

On Generalized ϕ -Recurrent Para-Sasakian Manifolds

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Abstract: The purpose of this paper is to study generalized ϕ -recurrent para-sasakian manifolds.

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1. Introduction

The notion of ϕ -recurrent Sasakian manifolds was introduced by U. C. De, A. A. Shaikh and Sudipta Biswas¹. This notion generalizes the notion of locally ϕ -symmetric Sasakian manifolds. A Sasakian manifold is said to be a ϕ -recurrent manifold if there exists a nonzero 1-form A such that

$$\phi^2((\nabla_X R)(Y, Z)W) = A(X)R(Y, Z)W$$

for arbitrary vector fields X, Y, Z, W . If the 1-form A vanishes, then the manifold reduces to a ϕ -symmetric manifold.

The notion of generalized recurrent manifolds was introduced by U. C. De and N. Guha³. A Riemannian manifold (M^n, g) is called generalized recurrent if its curvature tensor R satisfies the condition

$$(\nabla_X R)(Y, Z)W = A(X)R(Y, Z)W + B(X)[g(Z, W)Y - g(Y, W)Z]$$

where, A and B are two 1-forms, B is non-zero and these are defined by

$$A(X) = g(X, \rho_1) \quad B(X) = g(X, \rho_2)$$

ρ_1 and ρ_2 are vector fields associated with 1-forms A and B, respectively.

The notion of generalized ϕ -recurrent Kenmotsu manifolds was introduced by A.Basari and C.Murathan² and the study has been continued by the authors³. Motivated by the above studies, in this paper we extend the results to generalized ϕ -recurrent para-Sasakian manifolds.

The paper is organized as follows. In Preliminaries, we give a brief account of Para Sasakian manifolds. In section⁴, we study generalized ϕ -recurrent and shown that Para Sasakian manifold under this condition reduced to Einstein one. and also proved that generalized ϕ -recurrent para-Sasakian manifold is of constant negative curvature. in the last section it is shown that generalized quasi- conformally ϕ -recurrent para-Sasakian manifold is an Einstein manifold. and also proved that a 3-dimensional generalized quasi-conformally ϕ -recurrent para- Sasakian manifold is of constant curvature.

2. Preliminaries

Let (M^{2n+1}, g) be a Para Sasakian manifold with structure (ϕ, ξ, η, g) then the following relations hold ⁶

$$(2.1) \quad \phi^2 = I - \eta \otimes \xi,$$

$$(2.2) \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(X, \xi) = \eta(X),$$

$$(2.3) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(2.4) \quad R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi,$$

$$(2.5) \quad \nabla_X \xi = \phi X,$$

$$(2.6) \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$

$$(2.7) \quad R(\xi, X)\xi = X - \eta(X)\xi,$$

$$(2.8) \quad \eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X),$$

$$(2.9) \quad S(X, \xi) = -2n\eta(X),$$

$$(3.0) \quad S(\phi X, \phi Y) = S(X, Y) + 2n\eta(X)\eta(Y),$$

for all vector fields X, Y, Z where ∇ denotes the operator of covariant

differentiation with respect to g , ϕ is a $(1, 1)$ tensor field, S is the Ricci tensor of type $(0, 2)$ and R is the Riemannian curvature tensor of the manifold.

3. Generalized ϕ -Recurrent Para Sasakian Manifolds

Definition 3.1. Para Sasakian manifold (M^{2n+1}, g) is called generalized ϕ -recurrent if its curvature tensor R satisfies the condition

$$(3.1) \quad \phi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y],$$

where A and B are two 1-forms, B is non-zero and these are defined by:

$$(3.2) \quad A(W) = g(W, \rho_1), \quad B(W) = g(W, \rho_2)m,$$

and ρ_1, ρ_2 are vector fields associated with 1-forms A and B , respectively.

From (3.1), using (2.1) we have

$$(3.3) \quad \begin{aligned} &(\nabla_W R)(X, Y)Z - \eta((\nabla_W R)(X, Y)Z)\xi \\ &= A(W)R(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y], \end{aligned}$$

From which it follows that

$$(3.4) \quad \begin{aligned} &g((\nabla_W R)(X, Y)Z, U) - \eta((\nabla_W R)(X, Y)Z)\eta(U) \\ &= A(W)g(R(X, Y)Z, U) + B(W)[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)], \end{aligned}$$

Let $\{e_i\}, i = 1, 2, \dots, 2n+1$, be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (3.4) and taking summation over i , $1 \leq i \leq 2n+1$ we get

$$(3.5) \quad \begin{aligned} &(\nabla_W S)(Y, Z) - \sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i, Y)Z)\eta(e_i) \\ &= A(W)S(Y, Z) + B(W)[g(Y, Z)(n) - g(Y, Z)]. \end{aligned}$$

The second term of (3.5) is reduced to

$$\sum_{i=1}^{2n+1} \eta((\nabla_W R)(e_i, Y)Z)\eta(e_i) = g((\nabla_W R)(\xi, Y)Z, \xi).$$

Using (2.5) and (2.6), we get

$$g((\nabla_W R)(\xi, Y)Z, \xi) = 0.$$

So, the equation (3.5) has the following form

$$(\nabla_w S)(Y, Z) = A(W)S(Y, Z) + 2nB(W)g(Y, Z).$$

Replacing Z by ξ in (3.5) and using (2.9) we have

$$(3.6) \quad (\nabla_w S)(Y, \xi) = -A(W)2n\eta(Y) + B(W)2n\eta(Y).$$

Now we have

$$(\nabla_w S)(Y, \xi) = \nabla_w S(Y, \xi) - S(\nabla_w Y, \xi) - S(Y, \nabla_w \xi).$$

Using (2.5) and (2.9) in the above relation it follows that

$$(3.7) \quad (\nabla_w S)(Y, \xi) = -2ng(\phi W, Y) - S(Y, \phi W)$$

From (3.6) and (3.7) we obtain

$$(3.8) \quad -2ng(\phi W, Y) - S(Y, \phi W) = 2n\eta(Y)(B(W) - A(W)).$$

Replacing $Y = \xi$ in (3.8) and then using (2.9) and (2.2) we get

$$(3.9) \quad A(W) = B(W).$$

Using (3.9) in (3.8) we obtain

$$(3.10) \quad S(Y, W) = -2ng(Y, W).$$

This leads to following results:

Theorem 3.1. *A generalized ϕ -recurrent Para Sasakian manifold (M^{2n+1}, g) is an Einstein manifold. Two Vector fields P and Q are said to be co-directional if $P = fQ$ where f is a nonzero scalar, that is $g(P, X) = fg(Q, X)$ for all X . from (3.1) we have*

$$(3.11) \quad (\nabla_w R)(X, Y)Z = \eta((\nabla_w R)(X, Y)Z)\xi + A(W)R(X, Y)Z \\ + B(W)[g(Y, Z)X - g(X, Z)Y]$$

Then by the use of second Bianchi identity and (3.11) we get

$$(3.12) \quad A(W)\eta(R(X, Y)Z) + A(X)\eta(R(Y, W)Z) + A(Y)\eta(R(W, X)Z) \\ + B(W)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + B(X)[g(W, Z)\eta(Y) - g(Y, Z)\eta(W)] \\ + B(Y)[g(X, Z)\eta(W) - g(W, Z)\eta(X)] = 0.$$

By virtue of (2.8) we obtain from (3.12) that

$$(3.13) \quad \begin{aligned} & [A(W) - B(W)][g(X, Z)\eta(Y) - g(Y, Z)\eta(X)] \\ & + [A(X) - B(X)][g(Y, Z)\eta(W) - g(W, Z)\eta(Y)] \\ & + [A(Y) - B(Y)][g(W, Z)\eta(X) - g(X, Z)\eta(W)] = 0. \end{aligned}$$

Putting $Y = Z = e_i$ in (3.13) and taking summation over i $1 \leq i \leq 2n+1$ we get

$$(3.14) \quad [A(W) - B(W)]\eta(X) = [A(X) - B(X)]\eta(W),$$

for all vector fields X, W . Replacing X by ξ in (3.14) it follows that

$$(3.15) \quad [A(W) - B(W)] = [\eta(\rho_1) - \eta(\rho_2)]\eta(W),$$

for any vector field W ,

where $A(\xi) = g(\xi, \rho_1) = \eta(\rho_1)$ and $B(\xi) = g(\xi, \rho_2) = \eta(\rho_2)$

This leads the following result.

Theorem 3.2. *In a generalized ϕ -recurrent para-Sasakian manifold (M^{2n+1}, g) the characterstic vector field ξ and the vector field $\rho_1 - \rho_2$ associated to the one form $A - B$ are in the same direction.*

Now

$$(3.16) \quad \begin{aligned} (\nabla_W R)(X, Y)\xi &= \nabla_W R(X, Y)\xi - R(\nabla_W X, Y)\xi \\ &\quad - R(X, \nabla_W Y)\xi - R(X, Y)\nabla_W \xi. \end{aligned}$$

Using (2.6) and (2.5) we get from the above that

$$(3.17) \quad (\nabla_W R)(X, Y)\xi = (\nabla_W \eta)(X)Y - (\nabla_W \eta)(Y)X - R(X, Y)\phi W.$$

If we consider X, Y orthogonal to ξ , then in view of (2.8) we have

$\eta(\nabla_W R)(X, Y)\xi = 0$, Hence

$$(3.18) \quad \eta(\nabla_{\phi W} R)(X, Y)\xi = 0.$$

From (3.17) it follows that

$$(3.19) \quad (\nabla_{\phi W} R)(X, Y)\xi = (\nabla_{\phi W} \eta)(X)Y - (\nabla_{\phi W} \eta)(Y)X - R(X, Y)\phi^2 W.$$

By using $\phi^2 W = W - \eta(W)\xi$ we get

$$(3.20) \quad (\nabla_{\phi W} R)(X, Y)\xi = g(X, W)Y - g(Y, W)X - R(X, Y)W.$$

Suppose the manifold is generalized ϕ -recurrent .then using (3.3) we obtain from the above

$$(3.21) \quad \eta((\nabla_{\phi W} R)(X, Y)\xi)\xi + A(\phi W)R(X, Y)\xi + B(\phi W)[\eta(Y)X - \eta(X)Y] \\ = g(X, W)Y - g(Y, W)X - R(X, Y)W.$$

Using (3.18) and $A(\phi W) - B(\phi W) = 0$ we get from the above

$$(3.22) \quad R(X, Y)W = -[g(Y, W)X - g(X, W)Y]$$

This leads the following result.

Theorem 3.3. *A $(2n + 1)$ dimensional $(n > 1)$ generalized ϕ -recurrent Para Sasakian manifold is of constant negative curvature -1 .*

4. Generalized Quasi-conformally f-recurrent Para-Sasakian Manifold

Let (M^{2n+1}, g) , $n \geq 1$, be Riemannian manifold. The notion of the quasi-conformal curvature tensor was introduced by Yano and Sawaki⁵. According to them a quasiconformal curvature tensor is defined by

$$(4.1) \quad C^*(X, Y)Z = aR(X, Y)Z \\ + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ - \frac{r}{2n+1} \left[\frac{a}{2n} + 2b \right] [g(Y, Z)X - g(X, Z)Y]$$

Definition 4.2. Para Sasakian manifold (M^{2n+1}, g) is called generalized quasi-conformal ϕ -recurrent if its curvature tensor C^* satisfies the condition

$$(4.2) \quad \phi^2((\nabla_W C^*)(X, Y)Z) = A(W)C^*(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y].$$

From (4.1) and (2.1) it follows that

$$(4.3) \quad a(g((\nabla_W R)(X, Y)Z, U)) + bg(X, U)(\nabla_W S)(Y, Z) \\ - bg(Y, U)(\nabla_W S)(X, Z) + bg(Y, Z)g((\nabla_W Q)X, U) \\ - bg(X, Z)g((\nabla_W Q)Y, U) - \frac{1}{2n+1} dr(W) \left[\frac{a}{2n} + 2b \right] \\ [g(Y, Z)g(X, U) - g(X, Z)g(Y, U)]$$

$$\begin{aligned}
& -a\eta((\nabla_w R)(X, Y)Z)\eta(U) - b(\nabla_w S)(Y, Z)\eta(U)\eta(X) \\
& + b(\nabla_w S)(X, Z)\eta(U)\eta(Y) - bg(Y, Z)\eta((\nabla_w Q)X)\eta(U) \\
& + bg(X, Z)\eta((\nabla_w Q)Y)\eta(U) + \frac{1}{2n+1}dr(W)\left[\frac{a}{2n} + 2b\right] \\
& [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\eta(U)] \\
& = A(W)[g(aR(X, Y)Z, U) + bS(Y, Z)g(X, U) - bS(X, Z)g(Y, U)] \\
& + A(W)[bg(Y, Z)S(X, U) - bg(X, Z)S(Y, U)] \\
& - A(W)\frac{r}{2n+1}\left[\frac{a}{2n} + 2b\right][g(Y, Z)g(X, U) - g(X, Z)g(Y, U)] \\
& + B(W)[g(Y, Z)g(X, U) - g(X, Z)g(Y, U)].
\end{aligned}$$

Putting $X = U = e_i$ where $e_i = 1, 2, \dots, 2n+1$ is an orthonormal basis of the tangent space at each point of the manifold, and taking the summation over i , we get

$$\begin{aligned}
(4.4) \quad & (a + (2n+1)b - 2b)(\nabla_w S)(Y, Z) \\
& + \left\{ bg((\nabla_w Q)e_i, e_i) - \frac{2n}{2n+1}dr(W)\left[\frac{a}{2n} + 2b\right] \right\} g(Y, Z) \\
& + \left\{ -b\eta((\nabla_w Q)e_i)\eta(e_i) + \frac{1}{2n+1}dr(W)\left[\frac{a}{2n} + 2b\right] \right\} g(Y, Z) - bg((\nabla_w Q)Y, Z) \\
& - a\eta((\nabla_w R)(e_i, Y)Z)\eta(e_i) + b(\nabla_w S)(\xi, Z)\eta(Y) + b\eta((\nabla_w Q)Y)\eta(Z) \\
& - \frac{1}{2n+1}dr(W)\left[\frac{a}{2n} + 2b\right]\eta(Y)\eta(Z) = A(W)[a + b(2n-1)]S(Y, Z) \\
& + A(W)\left\{ br - \frac{r}{2n+1}(a + 4nb) \right\} g(Y, Z) + B(W)[2ng(Y, Z)].
\end{aligned}$$

Putting $Z = \xi$ we obtain

$$\begin{aligned}
(4.5) \quad & (a + (2n+1)b - 2b)(\nabla_w S)(Y, \xi) \\
& + \eta(Y)\left\{ bdr(W) - \frac{2n}{2n+1}dr(W)\left[\frac{a}{2n} + 2b\right] \right\}
\end{aligned}$$

$$\begin{aligned}
& +\eta(Y)\left\{-b\eta((\nabla_w Q)e_i)\eta(e_i)+\frac{1}{2n+1}dr(W)\left[\frac{a}{2n}+2b\right]\right\} \\
& -a\eta((\nabla_w R)(e_i,Y)\xi)\eta(e_i)+b(\nabla_w S)(\xi,\xi)\eta(Y) \\
& -\frac{1}{2n+1}dr(W)\left[\frac{a}{2n}+2b\right]\eta(Y) \\
& =A(W)\{a+b(2n-1)\}S(Y,\xi)+A(W)\left\{br-\frac{r}{2n+1}(a+4nb)\right\} \\
& +B(W)[2ng(Y,\xi)].
\end{aligned}$$

Now

$$(4.6) \quad \eta((\nabla_w Q)e_i)\eta(e_i)=0,$$

$$(4.7) \quad \eta((\nabla_w R)(e_i,Y)\xi)\eta(e_i)=g((\nabla_w R)(e_i,Y)\xi,\xi)g(e_i,\xi).$$

Since $\{e_i\}$ is an orthonormal basis $\nabla_x e_i = 0$ and using (2.6) we find

$$g(R(e_i,\nabla_w Y)\xi,\xi)=0.$$

We have

$$g(\nabla_w R(e_i,Y)\xi,\xi)+g(R(e_i,Y)\xi,\nabla_w \xi)=0.$$

Using this we get

$$(4.8) \quad g(\nabla_w R(e_i,Y)\xi,\xi)=0.$$

By the use of (4.6) - (4.8), from (4.5) we obtain

$$\begin{aligned}
(4.9) \quad -(\nabla_w S)(Y,\xi)+\frac{1}{2n+1}dr(W)\eta(Y) & =-A(W)\{a+b(2n-1)\}2n\eta(Y) \\
& +A(W)\left\{br-\frac{r}{2n+1}(a+4nb)\right\}+B(W)2n\eta(Y).
\end{aligned}$$

provided $a+b(2n-1) \neq 0$ Here

$$(4.10) \quad (\nabla_w S)(Y,\xi)=-2ng(\phi W,Y)-S(Y,\phi W).$$

Using this in the equation (4.9) we get

$$\begin{aligned}
(4.11) \quad S(Y,\phi W)+2ng(\phi W,Y)+\frac{1}{2n+1}dr(W)\eta(Y) \\
= A(W)\left\{br-\frac{r}{2n+1}(a+4nb)\right\}+B(W)2n\eta(Y).
\end{aligned}$$

Putting $Y = \phi Y$ we get

$$(4.12) \quad S(Y, W) = -2ng(Y, W).$$

Hence we can state the following theorem

Theorem 4.4. *A generalized quasi-conformal ϕ -recurrent Para Sasakian manifold is an Einstein manifold. In a 3-dimensional para-Sasakian Manifold (M^3, g) we have*

$$(4.13) \quad R(X, Y)Z = \left(\frac{r}{2} + 2\right)[g(Y, Z)X - g(X, Z)Y] \\ - \left(3 + \frac{r}{2}\right)[g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y,]$$

$$(4.14) \quad S(X, Y) = \left(\frac{r}{2} + 1\right)g(X, Y) - \left(3 + \frac{r}{2}\right)\eta(X)\eta(Y)$$

$$(4.15) \quad QX = \left(\frac{r}{2} + 1\right)X - \left(3 + \frac{r}{2}\right)\eta(X)\xi.$$

Thus from (4.13), (4.14) and (4.15), we get

$$(4.16) \quad C^*(X, Y)Z = \left\{a\left(\frac{r}{2} + 2\right) + 2b\left(\frac{r}{2} + 1\right) - \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right\} \\ [g(Y, Z)X - g(X, Z)Y] \left[a\left(3 + \frac{r}{2}\right) + b\left(3 + \frac{r}{2}\right)\right] \\ [g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]$$

Differentiating the equation (4.16) covariantly, we get

$$(4.17) \quad (\nabla_w C^*)(X, Y)Z = \frac{dr(W)}{21}(10a + 15b)[g(Y, Z)X - g(X, Z)Y] \\ - \frac{dr(W)}{2}(a + b) \begin{bmatrix} g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \end{bmatrix}.$$

Noting that we may assume that all vector fields X, Y, Z, W are orthogonal to ξ and using (2.1), we get

$$(4.18) \quad (\nabla_w C^*)(X, Y)Z = \frac{dr(W)}{21}(10a + 15b)[g(Y, Z)X - g(X, Z)Y].$$

Applying ϕ^2 to the both sides of (4.18) and using (2.1) and (2.2), we get

$$(4.19) \quad \phi^2(\nabla_w C^*)(X, Y)Z = \frac{dr(W)}{21}(10a+15b)[g(Y, Z)X - g(X, Z)Y].$$

By (3.1) the equation (4.19) reduces to

$$\begin{aligned} A(W)C^*(X, Y)Z + B(W)[g(Y, Z)X - g(X, Z)Y] \\ = \frac{dr(W)}{21}(10a+15b)[g(Y, Z)X - g(X, Z)Y]. \end{aligned}$$

Putting $W = \{e_i\}$, where $\{e_i\}, i = 1, 2, 3$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over $i, 1 \leq i \leq 3$ we obtain

$$C^*(X, Y)Z = \lambda[g(Y, Z)X - g(X, Z)Y],$$

where $\lambda = \left[\frac{dr(e_i)}{21A(e_i)}(10a+15b) + \frac{B(e_i)}{A(e_i)} \right]$ is a scalar, since A is a non-zero 1-form. Then by Schur's theorem λ will be a constant on the manifold. Therefore, (M^3, g) is of constant curvature λ . Thus we get the following theorem:

Theorem 4.5. *A 3-dimensional generalized ϕ -recurrent Quasi-conformal para-Sasakian manifold is of constant curvature.*

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