

Exponential Change of Finsler Metric by h -Vector

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Abstract: The purpose of the present paper is to find the necessary and sufficient conditions under which an exponential change of Finsler metric by h -vector becomes a projective change. We have also found the condition under which this change of Finsler metric of Douglas space gives a Douglas space.

Keywords: Finsler metric, Exponential change, Projective change, Douglas space.

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1. Introduction

Let $F^n = (M^n, L)$ be an n -dimensional Finsler space on a differentiable manifold M^n , equipped with the fundamental function $L(x, y)$. In 1971, Matsumoto¹ introduced the transformation of Finsler metric

$$(1.1) \quad L^*(x, y) = L(x, y) + \beta(x, y),$$

where $\beta(x, y) = b_i(x)y^i$ and $b_i(x)$ are components of a covariant vector depending on positional coordinates only. If L is a metric function of Riemannian space then L^* reduces to the metric function of Randers space. Such a Finsler metric was first introduced by G. Randers² from the standpoint of general theory of relativity and applied to the theory of electron microscope by R. S. Ingarden³ who called a Finsler space with such metric

as Randers space. The geometrical properties of this space have been studied by various authors⁴⁻⁷.

In 1978, Numata⁸ studied the properties of $F^{*n} = (M^n, L^*)$ which is obtained from a Minkowskian space $F^n = (M^n, L)$ by the transformation (1.1). In 1984, Shibata⁹ studied the properties of Finsler space $\bar{F}^n = (M^n, \bar{L})$ whose metric function $\bar{L}(x, y)$ is obtained from $L(x, y)$ by the relation

$$\bar{L}(x, y) = f(L, \beta)$$

where f is positively homogeneous of degree one in L and β . This change of Finsler metric is called a β -change. The change (1.1) is a particular case of β -change called Randers change. Shukla, Prasad and Pandey¹⁰ introduced the transformation of Finsler metric

$$(1.2) \quad \bar{L} = L e^{\beta/L}, \quad \text{where } \beta = b_i(x) y^i,$$

in 2012. The change (1.2) is another particular case of β -change, called exponential change. The authors¹¹⁻¹⁵ have dealt with Kropina change, Matsumoto change, Kropina-Randers change, exponential-Randers change, first approximate Matsumoto change of Finsler metric respectively and obtained conditions for these changes of metric to be projective and also the condition under which the said change transforms a Douglas space into a space of the same kind.

In the present paper we have obtained the above conditions for the transformation (1.2) replacing $b_i(x)$ in β by h -vector $b_i(x, y)$. Then we call such change (1.2) an exponential change of Finsler metric by h -vector. This h -vector b_i is not only a function of coordinates, but it is also a function of directional arguments satisfying the relation $\partial b_i / \partial y^j = \rho h_{ij}$, where ρ is a scalar function of (x, y) and $h_{ij} (= g_{ij} - l_i l_j)$ are components of angular metric tensor. We shall use the following result obtained by Shukla, Pandey and Joshi¹⁶

$$(1.3) \quad \partial_k \rho = -\frac{\rho}{L} l_k, \quad n > 2.$$

Let $\bar{F}^n = (M^n, \bar{L})$ where the Finsler space $\bar{F}^n$ is obtained from F^n by an exponential change of Finsler metric by h -vector. The quantities corresponding to $\bar{F}^n$ are denoted by putting bar over those quantities. The fundamental metric tensor g_{ij} , the normalized element of support l_i and angular metric tensor h_{ij} of F^n are given by

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j}, \quad l_i = \frac{\partial L}{\partial y^i} \quad \text{and} \quad h_{ij} = L \frac{\partial^2 L}{\partial y^i \partial y^j} = g_{ij} - l_i l_j.$$

We shall denote the partial derivative with respect to x^i and y^i by ∂_i and $\dot{\partial}_i$ respectively and write

$$\text{Then } \begin{cases} L_i = \dot{\partial}_i L, & L_{ij} = \dot{\partial}_i \dot{\partial}_j L, & L_{ijk} = \dot{\partial}_i \dot{\partial}_j \dot{\partial}_k L, \\ L_i = l_i, & L^{-1} h_{ij} = L_{ij}. \end{cases}$$

The geodesic of F^n are given by the system of differential equations

$$\frac{d^2 x^i}{ds^2} + 2G^i \left(x, \frac{dx}{ds} \right) = 0,$$

where $G^i(x, y)$ are positively homogeneous of degree two in y^i and are given by

$$(1.4) \quad 2G^i = g^{ij} (y^r \dot{\partial}_j \partial_r F - \partial_j F), \quad F = \frac{L^2}{2}$$

where g^{ij} are the inverse of g_{ij} .

The well known Berwald connection $B\Gamma = (G_{jk}^i, G_j^i)$ of a Finsler space is constructed from the quantity G^i appearing in the equation of geodesic and is given by⁶

$$G_j^i = \dot{\partial}_j G^i, \quad G_{jk}^i = \dot{\partial}_k G_j^i.$$

The Cartan's connection $C\Gamma = (F_{jk}^i, G_j^i, C_{jk}^i)$ is constructed from the metric function L by the following five axioms¹⁷

$$\left\{ \begin{array}{l} (i) \ g_{ij|k} = 0, \ (ii) \ g_{ij} |_k = 0, \ (iii) \ F_{jk}^i = F_{kj}^i, \\ (iv) \ F_{0k}^i = G_k^i, \ (v) \ C_{jk}^i = C_{kj}^i. \end{array} \right.$$

where $|_k$ and $|_k$ denote h - and v -covariant derivatives with respect to $C\Gamma$. It is clear that the h -covariant derivative of L with respect to $B\Gamma$ and $C\Gamma$ is the same and vanishes identically. Furthermore, the h -covariant derivatives of L_i , L_{ij} with respect to $C\Gamma$ are also zero.

We shall write

$$(1.5) \quad 2r_{ij} = b_{i|j} + b_{j|i}, \quad 2s_{ij} = b_{i|j} - b_{j|i}.$$

2. Difference Tensor D^i

The exponential change of Finsler metric L by h -vector is given by

$$(2.1) \quad \bar{L} = Le^{\beta/L}, \quad \text{where} \quad \beta(x, y) = b_i(x, y)y^i.$$

We may put

$$(2.2) \quad \bar{G}^i = G^i + D^i.$$

Then $\bar{G}_j^i = G_j^i + D_j^i$ and $\bar{G}_{jk}^i = G_{jk}^i + D_{jk}^i$, where $D_j^i = \dot{\partial}_j D^i$ and $D_{jk}^i = \dot{\partial}_k D_j^i$. The tensors D^i , D_j^i and D_{jk}^i are positively homogeneous in y^i of degree two, one and zero respectively. To find D^i we deal with equation¹ $L_{ij|k} = 0$, i.e.

$$(2.3) \quad \partial_k L_{ij} - L_{ijr} G_k^r - L_{rj} F_{ik}^r - L_{ir} F_{jk}^r = 0.$$

Since $\dot{\partial}_i \beta = b_i$, from (2.1), we have

$$(2.4 \text{ a}) \quad \bar{L}_i = e^{\beta/L} \left[\left(1 - \frac{\beta}{L} \right) L_i + b_i \right],$$

$$(2.4 \text{ b}) \quad \bar{L}_{ij} = e^{\beta/L} \left[\left(1 - \frac{\beta}{L} + \rho L \right) L_{ij} + \frac{\beta^2}{L^3} L_i L_j \right. \\ \left. + \frac{1}{L} b_i b_j - \frac{\beta}{L^2} (L_i b_j + L_j b_i) \right],$$

$$(2.4 \text{ c}) \quad \partial_j \bar{L}_i = e^{\beta/L} \left[\frac{\beta}{L^3} (\beta L_i - L b_i) \partial_j L + \frac{1}{L^2} (L b_i - \beta L_i) \partial_j \beta \right. \\ \left. + \left(1 - \frac{\beta}{L} \right) \partial_j L_i + \partial_j b_i \right],$$

$$(2.4 \text{ d}) \quad \partial_k \bar{L}_{ij} = e^{\beta/L} \left[\left(1 - \frac{\beta}{L} + \rho L \right) \partial_k L_{ij} + \left\{ \frac{1}{L^3} (\beta^2 + \rho L^3 - \rho L^2 \beta) L_{ij} \right. \right. \\ \left. - \frac{\beta^2}{L^5} (3L + \beta) L_i L_j + \frac{\beta}{L^4} (2L + \beta) (L_i b_j + L_j b_i) \right. \\ \left. - \frac{(L + \beta)}{L^3} b_i b_j \right\} \partial_k L + \left\{ \frac{1}{L^2} (\rho L^2 - \beta) L_{ij} + \frac{\beta}{L^4} (2L + \beta) L_i L_j \right. \\ \left. - \frac{(L + \beta)}{L^3} (L_i b_j + L_j b_i) + \frac{1}{L^2} b_i b_j \right\} \partial_k \beta + \frac{\beta}{L^3} (\beta L_j - L b_j) \partial_k L_i \\ + \frac{\beta}{L^3} (\beta L_i - L b_i) \partial_k L_j + \frac{1}{L^2} (L b_j - \beta L_j) \partial_k b_i \\ \left. + \frac{1}{L^2} (L b_i - \beta L_i) \partial_k b_j + L L_{ij} \partial_k \rho \right]$$

and

$$(2.4 \text{ e}) \quad \bar{L}_{ijk} = e^{\beta/L} \left[\left(1 - \frac{\beta}{L} + \rho L \right) L_{ijk} + \frac{\beta}{L^3} (\beta - \rho L^2) \right. \\ (L_i L_{jk} + L_j L_{ik} + L_k L_{ij}) + \frac{1}{L^2} (\rho L^2 - \beta) \\ \left. (b_i L_{jk} + b_j L_{ik} + b_k L_{ij}) + \frac{\beta}{L^4} (2L + \beta) \right]$$

$$\begin{aligned}
& (L_i L_j b_k + L_i L_k b_j + L_j L_k b_i) - \frac{(L + \beta)}{L^3} \\
& (L_i b_j b_k + L_j b_k b_i + L_k b_i b_j) - \frac{\beta^2}{L^5} (3L + \beta) L_i L_j L_k \\
& + \frac{1}{L^2} b_i b_j b_k \Big].
\end{aligned}$$

Since $\bar{L}_{ij|k} = 0$ in $\bar{F}^n$, after using (2.2), we have

$$\partial_k \bar{L}_{ij} - \bar{L}_{ijr} (G_k^r + D_k^r) - \bar{L}_{rj} (F_{ik}^r + {}^c D_{ik}^r) - \bar{L}_{ir} (F_{jk}^r + {}^c D_{jk}^r) = 0,$$

where¹⁰ $\bar{F}_{jk}^i - F_{jk}^i = {}^c D_{jk}^i$.

Substituting in the above equation the values of $\partial_k \bar{L}_{ij}$, $\bar{L}_{ir}$ and $\bar{L}_{ijr}$ from (2.4 d), (2.4 b) and (2.4 e) respectively and using (2.3) and then contracting the equation thus obtained with y^k , we get

$$\begin{aligned}
(2.5) \quad & e^{\beta/L} \left[2 \left\{ \frac{1}{L} (L - \beta + \rho L^2) L_{ijr} + \frac{\beta}{L^3} (\beta - \rho L^2) (L_i L_{jr} + L_j L_{ir} + L_r L_{ij}) \right. \right. \\
& + \frac{1}{L^2} (\rho L^2 - \beta) (b_i L_{jr} + b_j L_{ir} + b_r L_{ij}) + \frac{\beta}{L^4} (2L + \beta) (L_i L_j b_r + L_i L_r b_j \\
& + L_j L_r b_i) - \frac{L + \beta}{L^3} (L_i b_j b_r + L_j b_r b_i + L_r b_i b_j) - \frac{\beta^2}{L^5} (3L + \beta) L_i L_j L_r \\
& \left. \left. + \frac{1}{L^2} b_i b_j b_r \right\} D^r + \left\{ \frac{1}{L} (L - \beta + \rho L^2) L_{jr} + \frac{\beta^2}{L^3} L_r L_j + \frac{1}{L} b_r b_j \right. \right. \\
& - \frac{\beta}{L^2} (L_j b_r + L_r b_j) \left. \right\} D_i^r + \left\{ \frac{1}{L} (L - \beta + \rho L^2) L_{ir} + \frac{\beta^2}{L^3} L_i L_r + \frac{1}{L} b_r b_i \right. \\
& \left. \left. - \frac{\beta}{L^2} (L_i b_r + L_r b_i) \right\} D_j^r - \frac{1}{L^2} (L b_j - \beta L_j) (r_{0i} + s_{0i}) - \frac{1}{L^2} (L b_i - \beta L_i) \right]
\end{aligned}$$

$$\times(r_{0j} + s_{0j}) - \left\{ \frac{1}{L^2} (\rho L^2 - \beta) L_{ij} + \frac{\beta}{L^4} (2L + \beta) L_i L_j + \frac{1}{L^2} b_i b_j \right. \\ \left. - \frac{(L + \beta)}{L^3} (L_i b_j + L_j b_i) \right\} r_{00} - L \partial_k \rho y^k - 2 \rho L_r L_{ij} G^r \Big] = 0.$$

where ‘0’ stands for contraction with respect to y^i viz. $r_{0k} = r_{ik} y^i$, $r_{00} = r_{ij} y^i y^j$ and we have use the fact that¹¹ $D_{jk}^i y^k = {}^c D_{jk}^i y^k = D_{ij}^i$.

Next, we deal with $\bar{L}_{i|j} = 0$, that is $\partial_j \bar{L}_i - \bar{L}_{ir} \bar{G}_j^r - \bar{L}_r \bar{F}_{ij}^r = 0$, then we have

$$(2.6) \quad \partial_j \bar{L}_i - \bar{L}_{ir} (G_j^r + D_j^r) - \bar{L}_r (F_{ij}^r + {}^c D_{ij}^r) = 0.$$

Putting the values of $\partial_j \bar{L}_i$, $\bar{L}_{ir}$ and $\bar{L}_r$ from (2.4 c), (2.4 b) and (2.4 a) respectively in (2.6), using equation $L_{i|j} = \partial_j L_i - L_{ir} G_j^r - L_r F_{ij}^r = 0$, and rearranging the terms, we get

$$b_{i|j} = \left\{ \left(1 - \frac{\beta}{L} + \rho L \right) L_{ir} + \frac{\beta^2}{L^3} L_i L_r - \frac{\beta}{L^2} (L_i b_r + L_r b_i) + \frac{1}{L} b_i b_r \right\} D_j^r \\ + \frac{1}{L^2} (\beta L_i - L b_i) (r_{0j} + s_{0j}) + \left\{ \left(1 - \frac{\beta}{L} \right) L_r + b_r \right\} D_{ij}^r,$$

which after using (1.5) gives

$$(2.7) \quad 2r_{ij} = \left\{ \frac{1}{L} (L - \beta + \rho L^2) L_{ir} + \frac{\beta^2}{L^3} L_i L_r \right. \\ \left. + \frac{1}{L} b_i b_r - \frac{\beta}{L^2} (L_i b_r + L_r b_i) \right\} D_j^r \\ + \left\{ \frac{1}{L} (L - \beta + \rho L^2) L_{jr} + \frac{\beta^2}{L^3} L_j L_r \right. \\ \left. + \frac{1}{L} b_j b_r - \frac{\beta}{L^2} (L_j b_r + L_r b_j) \right\} D_i^r$$

$$\begin{aligned}
& + \frac{1}{L^2} (\beta L_i - L b_i) (r_{0j} + s_{0j}) \\
& + \frac{1}{L^2} (\beta L_j - L b_j) (r_{0i} + s_{0i}) \\
& + 2 \left(\frac{L - \beta}{L} L_r + b_r \right) D_{ij}^r
\end{aligned}$$

and

$$\begin{aligned}
(2.8) \quad 2s_{ij} = & \left\{ \frac{1}{L} (L - \beta + \rho L^2) L_{ir} + \frac{\beta^2}{L^3} L_i L_r \right\} D_j^r \\
& + \left\{ \frac{1}{L} b_i b_r - \frac{\beta}{L^2} (L_i b_r + L_r b_i) \right\} D_j^r \\
& - \left\{ \frac{1}{L} (L - \beta + \rho L^2) L_{jr} + \frac{\beta^2}{L^3} L_j L_r \right\} D_i^r \\
& + \left\{ \frac{1}{L} b_j b_r - \frac{\beta}{L^2} (L_j b_r + L_r b_j) \right\} D_i^r \\
& + \frac{1}{L^2} (\beta L_i - L b_i) (r_{0j} + s_{0j}) \\
& - \frac{1}{L^2} (\beta L_j - L b_j) (r_{0i} + s_{0i}).
\end{aligned}$$

Subtracting (2.7) from (2.5) and contracting the resulting equation with y^i , we obtain

$$\begin{aligned}
(2.9) \quad & \left\{ -\frac{1}{L} (L - \beta + \rho L^2) L_{jr} + \frac{\beta}{L^2} (L_j b_r + L_r b_j) - \frac{\beta^2}{L^3} L_j L_r - \frac{1}{L} b_j b_r \right\} D^r \\
& - \left(\frac{\beta}{L^2} L_j - \frac{1}{L} b_j \right) \frac{r_{00}}{2} - \left(\frac{L - \beta}{L} L_r + b_r \right) D_j^r + r_{0j} = 0.
\end{aligned}$$

Contracting (2.9) with y^j , we get

$$(2.10) \quad 2 \left(\frac{L - \beta}{L} L_r + b_r \right) D^r = r_{00}.$$

Adding (2.8) and (2.5) and contracting the resulting equation with y^i , we get

$$(2.11) \quad \left\{ \frac{1}{L} (L - \beta + \rho L^2) L_{ir} + \frac{\beta^2}{L^3} L_i L_r + \frac{1}{L} b_i b_r - \frac{\beta}{L^2} (L_i b_r + L_r b_i) \right\} D^r \\ = \frac{1}{L^2} (L b_i - \beta L_i) r_{00} + s_{i0}.$$

In view of $LL_{ir} = g_{ir} - L_i L_r$, the equation (2.11) can be written as

$$(2.12) \quad \frac{1}{L^2} (L - \beta + \rho L^2) g_{ir} D^r + \frac{1}{L^3} \{ (\beta^2 - L^2 + \beta L - \rho L^3) L_i - L \beta b_i \} L_r D^r \\ + \frac{1}{L^2} (L b_i - \beta L_i) b_r D^r = \frac{1}{L^2} (L b_i - \beta L_i) \frac{r_{00}}{2} + s_{i0}.$$

Contracting (2.12) by $b^i = g^{ij} b_j$, we get

$$(2.13) \quad 2L \{ L(L - \beta) + L^2 b^2 - \beta^2 + \rho L^3 \} b_r D^r + 2 \{ L(L - \beta) \\ + \beta(L^2 b^2 - \beta^2) + \rho \beta L^3 \} L_r D^r = L(L^2 b^2 - \beta^2) r_{00} + 2L^4 s_0,$$

where we have written s_0 for $s_{r0} b^r$.

The equation (2.10) and (2.13) constitute the system of algebraic equations in $L_r D^r$ and $b_r D^r$ whose solution is given by

$$(2.14) \quad b_r D^r = \frac{(L^2 b^2 - 2\beta^2 + L\beta + \rho L^2 \beta) r_{00} + 2L^2 (L - \beta) s_0}{2[L(L - \beta) + L^2 b^2 - \beta^2 + L^3 \rho]}$$

and

$$(2.15) \quad L_r D^r = \frac{L(L - \beta + \rho L^2) r_{00} - 2L^3 s_0}{2[L(L - \beta) + L^2 b^2 - \beta^2 + L^3 \rho]}.$$

Contracting (2.12) by g^{ij} and putting the values of $b_r D^r$ and $L_r D^r$ from (2.14) and (2.15) respectively, we get

$$(2.16) \quad D^i = \frac{1}{2(L - \beta + \rho L^2)\{L(L - \beta) + L^2 b^2 - \beta^2 + L^3 \rho\}} \times \\ \left[(L - 2\beta + \rho L^2)\{(L - \beta + \rho L^2)r_{00} - 2L^2 s_0\}y^i + L^2\{(L - \beta \right. \\ \left. + \rho L^2)r_{00} - 2L^2 s_0\}b^i + 2L^2\{L(L - \beta) + L^2 b^2 - \beta^2 + L^3 \rho\}s_0^i \right],$$

where $l^i = \frac{y^i}{L}$.

Proposition 2.1. *The difference tensor $D^i = \bar{G}^i - G^i$ of exponential change of Finsler metric by h -vector is given by (2.16).*

3. Conditions to be Projective Change

The Finsler space $\bar{F}^n$ is said to be projective to Finsler space F^n if every geodesic of F^n is transformed to a geodesic of $\bar{F}^n$. It is well known that the change $L \rightarrow \bar{L}$ is projective if $\bar{G}^i = G^i + P(x, y)y^i$, where $P(x, y)$ is a homogeneous scalar function of degree one in y^i , called projective factor⁷. Thus from (2.2) it follows that $L \rightarrow \bar{L}$ is projective iff $D^i = Py^i$. Now we assume that the exponential change $L \rightarrow \bar{L} = Le^{\beta/L}$ is projective. Then from equation (2.16), we have

$$(3.1) \quad Py^i = \frac{1}{2(L - \beta + \rho L^2)\{L(L - \beta) + L^2 b^2 - \beta^2 + L^3 \rho\}} \times \\ \left[(L - 2\beta + \rho L^2)\{(L - \beta + \rho L^2)r_{00} - 2L^2 s_0\}y^i + L^2\{(L - \beta \right. \\ \left. + \rho L^2)r_{00} - 2L^2 s_0\}b^i + 2L^2\{L(L - \beta) + L^2 b^2 - \beta^2 + L^3 \rho\}s_0^i \right].$$

Contracting (3.1) with $y_i (= g_{ij}y^j)$ and using the fact that $s_0^i y_i = 0$ and $y_i y^i = L^2$, we get

$$(3.2) \quad P = \frac{(L - \beta + L^2 \rho)r_{00} - 2L^2 s_0}{2\{L^2(b^2 + 1 + \rho L) - \beta(L + \beta)\}}.$$

Putting the value of P from (3.2) in (3.1), we get

$$(3.3) \quad \beta\{(L - \beta + L^2 \rho)r_{00} - 2L^5 s_0\}y^i = L^2\{(L - \beta + L^2 \rho)r_{00} - 2L^2 s_0\}b^i \\ + 2L\{(b^2 + 1 + \rho L)L^2 - \beta(L + \beta)\}s_0^i.$$

Transvecting (3.3) by b_i , we get

$$(3.4) \quad r_{00} = -2L \cdot \frac{s_0}{\Delta}, \text{ where } \Delta = b^2 - \frac{\beta^2}{L^2} \neq 0.$$

Substituting the value of r_{00} from (3.4) in (3.2), we get

$$(3.5) \quad P = -\frac{s_0}{\Delta}.$$

Eliminating P and r_{00} from (3.5), (3.4) and (3.1), we get

$$(3.6) \quad s_0^i = \left(b^i - \frac{\beta}{L^2} y^i\right) \frac{s_0}{\Delta}.$$

The equations (3.4) and (3.6) give the necessary conditions under which an exponential change of Finsler metric by h -vector becomes a projective change.

Conversely, if conditions (3.4) and (3.6) are satisfied, then putting these conditions in (2.16), we get

$$D^i = -\frac{s_0}{\Delta} y^i \text{ i.e. } D^i = P y^i, \text{ where } P = -\frac{s_0}{\Delta}.$$

Thus $\bar{F}^n$ is projective to F^n .

Theorem 3.1. *The exponential change of a Finsler metric by h -vector is projective if and only if (3.4) and (3.6) hold, the projective factor P is given by equation (3.5).*

4. Exponential Change of Douglas Space

The Finsler space F^n is called a Douglas space if and only if $G^i y^j - G^j y^i$ is homogeneous polynomial of degree three¹⁸ in y^i . We shall write $hp(r)$ to denote a homogeneous polynomial in y^i of degree r . If we write $B^{ij} = D^i y^j - D^j y^i$, then from (2.16), we get

$$(4.1) \quad B^{ij} = \frac{L^2 \{(L - \beta + \rho L^2)r_{00} - 2L^2 s_0\}}{2(L - \beta + \rho L^2)\{L(L - \beta) + L^2 b^2 - \beta^2 + L^3 \rho\}} (b^i y^j - b^j y^i) \\ + \frac{L^2}{L - \beta + L^2 \rho} (s_0^i y^j - s_0^j y^i).$$

If a Douglas space is transformed to a Douglas space by the exponential change (2.1), then B^{ij} must be $hp(3)$ and vice-versa.

Theorem 4.1. *The exponential change of Finsler metric by h -vector of Douglas space leads to a Douglas space if and only if B^{ij} given by (4.1) is $hp(3)$.*

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