

Study of Hypersurfaces of Modified Finsler Space

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(Received January 27, 2021)

Abstract: M. Mastumoto¹ defined modified Finsler space $F^{*n} = (M^n, L^*)$ from $F^n = (M^n, L)$ by the following transformation $L^{*2} = L^2 + \beta^2$, where $\beta = b_i(x)y^i$ and $b_i(x)$ is covector and function of position co-ordinates only. In the present paper we have study the relation between hypersurfaces of F^n and F^{*n} taking normalized semi-parallel vector field $X_i(x)$ in place of $b_i(x)^2$.

Keywords: normalized semi-parallel vector field, Hyperplane.

2010 Subject Classification: 53C60, 53B40

1. Introduction

Let $F^n = (M^n, L)$ be n-dimensional Finsler space, where M^n is differential n-manifold and L is Finsler fundamental function. $(F_{jk}^i, N_j^i, C_{jk}^i)$ is Cartan connection coefficients of F^n . These coefficients are given by³

$$(1.1) \quad F_{jk}^i = \frac{1}{2} g^{ip} \left(\frac{\partial g_{jp}}{\partial x^k} + \frac{\partial g_{kp}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^p} \right),$$

$$(1.2) \quad N_j^i = F_{jk}^i y^k,$$

$$(1.3) \quad C_{jk}^i = \frac{1}{2} g^{ip} \left(\frac{\partial g_{jp}}{\partial x^k} + \frac{\partial g_{kp}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^p} \right),$$

where g_{ip} is fundamental function of F^n and g^{ip} is its inverse. The non-zero torsion tensors of F^n with respect to Cartan connection are given by³

$$(1.4) \quad (\text{v})\text{hv torsion tensor} \quad P_{jk}^i = \frac{\partial N_j^i}{\partial y^k} - F_{jk}^i,$$

$$(1.5) \quad (\text{v})\text{htorsion tensor} \quad R_{jk}^i = \delta_k N_j^i - \delta_j N_k^i.$$

Now, we shall consider¹ modified Finsler space $F^{*n} = (M^n, L^*)$ from $F^n = (M^n, L)$ in the following way

$$(1.6) \quad L^{*2} = L^2 + \beta^2,$$

where β is one form given as $\beta = b_i(x)y^i$, $b_i(x)$ is covector. The fundamental metric tensor of F^{*n} in term of fundamental metric tensor of F^n , can be derived from (1.6)

$$(1.7) \quad g_{ij}^* = g_{ij} + b_i b_j$$

and its inverse is given by

$$(1.8) \quad g^{*ij} = g^{ij} - \frac{b^i b^j}{1 + b^2}.$$

In the present paper, we take normalized semi-parallel vector field X_i (function of position co-ordinate only) in place of $b_i(x)$ ⁴, which satisfies the following condition

$$(1.9) \quad \begin{cases} (a) & X_i C_{jk}^i = 0 \\ (b) & X_{i|j} = \rho(g_{ij} - X_i X_j), \end{cases}$$

where ρ is scalar function of position co-ordinate only.

For normalized semi-parallel vector field, from (1.7) and (1.8), we have

$$(1.10) \quad g_{ij}^* = g_{ij} + X_i X_j$$

$$(1.11) \quad g^{*ij} = g^{ij} - \frac{X^i X^j}{1 + X^2},$$

where X is length of X_i .

Cartan connection coefficients of F^{*n} given as In view of (1.9a), (1.10) and (1.11), we have

$$(1.12) \quad C_{jk}^{*i} = C_{jk}^i.$$

In view of (1.1), (1.10) and (1.11), we have

$$(1.13) \quad F_{jk}^{*i} = F_{jk}^i + \frac{\rho X^i (g_{jk} - X_j X_k)}{2}$$

and in view of (1.2) and (1.13), we have

$$(1.14) \quad N_j^{*i} = N_j^i + \frac{\rho X^i (g_{jk} - X_j X_k) y^k}{2}.$$

In view of (1.4), (1.13) and (1.14) (v)hv torsion tensor P_{jk}^{*i} of F^{*n} in term of (v)hv torsion tensor P_{jk}^i of F^n is given by

$$(1.15) \quad P_{jk}^{*i} = P_{jk}^i.$$

2. Hypersurfaces of Finsler space

A hypersurface M^{n-1} of the manifold M^n can be parametrically represented by

$$(2.1) \quad x^i = x^i(u^\alpha),$$

where u^α is Gaussian co-ordinate on M^{n-1} and α varying from 1 to $(n-1)$.

From (2.1), we construct a matrix which element is $B_\alpha^i(u) = \frac{\partial x^i}{\partial u^\alpha}$ and

assume that the rank of this matrix is $(n-1)$. $B_\alpha^i(u) = \frac{\partial x^i}{\partial u^\alpha}$ is called Projection

factor. These $(n-1)$ vectors $B_\alpha = (B_\alpha^i)$ may be regarded as $(n-1)$ linearly

independent vector tangent to M^{n-1} at the point u^α , so these vector B_α form basis of tangent space of M^{n-1} at the point u^α , if vector X^i tangent to M^{n-1} at the point u^α , then it can be expressed as

$$(2.2) \quad X^i = B_\alpha^i X^\alpha,$$

where X^α is component of X^i with respect to u^α .

If the supporting element y^i at the point u^α of M^{n-1} assumed to be tangential to M^{n-1} , so we may write

$$(2.3) \quad y^i = B_\alpha^i v^\alpha,$$

where v^α is treated as supporting element of M^{n-1} at the point u^α .

Then the Finsler metric of M^{n-1} is given by

$$(2.4) \quad \bar{L}(u, v) = L(x(u), y(u, v)).$$

Thus, we get (n-1) dimensional Finsler space $F^{n-1} = (M^{n-1}, \bar{L}(u, v))$, which is Finslerian hypersurface of $F^n = (M^n, L)$, and $\bar{L}(u, v)$ is called induced metric on M^{n-1} . At each point u^α of F^{n-1} , the unit normal N^i is defined by

$$(2.5) \quad g_{ij} B_\alpha^i N^j = 0,$$

$$(2.6) \quad g_{ij} N^i N^j = 1.$$

If (B_i^α, N_i) is the inverse matrix of (B_α^i, N^i) , we have

$$(2.7) \quad B_i^\alpha B_\beta^i = \delta_\beta^\alpha, \quad N_i N^i = 1, \quad B_i^\alpha N^i = B_\alpha^i N_i = 0.$$

Further, we use the following notation,

$$(2.8) \quad B_{\alpha\beta}^{ij\dots} = B_\alpha^i B_\beta^j \dots, \quad B_{\alpha\beta}^i = \frac{\partial^2 x^i}{\partial u^\alpha \partial v^\beta}, \quad B_{0\beta}^i = B_{\alpha\beta}^i v^\alpha$$

$$(2.9) \quad M_{\alpha\beta} = C_{ijk} B_{\alpha\beta}^{ij} N^k, \quad M_\alpha = C_{ijk} B_\alpha^i N^j N^k, \quad g_{\alpha\beta} = g_{ij} B_{\alpha\beta}^{ij}$$

For Induced Cartan connection $IC \Gamma \left(F_{\beta\gamma}^\alpha, N_\beta^\alpha, C_{\beta\gamma}^\alpha \right)$ on F^{n-1} , the Normal curvature vector H_α and fundamental h-tensor are given by

$$(2.10) \quad H_\alpha = N_i \left(B_{0\alpha}^i + N_j^i B_\alpha^j \right)$$

$$(2.11) \quad H_{\alpha\beta} = N_i \left(B_{\alpha\beta}^i + F_{jk}^i B_{\alpha\beta}^{jk} \right) + M_\alpha H_\beta.$$

Suppose N^{*i} is unit normal to the hypersurface F^{*n-1} of F^{*n} . $B_\alpha^i(u)$ is invariant under this transformation.

$$(2.12) \quad g_{ij}^* B_\alpha^i N^{*j} = 0, \quad g_{ij}^* N^{*i} N^{*j} = 1.$$

Contracting (2.5) by v^α , we have

$$(2.13) \quad y_j N^j = 0.$$

Now, contracting (1.10) by $N^i N^j$ and using (2.6), we have

$$g_{ij}^* N^i N^j = 1 + \left(X_i N^i \right)^2$$

or

$$(2.14) \quad g_{ij}^* \left(\pm \frac{N^i}{\sqrt{1 + \left(X_i N^i \right)^2}} \right) \left(\pm \frac{N^j}{\sqrt{1 + \left(X_i N^i \right)^2}} \right) = 1.$$

To fix the orientation, we take positive sign comparing (2.13) and second equation of (2.11), we get

$$(2.15) \quad N^{*j} = + \frac{N^j}{\sqrt{1 + \left(X_i N^i \right)^2}},$$

contracting (1.10) by $B_\alpha^i N^{*j}$ and using first equation of (2.11) and (2.14), we have

$$(2.16) \quad (X_i B_\alpha^i)(X_j N^j) = 0.$$

Suppose

$$(2.17) \quad X_i B_\alpha^i = 0,$$

contracting (2.16) by v^α , we have

$$X_i y^i = 0 \quad \text{or} \quad \beta = 0,$$

which is not possible for transformation. Hence, we have

$$(2.18) \quad X_j N^j = 0.$$

In view of (2.17) and (2.14), we have

$$(2.19) \quad N^{*i} = N^i.$$

Lemma⁵ 2.1: *The induced Cartan (Berwald) connection of F^{n-1} coincide intrinsic Cartan (Berwald) connection of F^{n-1} iff*

$$M_{\alpha\beta} = 0 \quad \text{or} \quad H_\alpha = 0$$

3. Hyperplane

Definition⁵ 3.1: *If each path of a hypersurface F^{n-1} with respect to induced Cartan connection $IF\Gamma$ is a path of enveloping space F^n with respect to $F\Gamma$, then F^{n-1} is called hyperplane of first kind.*

A path with respect to Finsler connection $(F_{jk}^i, N_j^i, C_{jk}^i)$ is given by

$$(3.1) \quad \frac{d^2 x^i}{ds^2} + N_0^i \left(x, \frac{dx}{ds} \right) = 0.$$

The name Hyperplane of first kind has been given by Rapcsak⁶. Kikuchi⁷ said it totally extremal hypersurface.

Lemma⁵ 3.1: A hypersurfaces F^{n-1} is a hyperplane of first kind iff normal curvature vector H_α vanishes identically.

Theorem 3.1: Let X_i be normalized semi-parallel vector field with respect to Cartan connection on F^n . Then hypersurfaces F^{n-1} is a hyperplane of first kind iff hypersurfaces F^{*n-1} is a hyperplane of first kind.

Proof: The normal curvature vector of F^{*n-1} is given by

$$(3.2) \quad H_\alpha^* = N_i^* (B_{0\alpha}^i + N_j^* B_\alpha^j).$$

In view of (1.10), (1.14) and (2.19), equation (3.2) can be written as

$$(3.3) \quad H_\beta^* = H_\beta + \frac{\rho B_\beta^j (g_{jk} - X_j X_k) y^k}{2} X^i N_i.$$

In view of (2.18) and (3.3), we get

$$(3.8) \quad H_\beta^* = H_\beta.$$

From equation (3.8) and lemma 3.1, we have theorem 3.1.

Definition⁸ 3.2: If each h -path of a hypersurface F^{n-1} with respect to induced Cartan connection $IF\Gamma$ is a h -path of enveloping space F^n with respect to $F\Gamma$, then F^{n-1} is called hyperplane of second kind.

With respect to Finsler connection $(F_{jk}^i, N_j^i, C_{jk}^i)$ h -path is given by

$$(3.9) \quad \frac{dy^i}{ds} + N_j^i \frac{dx^j}{ds} = 0, \quad \frac{d^2 x^i}{ds^2} + F_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} = 0.$$

The name Hyperplane of second kind has been given by Rapcsak⁶. Kikuchi⁷ named it weakly geodesic hypersurface.

Lemma⁷ 3.2: With respect to Cartan connection a hypersurface is hyperplane of second kind iff $H_{\alpha\beta} = 0$.

Theorem 3.2: Let X_i be normalized semi-parallel vector field with respect to Cartan connection on F^n . Then hypersurfaces F^{n-1} is a

hyperplane of second kind iff hypersurfaces F^{*n-1} is a hyperplane of second kind.

Proof : Fundamental h-tensor of F^{*n-1} is given by

$$(3.10) \quad H_{\alpha\beta}^* = N_i^* \left(B_{\alpha\beta}^i + F_{jk}^{*i} B_{\alpha\beta}^{jk} \right) + H_{\beta}^* M_{\alpha}^*$$

$$(3.11) \quad M_{\alpha}^* = C_{ijk}^* B_{\alpha}^i N^{*j} N^{*k} = C_{ijk} B_{\alpha}^i N^j N^k = M_{\alpha}.$$

In view of (1.13) ,(2.19) , (3.8) and (3.11), equation (3.10) can be written as

$$(3.12) \quad H_{\alpha\beta}^* = H_{\alpha\beta} + X^i N_i \frac{\rho(g_{jk} - X_j X_k) B_{\alpha\beta}^{jk}}{2}.$$

In view of (2.19) and (3.12),we have

$$(3.13) \quad H_{\alpha\beta}^* = H_{\alpha\beta}.$$

From equation (3.13) and lemma 3.2,we have theorem 3.2

Definition⁵ 3.3: A hypersurface F^{n-1} of enveloping space F^n endowed with a Finsler connection $F\Gamma$, is called hyperplane of second kind if the unit normal vector N^i of F^{n-1} with respect to $L(x,y)$ is parallel along $(u^{\alpha}(s), v^{\alpha}(s))$ on F^{n-1} .

The name Hyperplane of third kind Rapcsak⁶. Kikuchi⁷ said it totally geodesic.

Lemma⁵ 3.3: A hypersurface is hyperplane of third kind with respect to Cartan connection iff $H_{\alpha\beta} = 0$ and $M_{\alpha\beta} = 0$.

Theorem 3.3: Let X_i be normalized semi-parallel vector field with respect to Cartan connection on F^n . Then hypersurfaces F^{n-1} is a hyperplane of third kind iff hypersurfaces F^{*n-1} is a hyperplane of third kind.

Proof: From equation (3.13) , we have

$$(3.14) \quad H_{\alpha\beta}^* = H_{\alpha\beta}$$

and $M_{\alpha\beta}^*$ of F^{*n-1} is given as

$$M_{\alpha\beta}^* = C_{ijk}^* B_{\alpha}^i B_{\beta}^j N^{*k} = C_{ijk} B_{\alpha}^i B_{\beta}^j N^k = M_{\alpha\beta}$$

or

$$(3.15) \quad M_{\alpha\beta}^* = M_{\alpha\beta}.$$

In view of (3.14), (3.15) and lemma 3.3, we have theorem 3.3.

4. Special Finsler Space

Definition⁵ 4.1: A Finsler space F^n is called Berwald space if the horizontal connection coefficient G_{jk}^i of Berwald connection is function of position co-ordinate only.

Theorem 4.1: Let X_i be normalized semi-parallel vector field with respect to Cartan connection on F^n . Then hyperplane F^{n-1} of first kind is Berwald space iff hyperplane F^{*n-1} of first kind is Berwald space, provided $B_i^{\alpha} X_i = 0$.

Proof: The non-linear connection of Berwald connection and cartan Connection are same³. The horizontal connection coefficient G_{jk}^{*i} of Berwald connection of F^{*n} is given as

$$(4.1) \quad G_{jk}^{*i} = \frac{\partial N_j^{*i}}{\partial y^k}.$$

In view of (1.14) and (4.1), we have

$$(4.2) \quad G_{jk}^{*i} = G_{jk}^i + \frac{\rho X^i (g_{jk} - X_j X_k)}{2}.$$

Induced Berwald connection of F^{*n-1} is given as

$$(4.3) \quad G_{\beta\gamma}^{*\alpha} = B_i^{\alpha} (B_{\beta\gamma}^{*i} + G_{jk}^i B_{\beta\gamma}^{jk}).$$

In view of (4.2) and (4.3), we have

$$G_{\beta\gamma}^{*\alpha} = G_{\beta\gamma}^{\alpha} + B_i^{\alpha} X^i \frac{\rho(g_{\beta\gamma} - X_{\beta} X_{\gamma})}{2}.$$

$$(4.4) \quad G_{\beta\gamma}^{*\alpha} = G_{\beta\gamma}^{\alpha} \quad \text{provided} \quad B_i^{\alpha} X^i = 0.$$

From (4.4), we have theorem 4.1.

Definition⁸ 4.2: A Finsler space F^n is called Landsberg space if (v)hv torsion tensor of Cartan connection P_{jk}^i vanishes identically.

Theorem 4.2: Let X_i be normalized semi-parallel vector field with respect to Cartan connection on F^n . Then hyperplane F^{n-1} of first kind is Landsberg space iff hyperplane F^{*n-1} of first kind is Landsberg space.

Proof: (v)hv torsion tensor of cartan connection P_{jk}^{*i} of F^{*n} is given by (1.15)

$$(4.5) \quad P_{jk}^{*i} = P_{jk}^i.$$

(v)hv torsion tensor of $P_{\beta\gamma}^{*\alpha}$ of F^{*n-1} is given as

$$(4.6) \quad P_{\beta\gamma}^{*\alpha} = P_{jk}^{*i} B_i^{\alpha} B_{\beta\gamma}^{jk}.$$

In view of (4.5) and (4.6), we have

$$(4.7) \quad P_{\beta\gamma}^{*\alpha} = P_{\beta\gamma}^{\alpha},$$

where $P_{\beta\gamma}^{\alpha}$ is (v)hv torsion tensor of F^{n-1} .

From equation (4.7), we have theorem 4.2.

Corollary 4.1: If we put $\rho = 0$ in (1.9b), we have⁸ parallel vector field X_i , in view of⁹ contravariant component X^i is function of position co-ordinate only. In case $\rho = 0$ these result is special case of⁴.

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