta|} \frac{\partial |\beta|}{\partial \eta^i} \frac{\partial |\beta|}{\partial \bar{\eta}^j} + L_{\gamma} \frac{\partial^2 \gamma}{\partial \eta^i \partial \bar{\eta}^j} + L_{|\beta|} \frac{\partial^2 |\beta|}{\partial \eta^i \partial \bar{\eta}^j}. \end{array} \right.$$

where $L_\gamma = \frac{\partial L}{\partial \gamma}$, $L_{|\beta|} = \frac{\partial L}{\partial |\beta|}$, $L_{\gamma\gamma} = \frac{\partial^2 L}{\partial \gamma^2}$, $L_{|\beta||\beta|} = \frac{\partial^2 L}{\partial |\beta|^2}$ and $L_{\gamma|\beta|} = \frac{\partial^2 L}{\partial \gamma \partial |\beta|}$.

Using (3. 2) and (3. 4) in (3. 7), we have

$$(3. 8) \quad \left\{ \begin{array}{l} g_{i\bar{j}} = \frac{2F}{m\gamma^{m-1}}(m-1)a_{i\bar{j}} - \frac{2(m-1)}{m^2\gamma^{2m-1}}\{F(m-1)-\gamma\}a_i a_{\bar{j}} \\ + \frac{1}{2}\left(1 + \frac{F}{|\beta|}\right)b_i b_{\bar{j}} - \frac{1}{m\gamma^{m-1}|\beta|}(\beta a_i b_{\bar{j}} + (m-1)\bar{\beta} b_i a_{\bar{j}}), \end{array} \right.$$

which may be written as

$$(3. 9) \quad g_{i\bar{j}} = \rho_0 a_{i\bar{j}} + \rho_{-2} a_i a_{\bar{j}} + \mu_{-2} (\beta a_i b_{\bar{j}} + (m-1)\bar{\beta} b_i a_{\bar{j}}) + \mu_0 b_i b_{\bar{j}}$$

where

$$(3.10) \quad \begin{cases} \rho_0 = \frac{2F}{m\gamma^{m-1}}(m-1), & \rho_{-2} = -\frac{2(m-1)}{m^2\gamma^{2m-1}}\{F(m-1)-\gamma\}, \\ \mu_{-2} = -\frac{1}{m\gamma^{m-1}|\beta|}, & \mu_0 = \frac{1}{2}\left(1 + \frac{F}{|\beta|}\right). \end{cases}$$

Using (3.5) (a) and (3.5) (b), we have

$$(3.11) \quad \eta_i \bar{\eta}_j = (m-1)\rho_0^2 a_i a_{\bar{j}} + \rho_0 \mu_0 \beta a_i b_{\bar{j}} + (m-1)b_i \bar{\beta} a_{\bar{j}} \mu_0 \rho_0 \\ + \mu_0^2 |\beta|^2 b_i b_{\bar{j}}.$$

A simple calculation shows that

$$(3.12) \quad (\beta a_i b_{\bar{j}} + (m-1)\bar{\beta} b_i a_{\bar{j}}) = \frac{1}{\mu_0 \rho_0} \eta_i \bar{\eta}_j - \frac{\mu_0}{\rho_0} |\beta|^2 b_i b_{\bar{j}} - \frac{(m-1)\rho_0}{\mu_0} a_i a_{\bar{j}}.$$

Substituting (3.12) in (3.9) we obtain a new expression for $g_{i\bar{j}}$ as

$$(3.13) \quad g_{i\bar{j}} = \rho_0 a_i a_{\bar{j}} + \rho'_{-2} a_i a_{\bar{j}} + \mu_0'' b_i b_{\bar{j}} + \mu_{-2}'' \eta_i \bar{\eta}_j,$$

where

$$(3.14) \quad \left\{ \rho'_{-2} = -\frac{2(m-1)^2 F}{m^2 \gamma^{2m-1}}, \mu_0'' = \frac{F}{2|\beta|}, \mu_{-2}'' = \frac{1}{2L} \right\}.$$

Theorem 3.1: *The expression for the fundamental metric tensor of the complex Finsler space (M, F) with generalized $(\gamma, |\beta|)$ -metric is given by (3.13).*

Our next aim is to find the formulas for the inverse as well as for the determinant of the fundamental metric tensor $g_{i\bar{j}}$. For this purpose we use the proposition given below¹³

Proposition 3.1: *Suppose*

1. $(Q_{i\bar{j}})$ is a non-singular $n \times n$ complex matrix with inverse $(Q^{\bar{j}i})$,
2. C_i and $C_{\bar{i}} = \bar{C}_i, i=1, 2, 3, \dots, n$, are complex numbers.
3. $C^i = Q^{\bar{j}i} C_{\bar{j}}$ and its conjugates, $C^2 = C^i C_i = \bar{C}^i C_{\bar{i}}; H_{i\bar{j}} = Q_{i\bar{j}} \pm C_i C_{\bar{j}}$.

Then,

$$(i) \det (H_{i\bar{j}}) = (1 \pm C^2) \det (Q_{i\bar{j}}),$$

(ii) whenever $1 \pm C^2 \neq 0$, the matrix $(H_{i\bar{j}})$ is invertible and in this case its inverse is

$$H^{\bar{j}i} = Q^{\bar{j}i} \mp \frac{1}{1 \pm C^2} C^i \bar{C}^j.$$

From (3.13) we may write

$$(3.15) \quad g_{i\bar{j}} = \frac{2F}{m\gamma^{m-1}}(m-1) \left\{ a_{i\bar{j}} - \frac{(m-1)}{m\gamma^m} a_i a_{\bar{j}} + \frac{m\gamma^{m-1}}{4LF(m-1)} \eta_i \eta_{\bar{j}} + \frac{m\gamma^{m-1}}{4|\beta|(m-1)} b_i b_{\bar{j}} \right\}.$$

Assuming $Q_{i\bar{j}} = a_{i\bar{j}}$ and $C_i = \sqrt{\frac{m-1}{m\gamma^m}} a_i$, and applying Proposition 3.1, we find

$$Q^{\bar{j}i} = a^{\bar{j}i}, C^i = \sqrt{\frac{m-1}{m\gamma^m}} \eta^i, C^2 = \frac{m-1}{m},$$

where $(a^{\bar{j}i})$ is the Hermitian inverse of $(a_{i\bar{j}})$.

Since $1 \pm C^2 \neq 0$, the matrix $H_{i\bar{j}} = a_{i\bar{j}} - \frac{m-1}{m\gamma^m} a_i a_{\bar{j}}$ is invertible with the inverse

$$H^{\bar{j}i} = a^{\bar{j}i} + \frac{m-1}{\gamma^m} \eta^i \bar{\eta}^j \quad \text{and} \quad \det(H_{i\bar{j}}) = \frac{1}{m} \det(a_{i\bar{j}}).$$

Taking $Q_{i\bar{j}} = a_{i\bar{j}} - \frac{m-1}{m\gamma^m} a_i a_{\bar{j}}$ and $C_i = \sqrt{\frac{m\gamma^{m-1}}{|\beta|(m-1)}} \frac{b_i}{2}$ and applying Proposition 3.1, we obtain

$$Q^{\bar{j}i} = a^{\bar{j}i} + \frac{m-1}{\gamma^m} \eta^i \bar{\eta}^j \quad \text{and} \quad C^i = \sqrt{\frac{m\gamma^{m-1}}{|\beta|(m-1)}} \frac{1}{2} \left(b^i + \frac{m-1}{\gamma^m} \eta^i \bar{\beta} \right),$$

where $b^i = a^{\bar{j}i} b_{\bar{j}}$ and $\|b\|^2 = a^{\bar{j}i} b_i b_{\bar{j}}$.

Therefore $C^2 = C^i C_i = \frac{m\gamma^{m-1}}{4|\beta|(m-1)} \left(\|b\|^2 + \frac{|\beta|^2}{\gamma^m} (m-1) \right)$, which gives

$$1 + C^2 = \frac{\sigma}{4|\beta|(m-1)\gamma},$$

where $\sigma = 4|\beta|(m-1)\gamma + m(\|b\|^2 \gamma^m + |\beta|^2 (m-1))$.

The inverse of $H_{i\bar{j}} = a_{i\bar{j}} - \frac{(m-1)}{m\gamma^m} a_i a_{\bar{j}} + \frac{m\gamma^{m-1}}{4|\beta|(m-1)} b_i b_{\bar{j}}$ exists and is given by

$$\begin{aligned} H^{\bar{j}i} &= a^{\bar{j}i} + \frac{m-1}{\gamma^m} \eta^i \bar{\eta}^j - \frac{1}{1+C^2} C^i C^{\bar{j}} \\ &= a^{\bar{j}i} + \frac{m-1}{\gamma^m} \eta^i \bar{\eta}^j - \frac{m\gamma^m}{\sigma} \left(b^i + \frac{m-1}{\gamma^m} \eta^i \bar{\beta} \right) \left(b^{\bar{j}} + \frac{m-1}{\gamma^m} \bar{\eta}^j \bar{\beta} \right). \end{aligned}$$

Also $\det(H_{i\bar{j}}) = (1+C^2) \det(Q_{i\bar{j}}) = \frac{\sigma}{4|\beta|m(m-1)\gamma} \det a_{i\bar{j}}$.

We set $Q_{i\bar{j}} = a_{i\bar{j}} - \frac{(m-1)}{m\gamma^m} a_i a_{\bar{j}} + \frac{m\gamma^{m-1}}{4|\beta|(m-1)} b_i b_{\bar{j}}$ and $C_i = \sqrt{\frac{m\gamma^{m-1}}{LF(m-1)}} \frac{1}{2} \eta_i$.

Then, in view of Proposition 3.1, we have

$$Q^{\bar{j}i} = a^{\bar{j}i} + \frac{m-1}{\gamma^m} \eta^i \bar{\eta}^j - \frac{m\gamma^m}{\sigma} \left(b^i + \frac{m-1}{\gamma^m} \eta^i \bar{\beta} \right) \left(b^{\bar{j}} + \frac{m-1}{\gamma^m} \bar{\eta}^j \bar{\beta} \right).$$

Also

$$(3.16) \quad \begin{cases} a^{\bar{j}i} \bar{\eta}_j = \frac{2F}{m\gamma^{m-1}} (m-1) \eta^i + \frac{F\beta}{|\beta|} b^i, b^i \eta_i = \frac{2F}{m\gamma^{m-1}} \bar{\beta} + \frac{F\bar{\beta}}{|\beta|} \|b\|^2, \\ b^{\bar{j}} \bar{\eta}_j = \frac{2F}{m\gamma^{m-1}} \beta (m-1) + \frac{F\beta}{|\beta|} \|b\|^2, \eta^i \eta_i = F \left(\frac{2\gamma}{3} + |\beta| \right), \\ \bar{\eta}^j \bar{\eta}_j = F \left(\frac{2\gamma}{3} + |\beta| \right). \end{cases}$$

Therefore

$$C^i = \sqrt{\frac{m\gamma^{m-1}}{LF(m-1)}} \frac{1}{2} \left[a^{\bar{j}i} \bar{\eta}_j + \frac{m-1}{\gamma^m} \eta^i \bar{\eta}^j \bar{\eta}_j - \frac{m}{\sigma} \gamma^m \bar{\eta}_j \left(b^i + \frac{m-1}{\gamma^m} \bar{\beta} \eta^i \right) \left(b^{\bar{j}} + \frac{m-1}{\gamma^m} \beta \bar{\eta}^j \right) \right].$$

Using (3. 16) in above expression of C^i and simplifying, we obtain

$$C^i = \sqrt{\frac{m\gamma^{m-1}}{LF(m-1)}} \frac{1}{2} \left[\left\{ \frac{2F}{m\gamma^{m-1}}(m-1) + \frac{2(m-1)^2 F}{m\gamma^{m-1}} + \frac{F(m-1)|\beta|}{\gamma^m} - \frac{2F|\beta|^2(m-1)^2}{\sigma\gamma^{m-1}} \right\} \eta^i \right. \\ \left. + \left\{ \frac{F|\beta|m(m-1)\|b\|^2}{\sigma} - \frac{2F(m-1)^3|\beta|^2}{\gamma^{m-1}} - \frac{Fm(m-1)^2|\beta|^3}{\gamma^m} \right\} b^i \right].$$

Therefore

$$C^2 = \frac{\eta_i}{4LF(m-1)\sigma|\beta|\gamma} \left[\left\{ \begin{aligned} &8m(m-1)F|\beta|^2\gamma^2 + 2Fm(m-1)\gamma^{m+1}|\beta|\|b\|^2 \\ &+ 2Fm(m-1)^2\gamma|\beta|^3 + 2Fm(m-1)^2|\beta|\|b\|^2\gamma^{m+1} \\ &+ 2m(m-1)^3F\gamma|\beta|^3 + 4Fm(m-1)^2\gamma|\beta|^3 \\ &+ Fm^2(m-1)^2|\beta|^4 - 8Fm(m-1)^4\gamma^2|\beta|^4 \\ &- 2Fm^2(m-1)^3\gamma^{m+1}|\beta|^3\|b\|^2 - 2Fm^2(m-1)^4\gamma|\beta|^5 \\ &- 4Fm^2(m-1)^3\gamma|\beta|^5 - Fm^3(m-1)^3|\beta|^5 \\ &- Fm^3(m-1)^2\gamma^m|\beta|^4\|b\|^2 - 2Fm(m-1)^2\gamma|\beta|^3 \end{aligned} \right\} \eta^i \right. \\ \left. + \left\{ 2Fm(m-1)\gamma^{m+1}|\beta| - 2Fm(m-1)^2\gamma^{m+1}|\beta|\beta \right\} b^i \right]$$

$$\text{Also } C^2 = \frac{\eta_i}{4LF(m-1)\sigma|\beta|\gamma} M,$$

where

$$M = \left[\begin{array}{l} \left\{ \begin{array}{l} 8m(m-1)F|\beta|^2\gamma^2 + 2Fm(m-1)\gamma^{m+1}|\beta||b|^2 \\ + 2Fm(m-1)^2\gamma|\beta|^3 + 2Fm(m-1)^2|\beta||b|^2\gamma^{m+1} \\ + 2m(m-1)^3F\gamma|\beta|^3 + 4Fm(m-1)^2\gamma|\beta|^3 \\ + Fm^2(m-1)^2|\beta|^4 - 8Fm(m-1)^4\gamma^2|\beta|^4 \\ - 2Fm^2(m-1)^3\gamma^{m+1}|\beta|^3|b|^2 - 2Fm^2(m-1)^4\gamma|\beta|^5 \\ - 4Fm^2(m-1)^3\gamma|\beta|^5 - Fm^3(m-1)^3|\beta|^5 \\ - Fm^3(m-1)^2\gamma^m|\beta|^4|b|^2 - 2Fm(m-1)^2\gamma|\beta|^3 \\ + \{2Fm(m-1)\gamma^{m+1}\beta|\beta| - 2Fm(m-1)^2\gamma^{m+1}\beta|\beta|\}b^i \end{array} \right\} \eta^i \end{array} \right]$$

Since $1 \pm C^2 \neq 0$, $H_{i\bar{j}} = a_{i\bar{j}} - \frac{(m-1)}{m\gamma^m} a_i a_{\bar{j}} + \frac{m\gamma^{m-1}}{4|\beta|(m-1)} b_i b_{\bar{j}} + \frac{m\gamma^{m-1}}{4LF(m-1)} \eta_i \bar{\eta}_j$ is invertible with the inverse

$$(3.17) \quad H^{\bar{j}i} = a^{\bar{j}i} + \frac{m-1}{\gamma^m} \eta^i \bar{\eta}^j - \frac{m\gamma^m}{\sigma} \left(b^i + \frac{m-1}{\gamma^m} \eta^i \bar{\beta} \right) \left(b^{\bar{j}} + \frac{m-1}{\gamma^m} \bar{\eta}^j \bar{\beta} \right) - \frac{1}{1+C^2} C^i C^{\bar{j}}.$$

also

$$(3.18) \quad \det(H_{i\bar{j}}) = M \gamma \left(1 + \frac{\alpha^2 \rho'_{-2}}{\rho_0} \right) \det(a_{i\bar{j}}) = \frac{M \gamma \rho'_{-2}}{R \rho_0} \det(a_{i\bar{j}}).$$

Since $g_{i\bar{j}} = \frac{2F(m-1)}{m\gamma^{m-1}} H_{i\bar{j}}$, the inverse of the fundamental metric tensor is given by

$$(3.19) \quad g^{\bar{j}i} = \frac{m\gamma^{m-1}}{2F(m-1)} H^{\bar{j}i},$$

where $H^{\bar{j}i}$ is given by (3.17).

Also, the determinant of the fundamental metric tensor is given by

$$(3.20) \quad \det(g_{i\bar{j}}) = \left(\frac{2F(m-1)}{m\gamma^{m-1}} \right)^n \det(H_{i\bar{j}}),$$

where $\det(H_{i\bar{j}})$ is given by (3.18).

Theorem 3.2: *The expressions for the inverse and determinant of the fundamental metric tensor of the complex Finsler space (M, F) with generalized $(\gamma, |\beta|)$ - metric are given by (3.19) and (3.20) respectively.*

Angular Metric Tensor

The angular metric tensor of the complex Finsler space (M, F) with generalized $(\gamma, |\beta|)$ - metric is given by

$$(3.21) \quad \left\{ \begin{aligned} k_{i\bar{j}} &= \frac{\partial^2 F}{\partial \eta^i \partial \bar{\eta}^j} \\ &= F_{\gamma\gamma} \frac{\partial \gamma}{\partial \eta^i} \frac{\partial \gamma}{\partial \bar{\eta}^j} + F_{\gamma|\beta|} \left(\frac{\partial \gamma}{\partial \eta^i} \frac{\partial |\beta|}{\partial \bar{\eta}^j} + \frac{\partial |\beta|}{\partial \eta^i} \frac{\partial \gamma}{\partial \bar{\eta}^j} \right) + F_{|\beta||\beta|} \frac{\partial |\beta|}{\partial \eta^i} \frac{\partial |\beta|}{\partial \bar{\eta}^j} \\ &\quad + F_{\gamma} \frac{\partial^2 \gamma}{\partial \eta^i \partial \bar{\eta}^j} + F_{|\beta|} \frac{\partial^2 |\beta|}{\partial \eta^i \partial \bar{\eta}^j}, \end{aligned} \right.$$

$$\text{where } F_{\gamma} = \frac{\partial F}{\partial \gamma}, F_{|\beta|} = \frac{\partial F}{\partial |\beta|}, F_{\gamma\gamma} = \frac{\partial^2 F}{\partial \gamma^2}, F_{|\beta||\beta|} = \frac{\partial^2 F}{\partial |\beta|^2}$$

$$\text{and } F_{\gamma|\beta|} = \frac{\partial^2 F}{\partial \gamma \partial |\beta|} = \frac{\partial^2 F}{\partial |\beta| \partial \gamma}.$$

On differentiating F with respect to γ and $|\beta|$, we obtain

$$(3.22) \quad F_{\gamma} = 1 = F_{|\beta|}, \quad F_{\gamma\gamma} = 0 = F_{|\beta||\beta|} = F_{\gamma|\beta|} = F_{|\beta|\gamma}.$$

On substituting (3.2) and (3.22) in (3.21), we obtain

$$K_{i\bar{j}} = \frac{(m-1)}{m\gamma^{m-1}} \left\{ a_{i\bar{j}} - \frac{m-1}{m\gamma^m} a_i a_{\bar{j}} \right\} + \frac{b_i b_{\bar{j}}}{4|\beta|},$$

which may be written as

$$(3.23) \quad K_{i\bar{j}} = \frac{(m-1)}{m\gamma^{m-1}} h_{i\bar{j}} + \frac{b_i b_{\bar{j}}}{4|\beta|},$$

where

$$(3.24) \quad h_{i\bar{j}} = a_{i\bar{j}} - \frac{m-1}{m\gamma^m} a_i a_{\bar{j}}.$$

Theorem 3.3: *Expression for the angular metric tensor of the complex Finsler space (M, F) with generalized $(\gamma, |\beta|)$ -metric is given by (3.23).*

To find out the inverse of the angular metric tensor, we will apply Proposition 3.1.

$$\text{From (3.23), } k_{i\bar{j}} = \frac{(m-1)}{m\gamma^{m-1}} \left\{ a_{i\bar{j}} - \frac{m-1}{m\gamma^m} a_i a_{\bar{j}} + \frac{m\gamma^m}{4(m-1)|\beta|} b_i b_{\bar{j}} \right\}.$$

Assuming $Q_{i\bar{j}} = a_{i\bar{j}}$ and $C_i = \sqrt{\frac{m-1}{m\gamma^m}} a_i$, and applying Proposition 3.1, we find

$$Q^{\bar{j}i} = a^{\bar{j}i}, C^i = \sqrt{\frac{m-1}{m\gamma^m}} \eta^i \text{ and } C^2 = \frac{m-1}{m},$$

Since $1 - C^2 = \frac{1}{m} \neq 0$, the matrix $H_{i\bar{j}} = a_{i\bar{j}} - \frac{m-1}{m\gamma^m} a_i a_{\bar{j}}$ is invertible with the inverse

$$H^{\bar{j}i} = a^{\bar{j}i} + \frac{m-1}{\gamma^m} \eta^i \bar{\eta}^j \text{ and } \det(H_{i\bar{j}}) = \frac{1}{m} \det(a_{i\bar{j}}).$$

Taking $Q_{i\bar{j}} = a_{i\bar{j}} - \frac{m-1}{m\gamma^m} a_i a_{\bar{j}}$ and $C_i = \sqrt{\frac{m\gamma^{m-1}}{(m-1)|\beta|}} \frac{1}{2} b_i$, Proposition 3.1 gives

$$Q^{\bar{j}i} = a^{\bar{j}i} + \frac{m-1}{\gamma^m} \eta^i \bar{\eta}^j \text{ and } C^i = \sqrt{\frac{m\gamma^{m-1}}{(m-1)|\beta|}} \frac{1}{2} \left(b^i + \frac{m-1}{\gamma^m} \bar{\beta} \eta^i \right).$$

Therefore $C^2 = C^i C_i = \frac{m\gamma^{m-1}}{(m-1)|\beta|} \left(\|b\|^2 + \frac{(m-1)}{\gamma^m} |\beta|^2 \right).$

Since $1 + C^2 = \frac{\sigma}{4(m-1)|\beta|\gamma} \neq 0$, the inverse of $H_{i\bar{j}} = a_{i\bar{j}} - \frac{m-1}{m\gamma^m} a_i a_{\bar{j}} + \frac{m\gamma^m}{4(m-1)|\beta|} b_i b_{\bar{j}}$ exists and is given by

$$(3.25) \quad H^{\bar{j}i} = a^{\bar{j}i} + \frac{m-1}{\gamma^m} \eta^i \bar{\eta}^j - \frac{m\gamma^m}{\sigma} \left(b^i + \frac{\bar{\beta}(m-1)}{\gamma^m} \eta^i \right) \left(b^{\bar{j}} + \frac{\beta(m-1)}{\gamma^m} \bar{\eta}^j \right),$$

and

$$(3.26) \quad \det(H_{i\bar{j}}) = \frac{\sigma}{4m(m-1)|\beta|\gamma} \det(a_{i\bar{j}}).$$

As $K_{i\bar{j}} = \frac{(m-1)}{m\gamma^{m-1}} H_{i\bar{j}}$, the inverse of the angular metric tensor is given by

$$(3.27) \quad K^{\bar{j}i} = \frac{m\gamma^{m-1}}{m-1} H^{\bar{j}i},$$

where $H^{\bar{j}i}$ is given by (3.25).

Also, the determinant of the angular metric tensor is given by

$$(3.28) \quad \det(K_{i\bar{j}}) = \frac{\sigma(m-1)^{n-1}}{4m^{n+1}\gamma^{n(m-1)+1}|\beta|} \det(a_{i\bar{j}}).$$

Theorem 3.2: The expressions for the inverse and determinant of the angular metric tensor of the complex Finsler space (M, F) with generalized $(\gamma, |\beta|)$ - metric are given by (3.27) and (3.28) respectively.

4. Euler - Lagrange Equations

Consider

$$(4.1) \quad L = F^2 = (\gamma + |\beta|)^2$$

Differentiating (4.1) with respect to t , we have

$$(4.2) \quad \frac{dL}{dt} = \frac{\partial L}{\partial z^i} \eta^i + \frac{\partial L}{\partial \eta^i} \frac{d\eta^i}{dt} + \frac{\partial L}{\partial \bar{z}^i} \bar{\eta}^i + \frac{\partial L}{\partial \bar{\eta}^i} \frac{d\bar{\eta}^i}{dt}.$$

Since L is homogeneous of degree 1 in η^i ,

$$(4.3) \quad \frac{\partial L}{\partial \eta^i} \eta^i = L.$$

On differentiating (4.3) with respect to t , we have

$$(4.4) \quad \frac{dL}{dt} = \frac{d}{dt} \left(\frac{\partial L}{\partial \eta^i} \right) \eta^i + \frac{\partial L}{\partial \eta^i} \frac{d\eta^i}{dt}.$$

From Euler - Lagrange equations, we obtain

$$(4.5) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \eta^i} \right) = \frac{\partial L}{\partial z^i}.$$

Substituting (4.5) in (4.4), we get

$$(4.6) \quad \frac{dL}{dt} = \frac{\partial L}{\partial z^i} \eta^i + \frac{\partial L}{\partial \eta^i} \frac{d\eta^i}{dt}.$$

On conjugating (4.6), we have

$$(4.7) \quad \frac{dL}{dt} = \frac{\partial L}{\partial \bar{z}^i} \bar{\eta}^i + \frac{\partial L}{\partial \bar{\eta}^i} \frac{d\bar{\eta}^i}{dt}.$$

Adding (4.6) and (4.7) and then using in (4.2), we obtain

$$(4.8) \quad \frac{dL}{dt} = 0.$$

Theorem 4.1: For a complex Finsler space with generalized $(\gamma, |\beta|)$ -metric, $\frac{dL}{dt} = 0 = \frac{dF}{dt}$ along an extremal curve $c(t)$ on $T'M$.

Differentiating (4.1) with respect to z^i and η^i respectively gives

$$\frac{\partial L}{\partial z^i} = L_\gamma \frac{\partial \gamma}{\partial z^i} + L_{|\beta|} \frac{\partial |\beta|}{\partial z^i}.$$

Using (3.4) in above expression, we have

$$(4.9) \quad \frac{\partial L}{\partial z^i} = 2F \left(\frac{\partial \gamma}{\partial z^i} + \frac{\partial |\beta|}{\partial z^i} \right).$$

Similary (4.1) and (3.4) gives

$$(4.10) \quad \frac{\partial L}{\partial \eta^i} = 2F \left(\frac{\partial \gamma}{\partial \eta^i} + \frac{\partial |\beta|}{\partial \eta^i} \right).$$

Further differentiation of (4.10) with respect to t yields

$$(4.11) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \eta^i} \right) = 2 \frac{dF}{dt} \left(\frac{\partial \gamma}{\partial \eta^i} + \frac{\partial |\beta|}{\partial \eta^i} \right) + 2F \left\{ \frac{d}{dt} \left(\frac{\partial \gamma}{\partial \eta^i} \right) + \frac{d}{dt} \left(\frac{\partial |\beta|}{\partial \eta^i} \right) \right\}.$$

In view of theorem 4.1, (4.11) gives

$$(4.12) \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \eta^i} \right) = 2F \left\{ \frac{d}{dt} \left(\frac{\partial \gamma}{\partial \eta^i} \right) + \frac{d}{dt} \left(\frac{\partial |\beta|}{\partial \eta^i} \right) \right\}.$$

In view of (2.2)

$$(4.13) \quad E_i(\gamma) = \frac{1}{m} \gamma^{-(m-1)} E_i(\gamma^m) + \frac{1}{\gamma^2} \gamma^{-(m-1)} \frac{d\gamma^{m-1}}{dt}$$

and

$$(4.14) \quad E_i(|\beta|) = \frac{1}{2|\beta|} E_i(|\beta|^2) + \frac{1}{|\beta|} \frac{\partial |\beta|}{\partial \eta^i} \frac{d|\beta|}{dt}.$$

Using (4.9) and (4.12) in (2.2), we have

$$(4.15) \quad E_i(L) = 2F \left(E_i(\gamma) + E_i(|\beta|) \right).$$

Substituting values of $E_i(\gamma)$ and $E_i(|\beta|)$ in (4.15) and simplifying, we obtain

$$(4.16) \quad \begin{cases} E_i(L) \equiv 2|\beta| E_i(\gamma^m) + m\gamma^{m-1} E_i(|\beta|^2) + 2m|\beta| \frac{d\gamma^{m-1}}{dt} \frac{\partial \gamma}{\partial \eta^i} \\ \quad + 2m\gamma^{m-1} \frac{d|\beta|}{dt} \frac{\partial |\beta|}{\partial \eta^i} = 0. \end{cases}$$

Theorem 4.2: In a complex Finsler space with generalized $(\gamma, |\beta|)$ -metric, Euler - Lagrange equations are given by (4.16).

In case of natural parametrization of the curve $c:t \in [0,1] \rightarrow z^i(t) \in M$, generalized $\gamma\left(z, \frac{dz}{dt}\right)=1$, the above theorem reduces to

Theorem 4. 3: *Euler - Lagrange equations in a complex Finsler space with generalized $(\gamma, |\beta|)$ - metric, in case of natural parametrization are given as*

$$(4.17) \quad E_i(L) \equiv 2|\beta|E_i(\gamma^m) + mE_i(|\beta|^2) + 2m\gamma^{m-1}\frac{d|\beta|}{dt}\frac{\partial|\beta|}{\partial\eta^i} = 0.$$

Assuming $|\beta|$ to be constant along the integral curve of the Euler - Lagrange equations with the natural parametrization, Euler - Lagrange equations are

$$(4.18) \quad E_i(L) \equiv 2|\beta|E_i(\gamma^m) + mE_i(|\beta|^2) = 0.$$

Theorem 4. 4: *Euler - Lagrange equations in a complex Finsler space with generalized $(\gamma, |\beta|)$ - metric, along the integral curve such that $|\beta|$ is constant are given by (4. 18).*

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