

Unified Fractional Calculus Results Related with $\bar{H}$ – Function

Jyoti Mishra

Department of Mathematics, GGITS, Jabalpur

Email: jyoti.mishra198109@gmail.com

(Received June 27, 2016)

Abstract: In the present work we introduce a composition formula of the caputo – type MSM fractional derivatives with $\bar{H}$ – function. The obtained results are in terms of $\bar{H}$ – function. MGM fractional derivative provide a natural framework for the discussion of numerous kinds of real problems by the help of fractional derivative. Certain special cases of the main results given in this paper generalize many known results obtained recently. Results which follows as special cases of our theorem are also cited.

Keywords: $\bar{H}$ – function, Caputo– type MSM fractional derivatives, Erdelyi-Kober fractional integral.

1. Introduction

The fractional calculus is a theory of integrals and derivatives of arbitrary real or even complex order. It is a generalization of classical calculus and has various applications. Fractional calculus has been used to model physical and engineering processes, which are found to be best represented by fractional differential equations. In the recent years, fractional calculus has played a very important role in various fields like mechanics, electricity, chemistry, biology, economics, notably control theory, and signal and image processing. Major topics include continuous time random walk, Levy statistics, fractional Brownian motion, abnormal diffusion, vibration and control, fractional nucleon purpose kinetic model, power law, Riesz potential, fractional, nonlocal phenomena, fractional wavelet, fractional predator-prey system, soft matter mechanics, fractional signal and image processing, history-dependent process, porous media, fractional filters, biomedical engineering, fractional phase-locked loops, fractional variational principles, fractional transforms, singularities

analysis and integral representations for fractional differential systems, special functions related to half calculus, non Fourier heat conduction, fluid dynamics, chaos, acoustic dissipation, geophysics relaxation, creep, viscoelasticity, rheology and groundwater problems.

Significant and importance of the fractional integral operators involving various special functions, have found applications in various sub-fields of applicable mathematical analysis. Since last four decades, a number of workers like Love¹, Marichev², Purohit, Suthar and Kalla³, Rao, Garg and Kalla⁴, Kalla and Saxena^{5,6}, Saigo⁷⁻⁹, Saigo and Maeda¹⁰, Kiryakova¹¹⁻¹², Srivastava, Lin and Wang¹³, Kataria and Vellaisamy¹⁴ etc. have studied the properties, applications and different extensions of various hypergeometric operators of fractional integration. A detailed account of such operators along with their properties and applications can be found in the research monographs by Ansari, Yilmaz, and Alouni¹⁵, Prathima, Nambisan, and Kumari¹⁶, Debnath and Bhatta¹⁷, Kilbas and Sebastian¹⁸⁻²⁰, Kilbas, Srivastava and Trujillo²¹ etc.

2. Preliminaries

In 1987, Inayat-Hussain²² was introduced generalization form of Fox's H- Function, which is popularly known as $\bar{H}$ -Function is defined and represented in the following manner

$$\bar{H}_{p,q}^{m,n}[z] = \bar{H}_{p,q}^{m,n} \left[z \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1,n} & (a_j, \alpha_j)_{n+1,p} \\ (b_j, \beta_j; B_j)_{1,m} & (b_j, \beta_j)_{m+1,q} \end{matrix} \right. \right] \quad (2.1)$$

$$= \frac{1}{2\pi i} \int_L z^\xi \bar{\phi}(\xi) d\xi, \quad (z \neq 0),$$

where

$$\bar{\phi}(\xi) = \frac{\prod_{j=1}^m \Gamma(b_j - \beta_j \xi) \prod_{j=1}^n \{\Gamma(1 - a_j + \alpha_j \xi)\}^{A_j}}{\prod_{j=m+1}^q \{\Gamma(1 - b_j + \beta_j \xi)\}^{B_j} \prod_{j=n+1}^p \Gamma(a_j - \alpha_j \xi)} \quad (2.2)$$

It may be noted that the $\bar{\phi}(\xi)$ contains fractional powers of some of the

gamma functions and m, n, p, q are integers such that $l \leq m \leq q$, $l \leq n \leq p$, $(\alpha_j)_{1,p}$, $(\beta_j)_{1,q}$ are positive real number and $(A_j)_{1,n}$, $(B_j)_{m+1,q}$ may take non-integer values, which we assume to be positive for standardization purpose. $(\alpha_j)_{1,p}$ and $(\beta_j)_{1,q}$ are complex numbers.

The nature of contour L , sufficient conditions of convergence of defining integral (2.1) and other details about the $\overline{H}$ – Function can be seen in the paper of Inayat- Hussain²². The behavior of the $\overline{H}$ – Function for the small values of $|z|$ follows easily from a result given by Rathie²³

$$(2.3) \quad \overline{H}_{p,q}^{m,n}[z] = O(|z|^\alpha),$$

where

$$(2.4) \quad \alpha = \min_{1 \leq j \leq m} \operatorname{Re} \left(\frac{b_j}{\alpha_j} \right), |z| \rightarrow 0.$$

where m, n, a_i, A_i, α_i and b_j, B_j, β_j appear in the definition of the $\overline{H}$ – function. (sec 1.1). The $\overline{H}$ – function is analytic if $\mu \geq 0$ and the integral in (1) converges absolutely if $|\arg z| < \frac{1}{2} \mu_1 \pi$, where μ_1 is given by equation

$$\mu_1 = \sum_{j=1}^m |B_j| + \sum_{j=1}^q |b_j B_j| - \sum_{j=1}^n |a_j A_j| - \sum_{j=n+1}^q |A_j| > 0, 0 < |z| < \infty.$$

Throughout our present investigation, we use the following standard notations:

The corresponding fractional differential operators¹⁰ have their respective form as

$$(2.5) \quad \left(D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} f \right)(x) = \left(\frac{d}{dx} \right)^{[\operatorname{Re}(\gamma)]+1} \left(I_{0+}^{-\alpha', -\alpha, -\beta' + [\operatorname{Re}(\gamma)]+1, -\beta, -\gamma + [\operatorname{Re}(\gamma)+1]} f \right)(x),$$

$$(2.6) \quad \left(D_-^{\alpha, \alpha', \beta, \beta', \gamma} f\right)(x) = \left(-\frac{d}{dx}\right)^{[\operatorname{Re}(\gamma)]+1} \left(I_-^{-\alpha', -\alpha, -\beta' + [\operatorname{Re}(\gamma)]+1, -\beta, -\gamma + [\operatorname{Re}(\gamma)+1]} f\right)(x),$$

where $m = [\operatorname{Re}(\gamma)] + 1$ and $[\operatorname{Re}(\gamma)]$ denotes the integer part of $\operatorname{Re}(\gamma)$.

Saigo⁷ introduced the fractional integral and differential operators involving Gauss hyper geometric function ${}_2F_1$ as the kernel. For $\alpha, \beta, \gamma \in \mathbb{C}$ and $x > 0$ with $\operatorname{Re}(\alpha) > 0$, the left- and right-hand sided Saigo fractional integral operators are defined by

$$(2.7) \quad \left(I_{0+}^{\alpha, \beta, \gamma} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha + \beta, -\gamma; 1 - \frac{t}{x}\right) f(t) dt$$

and

$$(2.8) \quad \left(I_-^{\alpha, \beta, \gamma} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha + \beta, -\gamma; \alpha; 1 - \frac{t}{x}\right) f(t) dt$$

respectively.

(2.9)

$$\left(I_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) f(t) dt$$

and

$$(2.10) \quad \left(I_-^{\alpha, \alpha', \beta, \beta', \gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) f(t) dt$$

respectively.

The third Appell function F_3 (also known as Horn function²³), is defined by

$$F_3(\alpha, \alpha', \beta, \beta'; \gamma; x, y) = \sum_{m, n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!}$$

such that $\max\{|x|, |y|\} < 1$, where $(z)_n$ is the Pochhammer symbol, which is defined as

$$(z)_n = \begin{cases} 1 & \text{if } n = 0 \\ z(z+1)(z+2)\dots(z+(n+1)) & \text{if } n \in \mathbb{N}, \end{cases} \quad \text{for } z \in \mathbb{C}.$$

The corresponding fractional differential operators are

$$(2.11) \quad (D_{0+}^{\alpha, \beta, \gamma} f)(x) = \left(\frac{d}{dx} \right)^{[\operatorname{Re}(\alpha)]+1} \left(I_{0+}^{-\alpha + [\operatorname{Re}(\alpha)]+1, -\beta - [\operatorname{Re}(\alpha)-1, \alpha + \gamma - [\operatorname{Re}(\alpha)-1]]} f \right)(x),$$

$$(2.12) \quad (D_-^{\alpha, \beta, \gamma} f)(x) = \left(-\frac{d}{dx} \right)^{[\operatorname{Re}(\alpha)]+1} \left(I_-^{-\alpha + [\operatorname{Re}(\alpha)]+1, -\beta - [\operatorname{Re}(\alpha)]-1, \alpha + \gamma} f \right)(x),$$

where now $m = [\operatorname{Re}(\alpha)] + 1$. For $\beta = -\alpha$ and $\beta = 0$ in (2.7)-(2.10), we get the corresponding Riemann – Liouville and Erdelyi – Kober fractional operators respectively⁷.

The Gauss hyper geometric function is related to third Appell function as

$$F_3(\alpha, \gamma - \alpha, \beta, \gamma - \beta, \gamma; x; y) = {}_2F_1(\alpha, \beta, \gamma; x + y - xy).$$

The MSM fractional operators (2.3)-(2.6) are connected to Saigo operators (2.7)-(2.10) by

$$(2.13) \quad (I_{0+}^{\alpha, 0, \beta, \beta', \gamma} f)(x) = (I_{0+}^{\gamma, \alpha - \gamma, -\beta} f)(x), (I_-^{\alpha, 0, \beta, \beta', \gamma} f)(x) = (I_{0+}^{\gamma, \alpha - \gamma, -\beta} f)(x)$$

and

$$(2.14) \quad (D_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} f)(x) = (D_{0+}^{\gamma, \alpha' - \gamma, \beta' - \gamma} f)(x), (D_-^{\alpha, \alpha', \beta, \beta', \gamma} f)(x) = (D_-^{\gamma, \alpha' - \gamma, \beta' - \gamma} f)(x).$$

The following are well known results for MSM integral operators of power functions⁴

Lemma 2.1: Let $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}$ such that $\operatorname{Re}(\gamma) > 0$.

(a) If $\operatorname{Re}(\rho) > \max\{0, \operatorname{Re}(\alpha' - \beta'), \operatorname{Re}(\alpha + \alpha' + \beta - \gamma)\}$, then

$$(2.15) \quad (I_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho-1})(x) = \frac{\Gamma(\rho)\Gamma(-\alpha' + \beta' + \rho)\Gamma(-\alpha - \alpha' - \beta + \gamma + \rho)}{\Gamma(\beta' + \rho)\Gamma(-\alpha - \alpha' + \gamma + \rho)\Gamma(-\alpha' - \beta + \gamma + \rho)} x^{-\alpha - \alpha' + \gamma + \rho - 1}.$$

(b) If $\operatorname{Re}(\rho) > \max\{\operatorname{Re}(\beta), \operatorname{Re}(-\alpha - \alpha' + \gamma), \operatorname{Re}(-\alpha - \beta' + \gamma)\}$, then

$$(2.16) \quad \left(I_{-}^{\alpha, \alpha', \beta, \beta', \gamma} t^{-\rho}\right)(x) = \frac{\Gamma(-\beta + \rho) \Gamma(\alpha + \alpha' - \gamma + \rho) \Gamma(\alpha + \beta' - \gamma + \rho)}{\Gamma(\rho) \Gamma(\alpha - \beta + \rho) \Gamma(\alpha + \alpha' + \beta' + \gamma + \rho)} x^{-\alpha - \alpha' + \gamma - \rho}.$$

In subsequent theorems, the conditions for the absolute convergence of the integral involved in (2.1) are assumed. Also, the contour C of integration is assumed to be the imaginary axis, i.e., $\operatorname{Re}(s) = 0$.

3. Marichev-Saigo-Maeda (MSM) Fractional Integration Representation related to the $\bar{H}$ - Function

Our first set of results are contained in Theorem 3.1 below which is related to the left-hand sided MSM fractional integration of the $\bar{H}$ - function.

Theorem 3.1: *The following Marichev-Saigo-Maeda (MSM) Fractional Integration representation formulas hold true for the $\bar{H}$ - function*

$$(3.1) \quad \left(I_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{\rho-1} \bar{H}_{p,q}^{m,n} \left[at^u \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,p} \end{matrix} \right| \right] \right) \right)(x) \\ = x^{-\alpha - \alpha' + \gamma + \rho - 1} \bar{H}_{p+3, q+3}^{m, n+3} \left[ax^\mu \left| \begin{matrix} (1-\rho, \mu, 1) & (1+\alpha' - \beta' - \rho, \mu, 1) \\ (b_j, B_j, \beta_j)_{1,q} & (1-\beta' - \rho, \mu, 1) \\ (1+\alpha + \alpha' + \beta - \gamma - \rho, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (1+\alpha + \alpha' - \gamma - \rho, \mu, 1) & (1+\alpha' + \beta - \gamma - \rho, \mu, 1) \end{matrix} \right| \right]$$

provided that $x > 0$ and each member of (3.1) exist i.e. $\alpha, \alpha', \beta, \beta', \gamma, \rho, a \in \mathbb{C}$ be such that $\operatorname{Re}(\gamma), \mu > 0$ and $\operatorname{Re}(\rho) > \max\{0, \operatorname{Re}(\alpha - \beta), \operatorname{Re}(\alpha + \alpha' + \beta - \gamma)\}$.

Proof. The left-hand side of (2.15) is given by

$$(3.2) \quad \left(I_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{\rho-1} \frac{1}{2\pi i} \int_c x(s) (at^\mu)^{-s} ds \right) \right)(x),$$

where $x(s)$ is defined as (2.2). Interchanging the order of integration and using (2.15) & (2.18) is equal to

$$\frac{1}{2\pi i} \int_c x(s) a^{-s} \left(I_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} t^{\rho - \mu s - 1} \right) (x) ds = x^{-\alpha - \alpha' + \gamma + \rho - 1} \frac{1}{2\pi i} \int_c x(s) x_1(s) (ax^\mu)^{-s} ds,$$

where

$$x_1(s) = \frac{\Gamma(\rho - \mu s) \Gamma(-\alpha' + \beta' + \rho - \mu s) \Gamma(-\alpha - \alpha' - \beta + \gamma + \rho - \mu s)}{\Gamma(\beta' + \rho - \mu s) \Gamma(-\alpha - \alpha' + \gamma + \rho - \mu s) \Gamma(-\alpha' - \beta + \gamma + \rho - \mu s)},$$

the result now follows from (2.1).

In view of (2.13), we have the following result for Saigo operators.

For the $\bar{H}$ -function

Corollary 3.1: Let $\alpha, \beta, \gamma, \rho, a \in \mathbb{C}$ be such that $\operatorname{Re}(\alpha), \mu > 0$ and $\operatorname{Re}(\rho) > \max\{0, \operatorname{Re}(\beta - \gamma)\}$. Then the left-hand sided generalized fractional integration $I_{0+}^{\alpha, \beta, \gamma}$ of the $\bar{H}$ -function is given for $x > 0$ by

$$\begin{aligned} & \left(I_{0+}^{\alpha, \beta, \gamma} \left(t^{\rho-1} \bar{H}_{p,q}^{m,n} \left[at^u \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,p} \end{matrix} \right| \right] \right) \right) (x) \\ &= x^{-\beta+\rho-1} \bar{H}_{p+2,q+2}^{m,n+2} \left[ax^\mu \left| \begin{matrix} (1-\rho, \mu-1) & (1+\beta-\gamma-\rho, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} & (1-\beta-\rho, \mu, 1) & (1+\alpha-\gamma-\rho, \mu, 1) \end{matrix} \right| \right]. \end{aligned}$$

The above corollary leads to Erdelyi-Kober fractional integral as follows.

Corollary 3.2:

$$\begin{aligned} & \left(I_{\gamma\alpha}^+ \left(t^{\rho-1} \bar{H}_{p,q}^{m,n} \left[at^u \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,p} \end{matrix} \right| \right] \right) \right) (x) \\ &= x^{\rho-1} \bar{H}_{p+1,q+1}^{m,n+1} \left[ax^\mu \left| \begin{matrix} (1-\gamma-\rho, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} & (1-\alpha-\gamma-\rho, \mu, 1) \end{matrix} \right| \right], \end{aligned}$$

provided that $\alpha, \gamma, \rho, a \in \mathbb{C}$ be such that $\operatorname{Re}(\alpha), \mu > 0$ and $\operatorname{Re}(\rho) > \max\{0, \operatorname{Re}(-\gamma)\}$. Then the left-hand sided Erdelyi-Kober fractional integration $I_{\gamma, \alpha}^+ (= I_{0+}^{\alpha, 0, \gamma})$ of the I -function is given for $x > 0$.

The following result corresponds to the right-hand sided MSM fractional integration of the $\bar{H}$ – function.

Theorem 3.2: Each of the following MSM fractional integration representation formulas hold true for the $\bar{H}$ – function

(3.3)

$$\left(I_-^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{-\rho} \bar{H}_{p,q}^{m,n} \left[at^{-\mu} \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,p} \end{matrix} \right| \right] \right) \right) (x)$$

$$= x^{-\alpha - \alpha' + \gamma - \rho} \bar{H}_{p+3, q+3}^{m, n+3} \left[ax^{-\mu} \left| \begin{matrix} (1 + \beta - \rho, \mu, 1) & (1 - \alpha - \alpha' + \gamma - \rho, \mu, 1) \\ (b_j, B_j, \beta_j)_{1,q} & (1 - \rho, \mu, 1) \\ (1 - \alpha - \beta' + \gamma - \rho, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (1 - \alpha + \beta - \rho, \mu, 1) & (1 - \alpha - \alpha' - \beta' + \gamma - \rho, \mu, 1) \end{matrix} \right| \right],$$

$$\left(I_-^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{-\rho} \bar{H}_{p,q}^{m,n} \left[at^{-\mu} \left| \begin{matrix} (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,p} \end{matrix} \right| \right] \right) \right) (x)$$

$$= x^{-\alpha - \alpha' + \gamma - \rho} \bar{H}_{p+3, q+3}^{m, n+3}$$

$$\left[ax^{-\mu} \left| \begin{matrix} (1 + \beta - \rho, \mu, 1) & (1 - \alpha - \alpha' + \gamma - \rho, \mu, 1) & (1 - \alpha - \beta' + \gamma - \rho, \mu, 1) & (a_i, A_i, \alpha_i)_{1,p} \\ (b_j, B_j, \beta_j)_{1,q} & (1 - \rho, \mu, 1) & (1 - \alpha + \beta - \rho, \mu, 1) & (1 - \alpha - \alpha' - \beta' + \gamma - \rho, \mu, 1) \end{matrix} \right| \right]$$

for $x > 0$ provided that $\alpha, \alpha', \beta, \beta', \gamma, \rho, a \in \mathbb{C}$ be such that $\text{Re}(\gamma), \mu > 0$ and $\text{Re}(\rho) > \max\{\text{Re}(\beta), \text{Re}(-\alpha - \alpha' + \gamma), \text{Re}(-\alpha - \beta' + \gamma)\}$.

Proof. Using (2.1), the left hand side of (3.3) is given by

$$(3.4) \quad \left(I_-^{\alpha, \alpha', \beta, \beta', \gamma} \left(t^{-\rho} \frac{1}{2\pi i} \int_C \mathcal{X}(s) (at^{-\mu})^{-s} ds \right) \right) (x),$$

where $x(s)$ is defined as (2.2). Interchanging the order of integration and using (2.16) & (3.4) reduces to

$$\frac{1}{2\pi i} \int_C \mathcal{X}(s) a^{-s} \left(I_-^{\alpha, \alpha', \beta, \beta', \gamma} t^{-(\rho - \mu s)} \right) (x) ds = x^{-\alpha - \alpha' + \gamma - \rho} \frac{1}{2\pi i} \int_C \mathcal{X}(s) \mathcal{X}_2(s) (ax^{-\mu})^{-s} ds,$$

where

$$\mathcal{X}_2(s) = \frac{\Gamma(-\beta + \rho - \mu s) \Gamma(\alpha + \alpha' - \gamma + \rho - \mu s) \Gamma(\alpha + \beta' - \gamma + \rho - \mu s)}{\Gamma(\rho - \mu s) \Gamma(\alpha - \beta + \rho - \mu s) \Gamma(\alpha + \alpha' - \beta' - \gamma + \rho - \mu s)}$$

The result now follows from (2.1).

The Saigo and Erdelyi-Kober fractional integration of the $\bar{H}$ -function follow as corollaries.

Corollary 3.3: Let $\alpha, \beta, \gamma, \rho, a \in \mathbb{C}$ be such that $\operatorname{Re}(\alpha), \mu > 0$ and $\operatorname{Re}(\rho) > \max\{\operatorname{Re}(-\beta), \operatorname{Re}(-\gamma)\}$. Then the right-hand sided generalized fractional integration $I_-^{\alpha, \beta, \gamma}$ of the $\bar{H}$ -function is given for $x > 0$ by

$$\begin{aligned} & \left(I_-^{\alpha, \beta, \gamma} \left(t^{-\rho} \bar{H}_{p, q}^{m, n} \left[\begin{matrix} (a_i, A_i, \alpha_i)_{1, p} \\ (b_j, B_j, \beta_j)_{1, p} \end{matrix} \right] \right) \right) (x) \\ &= x^{-\beta - \rho} \bar{H}_{p+2, q+2}^{m, n+2} \left[\begin{matrix} (1-\gamma-\rho, \mu, 1) & (1-\beta-\rho, \mu, 1) \\ (b_j, B_j, \beta_j)_{1, q} & (1-\rho, \mu, 1) \end{matrix} \right] \end{aligned}$$

Corollary 3.4: Let $\alpha, \gamma, \rho, a \in \mathbb{C}$ be such that $\operatorname{Re}(\alpha), \mu > 0$ and $\operatorname{Re}(\rho) > \max\{0, \operatorname{Re}(-\gamma)\}$. Then the right-hand sided Erdelyi-Kober fractional integration $K-\gamma, \alpha (= I-\alpha, 0, \gamma)$ of the $\bar{H}$ -function is given for $x > 0$ by

$$\begin{aligned} & \left(\bar{k}_{\gamma, \alpha} \left(t^{-\rho} \bar{H}_{p, q}^{m, n} \left[\begin{matrix} (a_i, A_i, \alpha_i)_{1, p} \\ (b_j, B_j, \beta_j)_{1, p} \end{matrix} \right] \right) \right) (x) \\ &= x^{-\rho} \bar{H}_{p+1, q+1}^{m, n+1} \left[\begin{matrix} (1-\gamma-\rho, \mu, 1) & (a_i, A_i, \alpha_i)_{1, p} \\ (b_j, B_j, \beta_j)_{1, q} & (1-\alpha-\gamma-\rho, \mu, 1) \end{matrix} \right]. \end{aligned}$$

References

1. E. R. Love, Some Integral Equations Involving Hypergeometric Functions, *Proc. Edin. Math. Soc.*, **15** (3) (1967) 169–198.
2. O. I. Marichev, Volterra Equation of Mellin convolution Type with a Horn Function in the Kernel, *Izvestiya Akademii Nauk SSSR*, **1** (1974) 128–129.
3. S. D. Purohit, D. L. Suthar, and S. L. Kalla, Saigo-Maeda Fractional Integration Operators of the Bessel Functions, *Le Matematiche*, **67** (2012) 21–32.
4. A. Rao, M. Garg, and S. L. Kalla, Caputo-Type Fractional Derivative of a Hypergeometric Intergral Operator, *Kuwait J. Sci. Eng*, **37(1A)** (2010) 15–29.
5. S.L.Kalla and R.K.Saxena, Integral Operators Involving Hypergeometric Functions, *Math. Z.*, **108** (1969) 231–234.
6. S. L. Kalla and R.K. Saxena, Integral Operators Involving Hypergeometric Functions II, *Univ. Nac. Tucum'an, Rev. Ser.*, **24(A)** (1974) 31–36.
7. M. Saigo, A Remark on Integral Operators Involving the Gauss Hypergeometric Functions, *Math. Rep. Kyushu Univ.*, **11(2)** (1978) 135–143.
8. M. Saigo, A Certain Boundary Value Problem for the Euler-Darboux Equation I, *Math. Japonica*, **24** (4) (1979) 377–385.
9. M. Saigo, A Certain Boundary Value Problem for the Euler-Darboux Equation II, *Math. Japonica*, **25(2)** (1980) 211–220.
10. M. Saigo and N. Maeda, More Generalization of Fractional Calculus, *Transform Methods and Special Function. Ed. by P. Rusev, I. Dimovski, and V. Kiryakova*, (1998) 386–400.
11. V. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman Scientific & Tech., Essex, 1994.
12. V. Kiryakova, A Brief Story About the Operators of the Generalized Fractional Calculus, *Fract. Calc. Appl. Anal.*, **11(2)** (2008) 203–220.
13. H. M. Srivastava, S. D. Lin, and P. Y. Wang, Some Fractional-Calculus Results for the H-function Associated with a Class of Feynman Integrals, *Russian Journal of Mathematical Physics*, **13(1)** (2006) 94–100.
14. K. K. Kataria and P. Vellaisamy, Some Fractional Calculus Results Associated with the I-function, *To appear in Le Matematiche*, (2015).
15. I. S. Ansari, F. Yilmaz, and M. S. Alouni, On the Sum of Squared η -random Variates with Application to the Performance of Wireless Communication Systems, *Vehicular Technology Conference (VTC Spring), 2013 IEEE 77th*, (2013) 1–6.
16. J. Prathima, T. M. V. Nambisan, and K. S. Kumari, A Study of I-function of Several Complex Variables, *International Journal of Engineering Mathematics*, **2014** (2014), Article ID 931395.
17. L. Debnath and D. Bhatta, *Integral Transforms and Their Applications*, Chapman and Hall/CRC Press, Boca Raton FL, 2006.

18. A. A. Kilbas and N. Sebastian, Fractional Integration of the Product of Bessel Functions of the First Kind, *Fract. Calc. and Appl. Analysis*, **13(2)** (2010) 159–175.
19. A. A. Kilbas and N. Sebastian, Generalized Fractional Differentiation of Bessel Function of the First Kind, *Mathematica Balkanica*, **22** (2008) 323–346.
20. A. A. Kilbas and N. Sebastian, Generalized Fractional Integration of Bessel Functions of the First Kind, *Integral Transform and Special Functions*, **19(12)** (2008) 869–883.
21. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North Holland, Elsevier, 2006.
22. A. A. Inayat-Hussain, New Properties of Hypergeometric Series Derivable from Feynman Integrals: II A Generalization of the H-function, *J. Phys. A: Math. Gen.*, **20** (1987) 4119–4128.
23. A. K. Rathie, A New Generalization of Generalized Hypergeometric Functions, *Le Mathematique*, **52** (1997) 297–310.
24. A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions*, Vol. I. New York, McGraw-Hill, 1953.
25. C. Fox, The G and H-functions as Symmetrical Fourier Kernels, *Trans. Amer. Math. Soc.*, **98** (1961) 395–429.
26. X. Minghua, W. Yik-Chung, and A. Sonia, Exact Outage Probability of Dual-hop CSI-assisted AF Relaying over Nakagami-m Fading Channels, *IEEE Transactions on Signal Processing*, **60(10)** (2012) 5578–5583.
27. A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev, *Integrals and Series More Special Functions*, Vol. 3. New York, Gordon and Breach, 1990.