

An Analytical Approach to the Problem of Dispersion of an Air Pollutant with Constant Wind Velocity and Constant Removal Rate

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Abstract : In this paper, an analytical approach to the problem of dispersion of an air pollutant with constant wind velocity and constant removal rate is proposed to study. Eddy diffusivities are also taken as constant. The Fourier transform technique has been used to solve the dispersion equation in steady state condition. It is shown that the concentration profile of an air pollutant decreases continuously with increasing downwind distance while it increases with increasing vertical distance. This increase in concentration with vertical distance is higher at lower values of downwind distance and there is negligible change in the concentration of air pollutant with increasing vertical distance at higher values of downwind distances.

Keywords : Dispersion, air-pollutant, constant wind velocity, constant removal rate, concentration profile.

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1. Introduction

There are several competing requirements in the design of an air pollution model. A model must capture the essential physics of the dispersion process and provide reasonable and repeatable estimates of downwind concentrations. This generally requires detailed knowledge of source characteristics and meteorological conditions, but it is also desirable to keep these input requirements to a minimum and simplicity is an important asset in any model. In choosing an air dispersion model, several levels of models are available with progressively increasing levels of

mathematical sophistication, input data requirements and user expertise required. At the low end of the scale are the gross screening models, which require only a hand-held calculator, nomograph or spreadsheet. They may treat only one source at a time (e.g. a single elevated stack) and provide some sort of worst-case prediction based on relatively primitive meteorological information. It is often wise to apply such a model prior to using the more advanced models, where the flow of information is more difficult to follow.

Although the input data requirements and level of sophistication increase with the more advanced models, a more complex model does not necessarily lead to predictions that are more accurate. As the number of input variables goes up in the advanced models, the room for input data error increases. In addition, the level of user understanding must increase to make proper use of model. It is often useful to perform a simple screening analysis before applying a more refined computer analysis. A gross screening analysis will quickly identify the order of magnitude of the expected concentrations and may even show that no problem exists, in which case more advanced modeling is unnecessary [Macdonald¹].

Thorough study of the analytical solutions allows valuable insights to be gained regarding the dispersion of air pollutant. Analytical solutions are useful for examining the accuracy and performance of the numerical models [Lin and Hildemann²]. Analytical solutions of the atmospheric dispersion equation with wind velocity and eddy diffusivities expressed as function of height have been found in the literature since the 1950's. Early solutions have been obtained either by dealing with only two dimensions or by neglecting the inversion effect [Rounds³, Smith⁴]. Yeh and Huang⁵ derived solutions for the three-dimensional advection-diffusion equation under boundary conditions corresponding to both bounded (inversion layer) and unbounded (infinite mixing layer) domains. Studies in this direction, have also been given by Demuth⁶, Tirabassi⁷, Sharan and Yadav⁸, Meeder and Neiuwstadt⁹, Agarwal and Shukla¹⁰ and many others including Sriram et al.¹¹ and Agarwal et al.¹².

In this paper, the dispersion of an air pollutant emitted from an elevated point source is investigated and analysed by taking constant wind velocity and constant removal rate. The eddy diffusivities are also taken as constants.

2. Mathematical Model

The dispersion of an air pollutant in the atmosphere under steady state condition by neglecting the diffusion in downwind direction, is described by partial differential equation

$$(2.1) \quad U \frac{\partial C}{\partial x} = K_y \frac{\partial^2 C}{\partial y^2} + K_z \frac{\partial^2 C}{\partial z^2} - \alpha C,$$

where U is the wind velocity taken to be constant, C is the concentration of the air pollutant, K_y and K_z are the eddy diffusivities in y - and z -directions respectively which are assumed constants, and α is the removal rate of the air pollutant due to some natural mechanism like chemical reaction, which is also taken to be constant. Typically $K_y > K_z$ in the atmosphere.

The boundary conditions for the equation (2.1) are taken as follows:

$$(2.2) \quad C(x, y, z) = \frac{Q \delta(y) \delta(z - h_s)}{U}, \quad x = 0, 0 \leq h_s \leq H,$$

$$(2.3) \quad C(x, y, z) = 0, \quad y \rightarrow \pm \infty,$$

$$(2.4) \quad C(x, y, z) = 0, \quad z = 0,$$

$$(2.5) \quad K_z \frac{\partial C}{\partial z} = v_d C, z = H,$$

where δ is the Dirac delta-function, Q is the emission strength of an elevated point source, h_s is the stack height and v_d is the deposition velocity of the air pollutant. Condition (2.2) states that the air pollutant is released from the elevated point source of strength Q . Condition (2.3) states that the concentration of the air pollutant is zero for $y \rightarrow \pm \infty$. Condition (2.4) states that the concentration of the air pollutant is zero at the ground surface and condition (2.5) states that there is some diffusion flux at the vertical height H from the ground surface.

3. Method of Solution

First of all, the partial differential equation (2.1) describing the dispersion of the air pollutant and the boundary conditions are made non-dimensional by introducing the following non-dimensional quantities:

$$x^* = \frac{K_{z0} x}{U_0 H^2}, \quad y^* = \frac{y}{H}, \quad z^* = \frac{z}{H}, \quad U^* = \frac{U}{U_0}, \quad C^* = \frac{U_0 H^2 C}{Q}, \quad \alpha^* = \frac{H^2 \alpha}{K_{z0}},$$

$$\beta^* = \frac{K_y}{K_{z_0}}, \quad \gamma^* = \frac{K_z}{K_{z_0}}, \quad \delta(y^*) = H \delta(y), \quad N^* = \frac{H v_d}{K_z},$$

where U_0 is the reference wind velocity and K_{z_0} is the reference diffusivity.

On dropping asterisk (*), the equation (2.1) and boundary conditions become

$$(3.1) \quad U \frac{\partial C}{\partial x} = \beta \frac{\partial^2 C}{\partial y^2} + \gamma \frac{\partial^2 C}{\partial z^2} - \alpha C,$$

$$(3.2) \quad C = \frac{\delta(y) \delta(z - h_s)}{U}, \quad x = 0,$$

$$(3.3) \quad C = 0, \quad y \rightarrow \pm \infty$$

$$(3.4) \quad C = 0, \quad z = 0$$

$$(3.5) \quad \frac{\partial C}{\partial z} = NC, \quad z = 1.$$

Now, we solve (3.1) by applying Fourier transform technique. Therefore, taking Fourier transform of (3.1) w.r. to y , we get

$$(3.6) \quad \frac{\partial \bar{C}}{\partial x} = -\left(\frac{p^2 \beta + \alpha}{U}\right) \bar{C} + \left(\frac{\gamma}{U}\right) \frac{\partial^2 \bar{C}}{\partial z^2}$$

where $\bar{C} \equiv \bar{C}(x, p, z)$ is the Fourier transform of $C \equiv C(x, y, z)$ w. r. to y and p is the corresponding Fourier transform parameter.

The boundary conditions are:

$$(3.7a) \quad \bar{C} = \frac{\delta(z - h_s)}{U}, \quad \leftarrow \quad x = 0,$$

$$(3.7b) \quad \bar{C} = 0, \quad \leftarrow \quad y \rightarrow \pm \infty$$

$$(3.7c) \quad \bar{C} = 0, \quad \leftarrow \quad z = 0$$

$$(3.7d) \quad \frac{\partial \bar{C}}{\partial z} = N \bar{C} \quad \leftarrow \quad z = 1.$$

Again, to solve equation (3.6), we shall use the method of separation of variables and therefore, we assume the following trial solution

$$(3.8) \quad \bar{C} = X(x) Z(z),$$

where $X(x)$ is a function of x only and $Z(z)$ is a function of z only.

Using (3.8) in equation (3.6), we get the following two ordinary differential equations :

$$(3.9) \quad Z'' + \lambda^2 Z = 0,$$

$$(3.10) \quad \left(\frac{U}{\gamma} \right) \frac{X'}{X} + \left(\frac{p^2 \beta + \alpha}{\gamma} \right) + \lambda^2 = 0$$

where λ^2 is a separation constant.

Solution of equations (3.9) and (3.10) are respectively given by

$$(3.11) \quad Z(z) = A \cos (\lambda z) + B \sin (\lambda z),$$

$$(3.12) \quad X(x) = M \exp \{ - (p^2 \beta + \alpha + \lambda^2 \gamma) \frac{x}{U} \},$$

where A , B and M are arbitrary constants.

Therefore, we have

$$(3.13) \quad \bar{C} = \{ A \cos (\lambda z) + B \sin (\lambda z) \} \exp \left[\frac{-\{p^2 \beta + \alpha + \lambda^2 \gamma\} x}{U} \right],$$

where M is taken to be 1, without any loss of generality.

Now, using the boundary conditions $\bar{C} = 0$, $z = 0$ and $\frac{\partial \bar{C}}{\partial z} = N \bar{C}$, $z = 1$, we get the following eigen value equation

$$(3.14) \quad \tan (\lambda_n) = \frac{\lambda_n}{N}, \text{ where } n = 1, 2, \dots, \left(\frac{n-1}{2} \right) \pi.$$

Again, using the boundary condition $\bar{C} = \frac{\delta(z-h_s)}{U}$, $x = 0$ and applying

$\int_0^1 \delta(z-h_s) f_n(z) dz = f_n(h_s)$ and $\int_0^1 z^p f_m(z) f_n(z) dz = 0$, $m \neq n$, the solution is given by

$$(3.15) \quad \bar{C} = \left(\frac{1}{U} \right) \exp \left[\frac{-\{p^2 \beta + \alpha + \lambda_n^2 \gamma\} x}{U} \right] \sum_{n=1}^{\infty} \left\{ \frac{\sin(\lambda_n z) \sin(\lambda_n h_s)}{P_n} \right\},$$

where $P_n = \int_0^1 \sin^2(\lambda_n z) dz$.

Finally, taking inverse Fourier transform of (3.15), we get

$$(3.16) \quad C = \left\{ \frac{0.28204}{\sqrt{\beta x U}} \right\} \exp \left[- \left\{ \frac{\alpha x}{U} + \frac{y^2 U}{4\beta x} \right\} \right] \sum_{n=1}^{\infty} \left\{ \frac{\sin(\lambda_n z) \sin(\lambda_n h_s)}{P_n} \right\} \exp \left\{ - \left(\frac{\gamma x}{U} \right) \lambda_n^2 \right\},$$

where $P_n = \int_0^1 \sin^2(\lambda_n z) dz$.

4. Results and Discussion

In order to study the effect of various parameters on dispersion of air pollutant, the concentration profile in non-dimensional form is calculated by using equation (3.16). The parametric values used in the analysis, are taken as :

$$\alpha = 2, \quad \beta = 10, \quad \gamma = 1, \quad h_s = 0.2, \quad U = 1.$$

Figure 1 shows the plot of concentration profile with downwind distance ($0 \leq x \leq 1$) for the two different values of vertical distance ($z = 0.2$ and $z = 0.4$) by taking the crosswind distance $y = 0$. Here, it is seen that the concentration profile decreases continuously with increasing downwind distance and it becomes negligible near $x = 1$. It is also observed that the concentration profile increases with increasing vertical distance, the increase in concentration with vertical distance being higher at lower values of downwind distance and at higher values of downwind distance, there is negligible change in the concentration of air pollutant with increasing vertical distance.

Figure 2 shows the plot of concentration profile again with downwind distance ($0 \leq x \leq 1$) for the two different values of vertical distance ($z = 0.2$ and $z = 0.4$) by taking the crosswind distance $y = 1$. Here, it is observed that the concentration profile follows the same pattern of variation as above mentioned.

5. Conclusion

It is seen that the concentration profile of air pollutant becomes high near the ground but as the distance from the ground increases, the concentration profile decreases regularly. This model is useful in studying the effect of removal mechanism (i.e. sink mechanism) to decrease the concentration of air pollutant from the atmosphere.

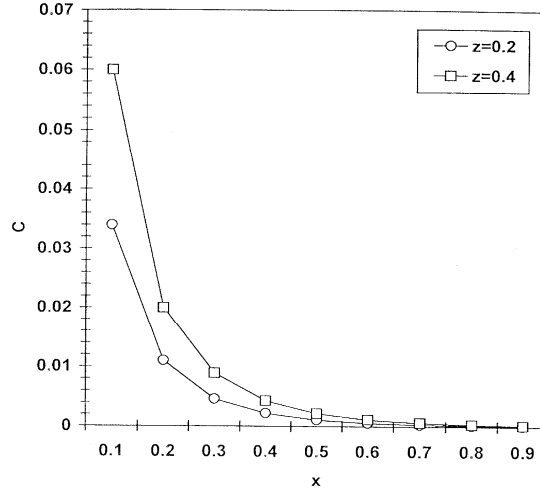


Fig. 1. Variation of concentration profile with downwind distance when crosswind distance $y = 0$

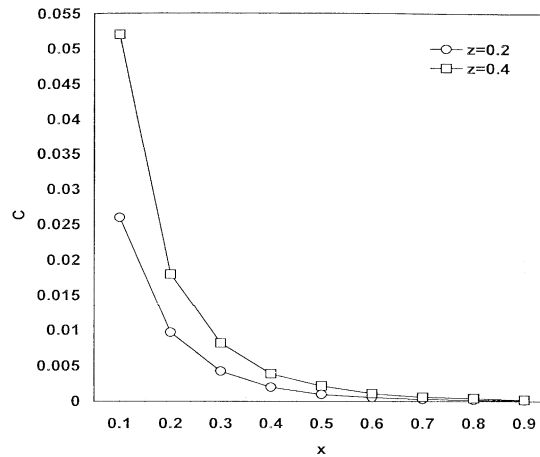


Fig. 2. Variation of concentration profile with downwind distance when crosswind distance $y = 1$

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