

Almost Pseudo Semi-conformally Symmetric Ricci-recurrent Space-time

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Abstract: The object of the present paper is to study almost pseudo semi-conformally symmetric Ricci-recurrent space-time. In this paper, it is proved that on an almost pseudo semi-conformally symmetric Ricci-recurrent manifold of dimension n ($n \geq 4$) if $r \neq 0$, then $r/2$ is an eigenvalue corresponding to the eigen vector μ . It is also observed that in an almost pseudo semi-conformally symmetric Ricci-recurrent perfect fluid space-time obeying Einstein's field equation with cosmological constant and non-zero constant scalar curvature having matter content as a perfect fluid whose velocity vector field is the vector field corresponding to the 1-form A , then the acceleration vector of fluid and expansion scalar must vanish. Finally, it is proved that an almost pseudo semi-conformally symmetric Ricci-recurrent dust fluid space-time obeying Einstein's field equation without cosmological constant is vacuum.

Keywords: Semi-conformal curvature tensor; Einstein field equations; Perfect fluid space-time; Dust fluid space-time; Vacuum.

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1. Introduction

The aim of the present work is to study certain investigations in general theory of relativity and cosmology by the coordinate free method of differential geometry. The basic difference between Riemannian and semi-Riemannian geometry are (i) the existence of null vector (i.e. $g(v, v) = 0$, for $v \neq 0$ where g is the metric tensor) in semi-Riemannian manifold but not

Riemannian manifold, (ii) the signature of metric tensor g in semi-Riemannian manifold is $(-, -, \dots, -, +, +, \dots, +)$ but in a Riemannian manifold the signature of g is $(+, +, \dots, +)$. Lorentzian manifold is a spacial case of semi-Riemannian manifold. The signature of metric tensor g in Lorentzian manifold is $(-, +, +, \dots, +)$. A Lorentzian manifold consists of three types of vectors such as time-like (i.e. $g(v, v) < 0$), spacelike (i.e. $g(v, v) > 0$) and null vector (i.e. $g(v, v) = 0$, for $v \neq 0$). In general, a Lorentzian manifold (M, g) may not have a globally time-like vector field. If (M, g) admits a globally time-like vector field, it is called time orientable Lorentzian manifold, physically known as space-time. The foundation of general relativity is based on a 4-dimensional space-time manifold which is the stage of present modelling of the physical world a torsionless, time-oriented Lorentzian manifold (M, g) .

In general relativity the matter content of the space-time is described by the energy momentum tensor. The matter content is assumed to be a fluid having density and pressure and possessing dynamical and kinematical quantities like velocity, acceleration, vorticity, shear and expansion. In a perfect fluid space-time¹, the energy momentum tensor T of type $(0, 2)$ is of the form

$$(1.1) \quad T(X, Y) = (\sigma + \rho)A(X)A(Y) + \rho g(X, Y),$$

where ρ is the isotropic pressure, σ is the energy density and A is a non-zero one-form such that $g(X, \mu) = A(X)$ for all X, μ and μ is the velocity vector field such that $g(\mu, \mu) = -1$. The fluid is called perfect because of the absence of heat conduction terms and stress terms corresponding to viscosity. Perfect- fluid space-times in a language of differential geometry are called quasi-Einstein spaces where A is metrically equivalent to a unit space-like vector field². If the isotropic pressure ρ is vanish in perfect fluid then it is said to be dust fluid. In a dust fluid space-time¹, the energy momentum tensor T of type $(0, 2)$ is of the form

$$(1.2) \quad T(X, Y) = \sigma A(X)A(Y),$$

The Einstein's field equation¹ with cosmological constant is given by

$$(1.3) \quad S(X, Y) - \frac{r}{2}g(X, Y) + \lambda g(X, Y) = kT(X, Y),$$

where S and r denotes the Ricci tensor and scalar curvature respectively, λ is the cosmological constant, T is the energy momentum tensor and $k \neq 0$. Einstein's field equation¹ without cosmological constant is given by

$$(1.4) \quad S(X, Y) - \frac{r}{2}g(X, Y) = kT(X, Y).$$

A symmetric $(0,2)$ type tensor field E on a semi-Riemannian manifold (M^n, g) is said to be a Codazzi tensor if it satisfies the Codazzi equation

$$(1.5) \quad (\nabla_U E)(V, X) = (\nabla_V E)(U, X),$$

for arbitrary vector fields U, V and X . The geometrical and topological consequences of the existence of a non-trivial Codazzi tensor on a Riemannian manifold have been studied by Derdzinski and Shen³.

2. Preliminaries

Kim⁴ introduced a curvature-like tensor named semi-conformal curvature tensor such that its components remain invariant under conharmonic transformation and obtained several results on semi-conformal curvature tensor and pseudo semi-conformally symmetric manifolds. The semi-conformal curvature tensor $\mathcal{P}$ of type $(0,4)$ on a Riemannian manifold is defined by

$$(2.1) \quad \begin{aligned} \mathcal{P}(U, V, X, Y) = & -(n-2)b\mathcal{C}(U, V, X, Y) \\ & + [a + (n-2)b]\mathcal{H}(U, V, X, Y), \end{aligned}$$

where a, b are constants not simultaneously zero, $\mathcal{C}$ is the conformal curvature tensor and $\mathcal{H}$ is the conharmonic curvature tensor. The Weyl conformal curvature tensor $\mathcal{C}$ is the traceless part of Riemann tensor given as

$$(2.2) \quad \mathcal{C}(U, V, X, Y) = \mathcal{R}(U, V, X, Y)$$

$$\begin{aligned}
& -\frac{1}{n-2} \left[S(V, X)g(U, Y) - S(U, X)g(V, Y) \right. \\
& \quad \left. + S(U, Y)g(V, X) - S(V, Y)g(U, X) \right] \\
& + \frac{r}{(n-1)(n-2)} [g(V, X)g(U, Y) - g(U, X)g(V, Y)],
\end{aligned}$$

where $\mathcal{R}$ is Riemannian curvature tensor and r denotes the scalar curvature. The number of algebraically independent components of the Ricci and the Weyl tensors equals that of the Riemannian curvature tensor. Since in general relativity, only the Ricci tensor is coupled to matter by the Einstein's fields equations, the conformal curvature tensor describes the pure gravity degrees of freedom. Divergence of conformal curvature tensor is given by

$$\begin{aligned}
(2.3) \quad (divC)(U, V)X &= \frac{n-3}{n-2} [(\nabla_U S)(V, X) - (\nabla_V S)(U, X) \\
& - \frac{1}{2(n-1)} \{g(V, X)dr(U) - g(U, X)dr(V)\}].
\end{aligned}$$

As a special subgroup of the conformal transformation group, Ishii⁵ introduced the notion of conharmonic transformation under which a harmonic function transforms into a harmonic function. The conharmonic curvature tensor $\mathcal{H}$ of type (0,4) on a Riemannian manifold (M^n, g) of dimension $n \geq 4$ defined as follows

$$\begin{aligned}
(2.4) \quad \mathcal{H}(U, V, X, Y) &= \mathcal{R}(U, V, X, Y) - \frac{1}{n-2} [S(V, X)g(U, Y) \\
& - S(U, X)g(V, Y) + S(U, Y)g(V, X) - S(V, Y)g(U, X)].
\end{aligned}$$

Shaikh and Hui⁶ showed that the conharmonic curvature tensor satisfies the symmetries and skew symmetries properties of the Riemannian curvature tensor as well as cyclic ones. Divergence of conharmonic curvature tensor is given by

$$\begin{aligned}
(2.5) \quad (div\mathcal{H})(U, V)X &= \frac{n-3}{n-2} [(\nabla_U S)(V, X) - (\nabla_V S)(U, X)] \\
& - \frac{1}{2(n-2)} [g(V, X)dr(U) - g(U, X)dr(V)].
\end{aligned}$$

In view of equations (2.2) and (2.4), equation (2.1) takes the form

$$(2.6) \quad \mathcal{P}(U, V, X, Y) = a\mathcal{C}(U, V, X, Y) - \frac{br}{n-1} [g(V, X)g(U, Y) - g(U, X)g(V, Y)]$$

In particular, if $a=1$ and $b=\frac{-1}{n-2}$, then the semi-conformal curvature tensor reduces to conformal curvature tensor whereas for $a=1$ and $b=0$, such a tensor turns into conharmonic curvature tensor. Also, equation (2.6) can be rewritten as

$$(2.7) \quad \mathcal{P}(U, V, X, Y) = a\mathcal{R}(U, V, X, Y) - \frac{a}{n-2} [S(V, X)g(U, Y) - S(U, X)g(V, Y) + S(U, Y)g(V, X) - S(V, Y)g(U, X)] - \frac{br}{(n-1)} [g(V, X)g(U, Y) - g(U, X)g(V, Y)].$$

De and Suh⁷ showed that the semi-conformal curvature tensor satisfies the symmetries and skew symmetries properties of the Riemannian curvature tensor as well as cyclic ones. Divergence of conharmonic curvature tensor is given by

$$(2.8) \quad (\operatorname{div} \mathcal{P})(U, V)X = -(n-2)b(\operatorname{div} \mathcal{C})(U, V)X + [a + (n-2)b](\operatorname{div} \mathcal{H})(U, V)X$$

In view of equations (2.3) and (2.5), equation (2.8) takes the form

$$(2.9) \quad (\operatorname{div} \mathcal{P})(U, V)X = \frac{n-3}{n-2} a[(\nabla_U S)(V, X) - (\nabla_V S)(U, X)] - \frac{(n-1)a + 2(n-2)b}{2(n-1)(n-2)} [g(V, X)dr(U) - g(U, X)dr(V)].$$

Also, from equation (2.7), we obtain

$$(2.10) \quad \sum_{i=1}^n \mathcal{P}(U, V, e_i, e_i) = 0 = \sum_{i=1}^n \mathcal{P}(e_i, e_i, U, V)$$

and

$$(2.11) \quad \sum_{i=1}^n \mathcal{P}(e_i, U, V, e_i) = \sum_{i=1}^n \mathcal{P}(U, e_i, e_i, V) = -\frac{\{a + (n-2)b\}r}{(n-2)} g(U, V).$$

Motivated by the study of on almost pseudo-conformally symmetric Ricci-recurrent manifolds with application to relativity by De⁸ and on pseudo semi-conformally symmetric manifolds by Kim⁴ in the present paper, we consider an almost pseudo semi-conformally symmetric Ricci-recurrent space-time. A Riemannian manifold (M^n, g) of dimension $n \geq 4$ is said to be almost pseudo semi-conformally symmetric manifold if its semi-conformal curvature tensor $\mathcal{P}$ of type (0,4) satisfies the relation

$$(2.12) \quad (\nabla_w \mathcal{P})(U, V, X, Y) = [A(W) + B(W)] \mathcal{P}(U, V, X, Y) \\ + A(U) \mathcal{P}(W, V, X, Y) + A(V) \mathcal{P}(U, W, X, Y) \\ + A(X) \mathcal{P}(U, V, W, Y) + A(Y) \mathcal{P}(U, V, X, W),$$

where A and B are two non-zero 1-forms. If in equation (2.12), $A = B$, then the manifold reduces to a pseudo semi-conformally symmetric manifold.

Definition 2.1: A non-flat Riemannian manifold (M^n, g) of dimension n ($n > 2$) is said to be Ricci-recurrent manifold⁸ if its Ricci tensor $S(X, Y)$ is non-zero and satisfies the condition

$$(2.13) \quad (\nabla_U S)(X, Y) = \eta(U) S(X, Y),$$

where η is non-zero 1-form, called the 1-form of recurrence and defined by

$$(2.14) \quad g(U, \mu) = \eta(U).$$

If $\eta = 0$, then the manifold reduces to Ricci symmetric manifold. Contracting equation (2.13), with respect to U and Y , we obtain

$$(2.15) \quad dr(X) = \eta(X)r$$

and contraction of equation (2.13), with respect to Y gives

$$(2.16) \quad (\nabla_X L)(U) = \eta(X)LU,$$

where L is the symmetric endomorphism corresponding to the Ricci tensor $S(X, Y)$ that is,

$$(2.17) \quad S(X, Y) = g(LX, Y).$$

Again, contracting with respect to X in equation (2.16), we get

$$(2.18) \quad (divLU) = \eta(LU).$$

However, it is known that⁸ $(divL)U = \frac{1}{2}dr(U)$. Thus, in view of equation (2.15), we have

$$(2.19) \quad \eta(LU) = \frac{1}{2}dr(U) = \frac{r}{2}\eta(U).$$

Also, it is known that⁸ in a Ricci-recurrent manifold the following relations hold

$$(2.20) \quad S(X, LY) = \frac{r}{2}S(X, Y)$$

and

$$(2.21) \quad S(\mathcal{R}(U, V)X, Y) + S(\mathcal{R}(U, V)Y, X) = 0.$$

3. Almost Pseudo Semi-conformally Symmetric Ricci-recurrent Manifold

Let (M^n, g) be n -dimensional ($n \geq 4$) almost pseudo semi-conformally symmetric manifold, then contracting equation (2.12), with respect to W and Y , we obtain

$$\begin{aligned}
 (3.1) \quad (\nabla_w \mathcal{P})(U, V, X, W) &= [A(W) + B(W)]\mathcal{P}(U, V, X, W) \\
 &+ A(U)\mathcal{P}(W, V, X, W) + A(V)\mathcal{P}(U, W, X, W) \\
 &+ A(X)\mathcal{P}(U, V, W, W) + A(W)\mathcal{P}(U, V, X, W)
 \end{aligned}$$

Using equations (2.10) and (2.11) in above equation (3.1), we have

$$\begin{aligned}
 (3.2) \quad (\operatorname{div} \mathcal{P})(U, V)X &= 2A(\mathcal{P}(U, V)X) + B(\mathcal{P}(U, V)X) \\
 &+ \frac{\{a + (n-2b)\}r}{n-2} [g(U, X)A(V) - g(V, X)A(U)].
 \end{aligned}$$

In view of equation (2.9), equation (3.2) takes the form

$$\begin{aligned}
 (3.3) \quad &\frac{n-3}{n-2} a[(\nabla_U S)(V, X) - (\nabla_V S)(U, X)] \\
 &- \left\{ \frac{(n-1)a + 2(n-2)b}{2(n-1)(n-2)} \right\} r [g(V, X)dr(U)g(U, X)dr(V)] \\
 &= 2A(\mathcal{P}(U, V)X) + B(\mathcal{P}(U, V)X) \\
 &+ \frac{\{a + (n-2b)\}r}{n-2} [g(U, X)A(V) - g(V, X)A(U)].
 \end{aligned}$$

Using equation (2.13) and (2.15), in above equation (3.3), we have

$$\begin{aligned}
 (3.4) \quad &\frac{n-3}{n-2} a[\eta(U)S(V, X) - \eta(V)S(U, X)] \\
 &- \left\{ \frac{(n-1)a + 2(n-2)b}{2(n-1)(n-2)} \right\} r [g(V, X)\eta(U) - g(U, X)\eta(V)] \\
 &= 2A(\mathcal{P}(U, V)X) + B(\mathcal{P}(U, V)X) \\
 &+ \frac{\{a + (n-2b)\}r}{n-2} [g(U, X)A(V) - g(V, X)A(U)].
 \end{aligned}$$

Taking $X = \mu$ in above equation (3.4), we have

$$\begin{aligned}
 (3.5) \quad &\frac{n-3}{n-2} a[\eta(U)S(V, \mu) - \eta(V)S(U, \mu)] \\
 &- \left\{ \frac{(n-1)a + 2(n-2)b}{2(n-1)(n-2)} \right\} r [g(V, \mu)\eta(U) - g(U, \mu)\eta(V)] = 0,
 \end{aligned}$$

i.e.

$$\begin{aligned}
 (3.6) \quad & \eta(U) \left[\frac{n-3}{n-2} aS(V, \mu) - \frac{\{(n-1)a + 2(n-2)b\}r}{2(n-1)(n-2)} A(V) \right] \\
 & = \eta(V) \left[\frac{n-3}{n-2} aS(U, \mu) - \frac{\{(n-1)a + 2(n-2)b\}r}{2(n-1)(n-2)} A(U) \right].
 \end{aligned}$$

Putting $U = LU$ in above equation, we have

$$\begin{aligned}
 (3.7) \quad & \eta(LU) \left[\frac{n-3}{n-2} aS(V, \mu) - \frac{\{(n-1)a + 2(n-2)b\}r}{2(n-1)(n-2)} A(V) \right] \\
 & = \eta(V) \left[\frac{n-3}{n-2} aS(LU, \mu) - \frac{\{(n-1)a + 2(n-2)b\}r}{2(n-1)(n-2)} A(LU) \right].
 \end{aligned}$$

In view of equations (2.19) and (2.20), equation (3.7) takes the form

$$\begin{aligned}
 (3.8) \quad & r\eta(U) \left[\frac{n-3}{n-2} aS(V, \mu) - \frac{\{(n-1)a + 2(n-2)b\}r}{2(n-1)(n-2)} A(V) \right] \\
 & = r\eta(V) \left[\frac{n-3}{n-2} aS(U, \mu) - \frac{\{(n-1)a + 2(n-2)b\}r}{(n-1)(n-2)} A(LU) \right],
 \end{aligned}$$

which implies either $r = 0$ or

$$\begin{aligned}
 (3.9) \quad & \eta(U) \left[\frac{n-3}{n-2} aS(V, \mu) - \frac{\{(n-1)a + 2(n-2)b\}r}{2(n-1)(n-2)} A(V) \right] \\
 & = \eta(V) \left[\frac{n-3}{n-2} aS(U, \mu) - \frac{\{(n-1)a + 2(n-2)b\}r}{(n-1)(n-2)} A(LU) \right].
 \end{aligned}$$

Since $\eta \neq 0$, equations (3.6) and (3.9) together with the condition $r \neq 0$ gives $A(LU) = \frac{r}{2} A(U)$, that is

$$(3.10) \quad S(U, \mu) = \frac{r}{2} g(U, \mu).$$

Therefore, we can state as follows

Theorem 3.1: *In an almost pseudo semi-conformally symmetric Ricci-recurrent manifold of dimension n ($n \geq 4$) if $r \neq 0$, then $r/2$ is an eigenvalue corresponding to the eigen vector μ .*

4. Almost Pseudo semi-conformally Symmetric Ricci-recurrent Space-time

We consider a perfect fluid almost pseudo semi-conformally symmetric Ricci-recurrent space-time of non-zero scalar curvature with velocity vector field μ obeying Einstein's field equation without cosmological constant. In view of equations (1.1) and (1.4), we have

$$(4.1) \quad S(X, Y) - \frac{r}{2}g(X, Y) = k[(\sigma + \rho)A(X)A(Y) + \rho g(X, Y)].$$

Putting $Y = \mu$ in above equation (4.1), we have

$$(4.2) \quad S(X, \mu) - \frac{r}{2}g(X, \mu) = k[(\sigma + \rho)g(X, \mu)g(\mu, \mu) + \rho g(X, \mu)].$$

Using theorem 3.1 in above equation (4.2), we obtain

$$(4.3) \quad -k\sigma A(X) = 0, \quad \text{since } g(\mu, \mu) = -1.$$

Since $k \neq 0$ and $A \neq 0$, it follows that from equation (4.3) that $\sigma = 0$. But this is unacceptable when pure matter exist, because in case of existence of pure matter $\sigma > 0$. Hence the space-time under consideration cannot contain pure matter.

Next, we consider a perfect fluid almost pseudo semi-conformally symmetric Ricci-recurrent space-time of non-zero scalar curvature with velocity vector field μ obeying Einstein's field equation with cosmological constant λ . In view of equations (1.1) and (1.3), we have

$$(4.4) \quad S(X, Y) - \frac{r}{2}g(X, Y) + \lambda g(X, Y) = k[(\sigma + \rho)A(X)A(Y) + \rho g(X, Y)].$$

Putting $Y = \mu$ in above equation (4.4), we have

$$(4.5) \quad S(X, \mu) - \frac{r}{2} g(X, \mu) + \lambda g(X, \mu) = k[(\sigma + \rho)g(X, \mu)g(\mu, \mu) + \rho g(X, \mu)].$$

Using theorem 3.1 in above equation (4.5), we obtain $-k\sigma A(X) = \lambda A(X)$, since $g(\mu, \mu) = -1$ which implies

$$(4.6) \quad \sigma = \frac{\lambda}{-k}.$$

Taking a frame field and contracting equation (4.4) over X and Y , we get

$$(4.7) \quad 3k\rho = -r + k\sigma + 4\lambda.$$

From equations (4.6) and (4.7), we obtain

$$(4.8) \quad \rho = \frac{-r + 4\lambda}{3k}.$$

From equations (4.6) and (4.8) it follows that σ and ρ are constant respectively provided that r is constant. It is known that the energy equation and the force equation for a perfect fluid respectively are

$$(4.9) \quad \mu.\sigma = -(\sigma + \rho)\text{div}\mu$$

and

$$(4.10) \quad (\sigma + \rho)\nabla_{\mu}\mu = -\text{grad}\rho - (\mu.\rho)\mu.$$

Since μ is constant, it follows from (4.9) that $\text{div}\mu = 0$, $\sigma + \rho \neq 0$. Again since ρ is constant provided that r is constant it follows from equation (4.10) that $\nabla_{\mu}\mu = 0$. But $\text{div}\mu$ represents the expansion scalar and $\nabla_{\mu}\mu$ represents the acceleration vector respectively. Thus, in this case both the expansion scalar and acceleration vector are zero. Thus, we can state as follows

Theorem 4.1: *If in an almost pseudo semi-conformally symmetric Ricci-recurrent perfect fluid space-time obeying Einstein's field equation with cosmological constant and non-zero constant scalar curvature the matter content is a perfect fluid whose velocity vector field is the vector*

field corresponding to the 1-form A , then the acceleration vector of fluid and expansion scalar must vanish.

Again, we consider a dust fluid almost pseudo semi-conformally symmetric Ricci-recurrent space-time of non-zero scalar curvature with velocity vector field μ obeying Einstein's field equation without cosmological constant. In view of equations (1.2) and (1.4), we have

$$(4.11) \quad S(X, Y) - \frac{r}{2}g(X, Y) = k\sigma A(X)A(Y).$$

Putting $Y = \mu$ in above equation (4.11), we have

$$(4.12) \quad S(X, \mu) - \frac{r}{2}g(X, \mu) = k\sigma g(X, \mu)g(\mu, \mu).$$

Using theorem 3.1 in above equation (4.12), we obtain $-k\sigma A(X) = 0$, since $g(\mu, \mu) = -1$ which implies

$$(4.13) \quad \sigma = 0.$$

From equations (1.2) and (4.13), we obtain

$$(4.14) \quad T(X, Y) = 0.$$

This means the space-time is free from matter. Thus, we can state as follows

Theorem 4.2: *If in an almost pseudo semi-conformally symmetric Ricci-recurrent dust fluid space-time obeying Einstein's field equation without cosmological constant is vacuum.*

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