

Four Terms Representation of Ramanujan's Continued Fraction

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Abstract : The m-dissection of the power series $P = \sum_{n=0}^{\infty} a_n q^n$ is the representation of P as $P = P_0 + P_1 + \dots + P_{m-1}$, where $P_k = \sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}$. Some continued fractions can be represented in terms of power series and infinite products. In this paper, an attempt is made to obtain four terms representation of Ramanujan's continued fraction.

1. Introduction

In the beginning of 20th century the great Indian Mathematician Srinivasa Ramanujan discovered a series of results involving continued fractions. The oldest and the most famous theorem associated with Ramanujan's career is the Rogers Ramanujan continued fraction given by

$$C(q) = 1 + \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+} \dots = \frac{(q^2, q^3; q^5)_{\infty}}{(q, q^4; q^5)_{\infty}},$$

The m-dissection of the power series $P = \sum_{n=0}^{\infty} a_n q^n$ is the representation of

P as $P = P_0 + P_1 + \dots + P_{m-1}$, where $P_k = \sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}$.

Andrews¹ and Hirschhorn² have given the 2 dissection and 5 dissection of the continued fractions $C(q)$ and $C(q)^{-1}$.

Ramanujan's³ continued fraction of order 6 is,

$$R(q) = 1 + \frac{q(1+q)}{1+} \frac{q^2(1+q^2)}{1+} \frac{q^3(1+q^3)}{1+} \dots\dots\dots$$

Ramanujan's³ gave the following result,

$$(1.1) \quad R(q) = \frac{(q^3, q^3; q^6)_\infty}{(q, q^5; q^6)_\infty},$$

Neera A Bhagirathi⁴ has also obtained the above result as a special case.

The m-dissection of the power series $P = \sum_{n=0}^{\infty} a_n q^n$ is the representation of P

as $P = P_0 + P_1 + \dots\dots\dots + P_{m-1}$, where $P_k = \sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}$.

Here an attempt has been made to obtain the 4-dissection for the continued fractions $R(q)$ and $R(q)^{-1}$.

2. Notations

Suppose that $|q| < 1$, where q is non-zero complex number, this condition ensures that all the infinite products that we use will converge. We will use the notation,

$$(2.1) \quad (z; q)_\infty = \prod_{n=0}^{\infty} (1 - zq^n),$$

$$(2.2) \quad [z; q]_\infty = (z; q)_\infty (z^{-1}q; q)_\infty, \quad (\text{for } z \neq 0)$$

and often we write

$$(2.3) \quad [z_1, z_2, \dots\dots\dots z_n; q]_\infty = [z_1; q]_\infty [z_2; q]_\infty \dots\dots\dots [z_n; q]_\infty.$$

The following facts can be easily verified;

$$(2.4) \quad [z^{-1}; q]_\infty = -z^{-1} [z; q]_\infty = [zq; q]_\infty,$$

$$(2.5) \quad [z, zq; q^2]_\infty = [z; q]_\infty,$$

$$(2.6) \quad [z, -z; q]_\infty = [z^2; q^2]_\infty,$$

$$(2.7) \quad [z^{-1}q; q]_{\infty} = [z; q]_{\infty},$$

$$(2.8) \quad [-1; q]_{\infty} [q; q^2]_{\infty} = 2,$$

Also, we have the following general relations;

Suppose $a_1, a_2, \dots, a_n; b_1, b_2, \dots, b_n \in C \setminus \{0\}$ satisfy

$$\text{i) } a_i \neq q^n a_j \text{ for } i \neq j \text{ and any } n \in \mathbb{Z}$$

$$\text{ii) } a_1 a_2 \dots a_n = b_1 b_2 \dots b_n$$

Then

$$(2.9) \quad \sum_{i=1}^n \frac{\prod_{j=1}^n [a_i^{-1} b_j; q]_{\infty}}{\prod_{j=1, j \neq i}^n [a_i^{-1} a_j; q]_{\infty}} = 0,$$

This theorem appears without proof as given by Slater⁵ and with a proof as given by Lewis⁶.

3. Main Results

$$(3.1) \quad R(q) = \frac{[q^2, q^4; q^{12}]_{\infty} [q^6; q^{24}]_{\infty}}{[q^6; q^{12}]_{\infty} [q^2, q^4, q^{12}; q^{24}]_{\infty}} + \frac{q^2 [q^2, q^4; q^{12}]_{\infty} [q^6; q^{24}]_{\infty}}{[q^6; q^{12}]_{\infty} [q^4, q^{12}, q^{14}; q^{24}]_{\infty}} \\ - \frac{q [-q^4, -q^4, q^{12}, q^8; q^{12}]_{\infty} [q^2; q^{24}]_{\infty}}{[-q, q, -q^2; q^{12}]_{\infty} [q^{12}; q^{24}]_{\infty}} \\ + \frac{q [q^2, q^2, -q^{10}, -q^6; q^{12}]_{\infty} [q^2; q^{24}]_{\infty}}{[q, -q, -q^2; q^{12}]_{\infty} [q^{12}; q^{24}]_{\infty}},$$

$$\begin{aligned}
(3.2) \quad R(q)^{-1} &= \frac{\left[q^2, q^4; q^{12} \right]_{\infty} \left[q^2, q^{14}; q^{24} \right]_{\infty}}{\left[q^6; q^{12} \right]_{\infty} \left[q^2, q^4, q^6, q^{12}; q^{24} \right]_{\infty}} + \frac{q^2 \left[q^2, q^4; q^{12} \right]_{\infty} \left[q^2, q^{14}; q^{24} \right]_{\infty}}{\left[q^6; q^{12} \right]_{\infty} \left[q^4, q^6, q^{12}, q^{14}; q^{24} \right]_{\infty}} \\
&+ \frac{q \left[-q^4, -q^4, q^{12}, q^8; q^{12} \right]_{\infty} \left[q^2, q^2, q^{14}; q^{24} \right]_{\infty}}{\left[-q, q, -q^2; q^{12} \right]_{\infty} \left[q^6, q^{12}, q^{18}; q^{24} \right]_{\infty}} \\
&- \frac{q \left[q^2, q^2, -q^{10}, -q^6; q^{12} \right]_{\infty} \left[q^2, q^2, q^{14}; q^{24} \right]_{\infty}}{\left[-q, q, -q^2; q^{12} \right]_{\infty} \left[q^6, q^{12}, q^{18}; q^{24} \right]_{\infty}},
\end{aligned}$$

Proof: To prove (3.1) by making use of (2.1), (2.2) and (2.5), we get

$$(3.3) \quad R(q) = \frac{\left[q^3, q^9; q^{12} \right]_{\infty}}{\left[q, q^7; q^{12} \right]_{\infty}},$$

Now, setting $(a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4) = (1, -1, q, q^7; q^3, q^9, -q^{-2}, q^{-2})$ and taking q^{12} for q in (2.9), we get

$$\begin{aligned}
&\frac{\left[q^3, q^9, -q^{-2}, q^{-2}; q^{12} \right]_{\infty}}{\left[-1, q, q^7; q^{12} \right]_{\infty}} + \frac{\left[-q^3, -q^9, q^{-2}, -q^{-2}; q^{12} \right]_{\infty}}{\left[-1, -q, -q^7; q^{12} \right]_{\infty}} \\
&\frac{\left[q^2, q^8, -q^{-3}, q^{-3}; q^{12} \right]_{\infty}}{\left[q^{-1}, -q^{-1}, q^6; q^{12} \right]_{\infty}} + \frac{\left[q^{-4}, q^2, -q^{-9}, q^{-9}; q^{12} \right]_{\infty}}{\left[q^{-7}, -q^{-7}, q^{-6}; q^{12} \right]_{\infty}} = 0,
\end{aligned}$$

By applying (2.4) to (2.8) we get

$$\begin{aligned}
&\frac{\left[q^3, q^9; q^{12} \right]_{\infty}}{\left[q, q^7; q^{12} \right]_{\infty}} + \frac{\left[-q^3, -q^9; q^{12} \right]_{\infty}}{\left[-q, -q^7; q^{12} \right]_{\infty}} \\
&= \frac{2}{\left[q^{12}; q^{24} \right]_{\infty} \left[q^4; q^{24} \right]_{\infty} (-q^{-4})} \\
&\left[-\frac{\left[q^2, q^8; q^{12} \right]_{\infty}}{\left[q^6; q^{12} \right]_{\infty}} \frac{\left[q^6; q^{24} \right]_{\infty}}{\left[q^2; q^{24} \right]_{\infty}} (q^{-4}) - \frac{\left[q^4, q^2; q^{12} \right]_{\infty}}{\left[q^6; q^{12} \right]_{\infty}} \frac{\left[q^{18}; q^{24} \right]_{\infty}}{\left[q^{14}; q^{24} \right]_{\infty}} q^{-2} \right].
\end{aligned}$$

By using (2.2) we get

$$(3.4) \quad R(q) + R(-q) = \frac{2[q^2, q^4; q^{12}]_\infty [q^6; q^{24}]_\infty}{[q^6; q^{12}]_\infty [q^4, q^{12}; q^{24}]_\infty} \left[\frac{1}{[q^2; q^{24}]_\infty} + \frac{q^2}{[q^{14}; q^{24}]_\infty} \right],$$

$$(3.5) \quad \alpha_1(q^2) = \frac{1}{2} [R(q) + R(-q)] = \frac{[q^2, q^4; q^{12}]_\infty [q^6; q^{24}]_\infty}{[q^6; q^{12}]_\infty [q^2, q^4, q^{12}; q^{24}]_\infty} + \frac{q^2 [q^2, q^4; q^{12}]_\infty [q^6; q^{24}]_\infty}{[q^6; q^{12}]_\infty [q^4, q^{12}, q^{14}; q^{24}]_\infty},$$

Again setting $(a_1, a_2, a_3, a_4; b_1, b_2, b_3, b_4) = (1, -1, q^{11}, -q^{13}; q^9, q^9, -q, -q^5; q^{12})$ and taking q^{12} for q in (2.9), we get

$$\frac{[q^9, q^9, -q, -q^5; q^{12}]_\infty}{[-1, q^{11}, -q^{13}; q^{12}]_\infty} + \frac{[-q^9, -q^9, q, q^5; q^{12}]_\infty}{[-1, -q^{11}, q^{13}; q^{12}]_\infty} + \frac{[q^{-2}, q^{-2}, -q^{-10}, -q^{-6}; q^{12}]_\infty}{[q^{-11}, -q^{-11}, -q^2; q^{12}]_\infty} + \frac{[-q^{-4}, -q^{-4}, q^{-12}, q^{-8}; q^{12}]_\infty}{[-q^{-13}, q^{-13}, -q^{-2}; q^{12}]_\infty} = 0$$

Dividing by $[-q, q, q^7, -q^7; q^{12}]_\infty$ and using (2.2) to (2.7)

$$\left[\frac{[q^3, q^9; q^{12}]_\infty}{[q, q^7; q^{12}]_\infty} - \frac{[-q^3, -q^9; q^{12}]_\infty}{[-q, -q^7; q^{12}]_\infty} \right] = \frac{2q^{-1} [-q^4, -q^4, q^{12}, q^8; q^{12}]_\infty [q^2; q^{24}]_\infty}{[-q^{-1}, q^{-1}, -q^2; q^{12}]_\infty [q^{12}; q^{24}]_\infty} + \frac{2q [q^2, q^2, -q^{10}, -q^6; q^{12}]_\infty [q^2; q^{24}]_\infty}{[q, -q, -q^2; q^{12}]_\infty [q^{12}; q^{24}]_\infty}$$

$$(3.6) \quad R(q) - R(-q) = -\frac{2q [-q^4, -q^4, q^{12}, q^8; q^{12}]_\infty [q^2; q^{24}]_\infty}{[-q, q, -q^2; q^{12}]_\infty [q^{12}; q^{24}]_\infty} + \frac{2q [q^2, q^2, -q^{10}, -q^6; q^{12}]_\infty [q^2; q^{24}]_\infty}{[q, -q, -q^2; q^{12}]_\infty [q^{12}; q^{24}]_\infty},$$

$$\begin{aligned}
(3.7) \quad \beta_1(q^2) &= \frac{1}{2q} [R(q) - R(-q)] \\
&= -\frac{[-q^4, -q^4, q^{12}, q^8; q^{12}]_\infty [q^2; q^{24}]_\infty}{[-q, q, -q^2; q^{12}]_\infty [q^{12}; q^{24}]_\infty} + \frac{[q^2, q^2, -q^{10}, -q^6; q^{12}]_\infty [q^2; q^{24}]_\infty}{[q, -q, -q^2; q^{12}]_\infty [q^{12}; q^{24}]_\infty} \\
&= -\frac{[-q^4, -q^4, q^{12}, q^8; q^{12}]_\infty [q^2; q^{24}]_\infty}{[-q, q, -q^2; q^{12}]_\infty [q^{12}; q^{24}]_\infty} + \frac{[q^2, q^2, -q^{10}, -q^6; q^{12}]_\infty [q^2; q^{24}]_\infty}{[q, -q, -q^2; q^{12}]_\infty [q^{12}; q^{24}]_\infty},
\end{aligned}$$

By using (3.5) and (3.7), we get (3.1)

$$\begin{aligned}
R(q) &= \alpha_1(q^2) + q\beta_1(q^2) \\
&= \frac{[q^2, q^4; q^{12}]_\infty [q^6; q^{24}]_\infty}{[q^6; q^{12}]_\infty [q^2, q^4, q^{12}; q^{24}]_\infty} + \frac{q^2 [q^2, q^4; q^{12}]_\infty [q^6; q^{24}]_\infty}{[q^6; q^{12}]_\infty [q^4, q^{12}, q^{14}; q^{24}]_\infty} \\
&\quad - \frac{q [-q^4, -q^4, q^{12}, q^8; q^{12}]_\infty [q^2; q^{24}]_\infty}{[-q, q, -q^2; q^{12}]_\infty [q^{12}; q^{24}]_\infty} + \frac{q [q^2, q^2, -q^{10}, -q^6; q^{12}]_\infty [q^2; q^{24}]_\infty}{[q, -q, -q^2; q^{12}]_\infty [q^{12}; q^{24}]_\infty},
\end{aligned}$$

Now, To prove (3.2), we consider (3.4)

$$R(q) + R(-q) = \frac{2[q^2, q^4; q^{12}]_\infty [q^6; q^{24}]_\infty}{[q^6; q^{12}]_\infty [q^4, q^{12}; q^{24}]_\infty} \left[\frac{1}{[q^2; q^{24}]_\infty} + \frac{q^2}{[q^{14}; q^{24}]_\infty} \right],$$

Multiplying (3.4) by

$$\frac{[q^2, q^{14}; q^{24}]_\infty}{[q^6; q^{18}; q^{24}]_\infty},$$

we get

$$\begin{aligned}
(3.8) \quad R(q)^{-1} + R(-q)^{-1} &= \frac{2[q^2, q^4; q^{12}]_\infty [q^6; q^{24}]_\infty [q^2, q^{14}; q^{24}]_\infty}{[q^6; q^{12}]_\infty [q^4, q^{12}; q^{24}]_\infty [q^6, q^{18}; q^{24}]_\infty} \\
&\quad \times \left[\frac{1}{[q^2; q^{24}]_\infty} + \frac{q^2}{[q^{14}; q^{24}]_\infty} \right],
\end{aligned}$$

$$\begin{aligned}
(3.9) \quad \alpha_2(q^2) &= \frac{1}{2} [R(q)^{-1} + R(-q)^{-1}] \\
&= \frac{[q^2, q^4; q^{12}]_\infty [q^2, q^{14}; q^{24}]_\infty}{[q^6; q^{12}]_\infty [q^2, q^4, q^6, q^{12}; q^{24}]_\infty} \\
&\quad + \frac{q^2 [q^2, q^4; q^{12}]_\infty [q^2, q^{14}; q^{24}]_\infty}{[q^6; q^{12}]_\infty [q^4, q^6, q^{12}, q^{14}; q^{24}]_\infty},
\end{aligned}$$

Now, we consider (3.6)

$$\begin{aligned}
R(q) - R(-q) &= -\frac{2q [-q^4, -q^4, q^{12}, q^8; q^{12}]_\infty [q^2; q^{24}]_\infty}{[-q, q, -q^2; q^{12}]_\infty [q^{12}; q^{24}]_\infty} \\
&\quad + \frac{2q [q^2, q^2, -q^{10}, -q^6; q^{12}]_\infty [q^2; q^{24}]_\infty}{[q, -q, -q^2; q^{12}]_\infty [q^{12}; q^{24}]_\infty},
\end{aligned}$$

Multiplying (3.6) by

$$\frac{[q^2, q^{14}; q^{24}]_\infty}{[q^6; q^{18}; q^{24}]_\infty}$$

we get

$$\begin{aligned}
(3.10) \quad R(q)^{-1} - R(-q)^{-1} &= \frac{2q [-q^4, -q^4, q^{12}, q^8; q^{12}]_\infty [q^2, q^2, q^{14}; q^{24}]_\infty}{[-q, q, -q^2; q^{12}]_\infty [q^6, q^{12}, q^{18}; q^{24}]_\infty} \\
&\quad - \frac{2q [q^2, q^2, -q^{10}, -q^6; q^{12}]_\infty [q^2, q^2, q^{14}; q^{24}]_\infty}{[-q, q, -q^2; q^{12}]_\infty [q^6, q^{12}, q^{18}; q^{24}]_\infty},
\end{aligned}$$

$$\begin{aligned}
(3.11) \quad \beta_2(q^2) &= \frac{1}{2q} [R(q)^{-1} - R(-q)^{-1}] \\
&= \frac{[-q^4, -q^4, q^{12}, q^8; q^{12}]_\infty [q^2, q^2, q^{14}; q^{24}]_\infty}{[-q, q, -q^2; q^{12}]_\infty [q^6, q^{12}, q^{18}; q^{24}]_\infty} \\
&\quad - \frac{[q^2, q^2, -q^{10}, -q^6; q^{12}]_\infty [q^2, q^2, q^{14}; q^{24}]_\infty}{[-q, q, -q^2; q^{12}]_\infty [q^6, q^{12}, q^{18}; q^{24}]_\infty},
\end{aligned}$$

By using (3.9) and (3.11), we get (3.2)

$$\begin{aligned}
 R(q)^{-1} &= \alpha_2(q^2) + q\beta_2(q^2) \\
 &= \frac{\left[q^2, q^4; q^{12} \right]_{\infty} \left[q^2, q^{14}; q^{24} \right]_{\infty}}{\left[q^6; q^{12} \right]_{\infty} \left[q^2, q^4, q^6, q^{12}; q^{24} \right]_{\infty}} \\
 &\quad + \frac{q^2 \left[q^2, q^4; q^{12} \right]_{\infty} \left[q^2, q^{14}; q^{24} \right]_{\infty}}{\left[q^6; q^{12} \right]_{\infty} \left[q^4, q^6, q^{12}, q^{14}; q^{24} \right]_{\infty}} \\
 &\quad + \frac{q \left[-q^4, -q^4, q^{12}, q^8; q^{12} \right]_{\infty} \left[q^2, q^2, q^{14}; q^{24} \right]_{\infty}}{\left[-q, q, -q^2; q^{12} \right]_{\infty} \left[q^6, q^{12}, q^{18}; q^{24} \right]_{\infty}} \\
 &\quad - \frac{q \left[q^2, q^2, -q^{10}, -q^6; q^{12} \right]_{\infty} \left[q^2, q^2, q^{14}; q^{24} \right]_{\infty}}{\left[-q, q, -q^2; q^{12} \right]_{\infty} \left[q^6, q^{12}, q^{18}; q^{24} \right]_{\infty}}.
 \end{aligned}$$

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