

On Semi-symmetric Non-metric Connection on an Almost Hermitian Manifold

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Abstract: In the present paper, an almost Hermitian manifold equipped with a semi-symmetric non-metric connection has been studied. The properties of a contravariant almost analytic vector field with semi-symmetric non-metric connection have also been studied.

Keywords and Phrases: Semi-symmetric non-metric connection, Hermitian manifold, Kähler manifold, Nijenhuis tensor, Contravariant almost analytic vector field.

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1. Introduction

The idea of semi-symmetric linear connection on a differentiable manifold was introduced by S. Golab¹. Various properties of quarter-symmetric metric connections have been studied by several geometers²⁻⁷. A Riemannian manifold equipped with a semi-symmetric metric connection has been studied by O.C. Andonie⁸, M.C. Chaki and A. Konar⁹, U.C. De¹⁰ etc. While a Kähler manifold equipped with semi-symmetric non-metric connection, and an almost Hermitian manifold with semi-symmetric recurrent connection have been studied by P.N. Pandey and B. B. Chaturvedi¹¹⁻¹². Nirmala S. Agashe and Mangala R. Chafle¹³ studied semi-symmetric non-metric connection on a Riemannian manifold.

Recently, Chaubey and Ojha¹⁴⁻¹⁵ have studied a quarter symmetric non-metric connection on an almost Hermitian manifold and B. Prasad and R. K. Verma¹⁶ studied a type of semi-symmetric non-metric connection on a Riemannian manifold. In the present paper, we study a semi-symmetric non-metric connection on an almost Hermitian manifold and obtain some

interesting results. It has been also proved that a contravariant almost analytic vector field V with respect to the Riemannian connection D is also contravariant almost analytic with respect to the semi-symmetric non-metric connection ∇ on a Kähler manifold.

2. Preliminaries

If on an even dimensional differentiable manifold V_n , $n = 2m$, of differentiability class C^{r+1} , there exists a vector valued real linear function F of differentiability class C^r , satisfying

$$(2.1) \quad F^2 X + X = 0,$$

for arbitrary vector field X , then V_n is said to be an almost complex manifold and $\{F\}$ is called an almost complex structure on V_n ¹⁷

If g is a non singular Hermitian metric of type $(0, 2)$ which satisfies

$$(2.2) \quad g(FX, FY) = g(X, Y),$$

for arbitrary vector fields X and Y on V_n , then an almost complex manifold V_n endowed with Hermitian metric g is called an almost Hermitian manifold and the system $\{F, g\}$ is called an almost Hermitian structure¹⁷.

An almost Hermitian manifold V_n is called

(a) a Kähler manifold if

$$(2.3) \quad (D_X F)Y = 0.$$

(b) a nearly Kähler manifold if

$$(2.4) \quad (D_X F)Y + (D_Y F)X = 0.$$

(c) an almost Kähler manifold if

$$(2.5) \quad (D_X F)Y + (D_Y F)Z + (D_Z F)X = 0.$$

(d) a quasi Kähler manifold if

$$(2.6) \quad (D_X F)Y + (D_{FX} F)FY = 0.$$

for arbitrary vector fields X, Y, Z , on V_n .

If we define

$$(2.7) \quad 'F(X, Y) = g(FX, Y),$$

for arbitrary vector fields X and Y, then

$$(2.8) \quad 'F(FX, FY) = 'F(X, Y).$$

2. Semi-Symmetric Non-Metric Connection

A linear connection ∇ on (V_n, g) defined as

$$(3.1) \quad \nabla_X Y = D_X Y + u(Y)X + a(X)Y,$$

for arbitrary vector fields X, Y is said to be semi symmetric non metric connection¹⁶ where u, a are 1- forms, associated with vector fields U and A on M by

$$(3.2) \quad u(X) = g(U, X),$$

and

$$(3.3) \quad a(X) = g(A, X),$$

we put

$$(3.4) \quad H(X, Y) = u(Y)X + a(X)Y,$$

then equation (3.1) becomes

$$(3.5) \quad \nabla_X Y = D_X Y + H(X, Y).$$

The torsion tensor $S(X, Y)$ of the connection ∇ and the metric g are given by¹⁶

$$(3.6) \quad S(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = u(Y)X - u(X)Y + a(X)Y - a(Y)X,$$

$$(3.7) \quad (\nabla_X g)(Y, Z) = -u(Y)g(X, Z) - u(Z)g(Y, X) - 2a(X)g(Y, Z).$$

Remark. In particular if $a=0$, then equation (3.1) reduces to

$$(3.8) \quad \nabla_X Y = D_X Y + u(Y)X.$$

A Riemannian manifold with above connection was studied by Agashe and Chafle¹³. So, we have

$$(3.9) \quad (\nabla_X g)(Y, Z) = -u(Y)g(X, Z) - u(Z)g(X, Y),$$

$$(3.10) \quad S(X, Y) = u(Y)X - u(X)Y.$$

For arbitrary vector fields X and Y .

Theorem 3.1. *If an almost Hermitian manifold V_n admits a semi-symmetric non-metric connection ∇ , then a necessary and sufficient condition for an almost Hermitian manifold to be a Hermitian manifold is that*

$$(\nabla_{FX} F)(FY) = (\nabla_X F)(Y).$$

Proof. Taking covariant derivative of FY with respect to the connection ∇ , we have

$$(\nabla_X F)(Y) + F(\nabla_X Y) = \nabla_X FY.$$

In consequence of (3.5) the above equation becomes

$$(3.11) \quad (\nabla_X F)(Y) = (D_X F)(Y) - F(H(X, Y)) + H(X, FY).$$

By equation (3.4), we have

$$H(X, Y) = u(Y)X + u(X)Y.$$

In view of equation (3.4), (3.11) gives

$$(3.12) \quad (\nabla_X F)(Y) = (D_X F)(Y) - u(Y)FX + u(FY)X.$$

Replacing X by FX and Y by FY in (3.12) and using (2.1), we obtain

$$(3.13) \quad (\nabla_{FX} F)(FY) = (D_{FX} F)(FY) + u(FY)X - u(Y)FX.$$

On subtracting (3.12) from (3.13), we have

$$(3.14) \quad (\nabla_{FX} F)(FY) - (\nabla_X F)(Y) = (D_{FX} F)(FY) - (D_X F)(Y).$$

We know that a necessary and sufficient condition for an almost Hermitian manifold to be a Hermitian manifold is ¹⁷

$$(3.15) \quad (D_{FX} F)(FY) = (D_X F)(Y).$$

The proof follows in view of (3.14) and (3.15).

Remark. *This theorem is also true for Agashe and Chafle connection¹³.*

Theorem 3.2. *If 'F is closed in an almost Hermitian manifold with a semi-symmetric non-metric connection ∇ and Riemannian connection D then*

$$\begin{aligned} & (\nabla_X 'F)(Y, Z) + (\nabla_Y 'F)(Z, X) + (\nabla_Z 'F)(X, Y) \\ & + 2\{a(X)g(FY, Z) + a(Y)g(FZ, X) + a(Z)g(FX, Y) \\ & + u(X)g(FZ, Y) + u(Y)g(FX, Z) + u(Z)g(FY, X)\} = 0. \end{aligned}$$

Proof. We have

$$(\nabla_X 'F)(Y, Z) = (D_X 'F)(Y, Z) - 'F(\nabla_X Y - D_X Y, Z) - 'F(Y, \nabla_X Z - D_X Z).$$

By using equation (3.5) in the above equation, we have

$$(3.16) \quad (\nabla_X 'F)(Y, Z) = (D_X 'F)(Y, Z) - 'F(H(X, Y), Z) - 'F(Y, H(X, Z)).$$

In view of (2.7) and (3.16), we have

$$(3.17) \quad \begin{aligned} (\nabla_X 'F)(Y, Z) &= (D_X 'F)(Y, Z) - g(F(H(X, Y)), Z) \\ &\quad - g(FY, H(X, Z)). \end{aligned}$$

By equation (3.1) (3.5) and (3.17), we have

$$(3.18) \quad \begin{aligned} (\nabla_X 'F)(Y, Z) &= (D_X 'F)(Y, Z) - u(Y)g(FX, Z) \\ &\quad - u(Z)g(FY, X). \end{aligned}$$

Taking cyclic sum of (3.18) in X, Y, Z, we obtain

$$\begin{aligned} (3.19) \quad & (\nabla_X 'F)(Y, Z) + (\nabla_Y 'F)(Z, X) + (\nabla_Z 'F)(X, Y) \\ &= (D_X 'F)(Y, Z) + (D_Y 'F)(Z, X) + (D_Z 'F)(X, Y) \\ &\quad - 2\{a(X)g(FY, Z) + a(Y)g(FZ, X) + a(Z)g(FX, Y) \\ &\quad + u(X)g(FZ, Y) + u(Y)g(FX, Z) + u(Z)g(FY, X)\}. \end{aligned}$$

If 'F is closed with respect to the connection D, then

$$(D_X 'F)(Y, Z) + (D_Y 'F)(Z, X) + (D_Z 'F)(X, Y) = 0.$$

Now the proof follows from (3.19).

Theorem 3.3. *If an almost Hermitian manifold admits a semi-symmetric non-metric connection ∇ , then the Nijenhuis tensors of D and ∇ coincide.*

Proof. The Nijenhuis tensor on an almost Hermitian manifold is given as¹⁷

$$\begin{aligned} N(X, Y) &= [FX, FY] + F^2[X, Y] - F[FX, Y] - F[X, FY], \\ &= (D_{FX}F)Y + F(D_{FX}Y) - (D_{FY}F)X - F(D_{FY}X) \\ &\quad - D_XY + D_YX - F(D_{FX}Y) + F\{(D_YF)X \\ &\quad + F(D_YX)\} - F\{(D_XF)Y + F(D_XY) - D_{FY}X\}, \end{aligned}$$

$$(3.20) \quad N(X, Y) = (D_{FX}F)Y - (D_{FY}F)X + F((D_YF)X) - F((D_XF)Y).$$

The equation (3.12) gives

$$(3.21) \quad (D_XF)(Y) = (\nabla_XF)(Y) + u(Y)FX - u(FY)X.$$

By operating F on both sides of equation (3.21) and using (2.1), we have

$$(3.22) \quad F((D_XF)(Y)) = F((\nabla_XF)(Y)) - u(Y)X - u(FY)FX.$$

Interchanging X and Y in (3.22), we get

$$(3.23) \quad F((D_YF)(X)) = F((\nabla_YF)(X)) - u(X)Y - u(FX)FY.$$

Replacing X by FX in equation (3.21), we have

$$(3.24) \quad (D_{FX}F)(Y) = (\nabla_{FX}F)(Y) - u(Y)X - u(FY)(FX).$$

Again interchanging X and Y in above equation, we obtain

$$(3.25) \quad (D_{FY}F)(X) = (\nabla_{FY}F)(X) - u(X)Y - u(FX)FY.$$

By using equations (3.22), (3.23), (3.24), (3.25) in (3.20), we have

$$N(X, Y) = N^*(X, Y),$$

where

$$N^*(X, Y) = (\nabla_{FX}F)(Y) - (\nabla_{FY}F)(X) - F((\nabla_XF)(Y)) + F((\nabla_YF)(X)),$$

which is the Nijenhuis tensor of the connection ∇ .

Now, we can state

Corollary 3.1. *An almost Hermitian manifold V_n with a semi-symmetric non-metric connection is a Hermitian manifold if the Nijenhuis tensor of connection ∇ vanishes, i.e. $N^*(X, Y) = 0$.*

Corollary 3.2. *On a Kähler manifold, Nijenhuis tensor with respect*

to semi-symmetric non-metric connection vanishes, i.e. $N^*(X, Y) = 0$.

Theorem 3.4. A Kähler manifold with a semi symmetric non-metric connection ∇^{I3} satisfies the relations

$$(3.26) \quad (a) \quad u(FY)X - u(Y)FX = 0.$$

i.e. $(\nabla_X F)(Y)$ is hybrid in both the slots.

$$(b) \quad (\nabla_X F)(Y) = 0 \Leftrightarrow u(Y) = 0.$$

Proof. By equations (3.11) and $H(X, Y) = u(Y)X$, we get

$$(3.27) \quad (\nabla_X F)(Y) = (D_X F)(Y) - u(Y)FX + u(FY)X.$$

On a Kähler manifold, we have

$$(D_X F)Y = 0.$$

So equation (3.27) becomes

$$(3.28) \quad (\nabla_X F)(Y) = u(FY)X - u(Y)FX.$$

Substituting FX in place of X and FY in place of Y in above equation and using (2.1), we have

$$(3.29) \quad (\nabla_{FX} F)(FY) = -u(Y)FX + u(FY)X.$$

In view of (3.28) and (3.29), we obtain

$$(3.30) \quad (\nabla_{FX} F)(FY) = (\nabla_X F)(Y),$$

again if

$$(\nabla_X F)(Y) = 0,$$

then (3.28) gives

$$u(FY)X - u(Y)FX = 0.$$

Since X and FX are linearly independent, we have

$$u(Y) = 0,$$

which proves the first part of the theorem. The proof of second part is trivial.

4. Contravariant Almost Analytic Vector Fields on a Kähler Manifold

If the Lie-derivative of F with respect to a vector field V vanishes identically for all X , i.e.

$$(4.1) \quad (L_V F)(X) = 0,$$

then V is said to be a contravariant almost analytic vector field¹⁷. The equation (4.1) is equivalent to

$$(4.2) \quad [V, FX] = F[V, X].$$

In view of equation (4.2), on a Kähler manifold, we have

$$(4.3) \quad (D_{FX} V) - F(D_X V) = 0 \Leftrightarrow F(D_{FX} V) + D_X V = 0.$$

Now, we have

Theorem 4.1. *On a Kähler manifold, a contravariant almost analytic vector field V with respect to the Riemannian connection D is also contravariant almost analytic with respect to semi-symmetric non-metric connection ∇ .*

Proof By equation (3.1), we have

$$(4.4) \quad \nabla_X Y = D_X Y + u(Y)X + a(X)Y.$$

Replacing Y by V in equation (4.4), we get

$$(4.5) \quad \nabla_X V = D_X V + u(V)X + a(X)V.$$

On replacing X by FX , the equation (4.5) becomes

$$(4.6) \quad \nabla_{FX} V = D_{FX} V + u(V)FX + a(FX)V.$$

Operating F on both sides of equation (4.5), we get

$$(4.7) \quad F(\nabla_X V) = F(D_X V) + u(V)FX + a(X)FV.$$

Subtracting the equation (4.7) from (4.6), we have

$$\nabla_{FX} V - F(\nabla_X V) = D_{FX} V - F(D_X V) + a(FX)V - a(X)FV.$$

Since V is contravariant almost with respect to connection D , we have

$$D_{FX} V - F(D_X V) = 0,$$

which gives

$$(4.8) \quad \nabla_{FX} V - F(\nabla_X V) = a(FX)V - a(X)FV.$$

Therefore V is contravariant almost analytic with respect to connection ∇ . The proof follows from (4.8).

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