

New Fixed Point Theorems in b-S-Multiplicative Metric Spaces

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Abstract: In this study, b-S-multiplicative metric space is introduced and some fixed point theorems are stated and proven in this spaces. We also show that some fixed point results for various b-S-multiplicative metric spaces are equivalent to those of corresponding fixed point results in S-multiplicative metric spaces. Some examples are offered to validate the unfamiliarity and implication of our results.

Keywords: Fixed point, b-S-multiplicative metric space, b-metric space.

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1. Introduction

In 1922, the Banach¹ contraction principle was introduced. This was forego by the presentation of metric spaces by Frechet² in 1906. The Banach contraction principle was used as an alternative method to prove the existence and uniqueness of solutions of ordinary differential equation^{1,3}. Since then, metric space had been a compelling device in functional analysis, nonlinear analysis and topology. The topological structure of this space with its utility in fixed-point theory has inspired the attention of many mathematicians⁴⁻⁵. In recent years, different applications of fixed-point theorems have inspired researchers to introduce different generalizations of metric spaces. These generalized spaces include complex-valued S_b -metric spaces, 2-metric spaces, D^* -metric spaces, G -metric spaces, b-metric spaces, quasi metric spaces, D-metric spaces, G_b -metric spaces, complex-valued G_b -metric spaces, quaternion-valued G -metric spaces, S-metric spaces, S_b -metric spaces, A-metric spaces, γ -generalized quasi-metric spaces, S_p -metric spaces and G_b -multiplicative metric spaces, A_p -metric spaces, multiplicative metric spaces, and most recently, rectangular S-metric spaces^{4,6-13}. Frechet² defined metric spaces as follows:

Definition 1.1.² For a nonempty set X and a function $d: X^2 \rightarrow [0, \infty)$ pleasing the succeeding properties

- (i) $d(x, y) \geq 0 \forall x, y \in X$
- (ii) $d(x, y) = 0$ if and only if $x = y \forall x, y \in X$
- (iii) $d(x, y) = d(y, x), \forall x, y \in X$
- (iv) $d(x, y) \leq d(x, z) + d(z, y) \forall x, y, z \in X$.

d is called a metric on X and (X, d) is called a rectangular metric space.

Sedghi *et al.*¹⁴ acquainted the concept of an S -metric space as entourages:

Definition 1.2.¹⁴ Let X be a non-empty set and $\bar{S}: X^3 \rightarrow \mathbb{R}^+$, a function amusing the coming properties

- (i) $\bar{S}(x, y, z) = 0$ if and only if $x = y = z$
- (ii) $\bar{S}(x, y, z) \leq \bar{S}(x, x, a) + \bar{S}(y, y, a) + \bar{S}(z, z, a) \forall a, x, y, z \in X$.
(rectangle inequality).

Then, $(X, \bar{S})$ is called a S -metric-metric space.

The ensuring is the definition of S_p -metric spaces, a generalization of both S -metric spaces and S_b -metric spaces.

Definition 1.3.¹⁵ Let X be a non-empty set and $\bar{S}: X^3 \rightarrow \mathbb{R}^+$, a function with a strictly increasing continuous function, $\Omega: [0, \infty) \rightarrow [0, \infty)$ such that $\Omega(t) \geq t \forall t > 0$ and $\Omega(0) = 0$ feasting the ensuring properties:

- (i) $\bar{S}(x, y, z) = 0$ if and only if $x = y = z$;
- (ii) $\bar{S}(x, y, z) \leq \Omega(\bar{S}(x, x, a) + \bar{S}(y, y, a) + \bar{S}(z, z, a)) \forall a, x, y, z \in X$
(rectangle inequality).

Then, $(X, \bar{S})$ is called an S_p -metric-space.

In 2008, Bashirov *et al.* 10, introduced multiplicative metric spaces in the following way.

Definition 1.4.¹⁶ For a non-empty set X and a function $d: X^2 \rightarrow [0, \infty)$ satisfying the following properties

- (i) $d(x, y) \geq 1 \forall x, y \in X$;
- (ii) $d(x, y) = 1$ if and only if $x = y \forall x, y \in X$
- (iii) $d(x, y) = d(y, x), \forall x, y \in X$
- (iv) $d(x, y) \leq d(x, z) \cdot d(z, y) \forall x, y, z \in X$.

d is called a multiplicative metric on X and (X, d) is called a multiplicative metric space.

Definition 1.5.¹⁷ Let X be a non-empty set. A function $d: X \times X \rightarrow [0, \infty)$ is said to be b -metric space if the succeeding conditions are satisfied:

- (i) $0 \leq d(x, y) \forall x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$,
- (ii) $d(x, y) = d(y, x) \forall x, y \in X$, and
- (iii) there exists $s \geq 1$ such that $d(x, z) \leq s[d(x, y) + d(y, z)] \forall x, y, z \in X$.

In this case, the pair (X, d) is called a b -metric space with coefficient s .

Definition 1.6. Let X be a nonempty set and let $s \geq 1$ be a given real number. A mapping $m : X \times X \rightarrow [1, \infty)$ is called a b-multiplicative metric if the following conditions hold:

- (m_1) $m(x, y) > 1 \forall x, y \in X$ with $x \neq y$ and $m(x, y) = 1$ if and only if $x = y$;
- (m_2) $m(x, y) = m(y, x) \forall x, y \in X$;
- (m_3) $m(x, z) \leq m(x, y)^s \cdot m(y, z)^s \forall x, y, z \in X$.

The triplet (X, m, s) is called a b-multiplicative metric space."

In this paper, some fixed-point theorems are voiced and established in multiplicative S-metric spaces.

Definition 1.7. A topological space X is sequentially compact if every sequence of points in X has a convergent subsequence converging to a point in X . Every metric space is naturally a topological space, and for metric spaces, the impressions of compactness and sequential compactness are equivalent.

Definition 1.8.¹⁸ Let (X, d) be a complete b -metric space with constant $s \geq 1$, such that b -metric is a continuous functional. Let $T : X \rightarrow X$ be a contraction having contraction constant $k \in [0, 1)$ such that $ks < 1$. Then T has a unique fixed point.

2. Main Results

We introduce the following definitions:

Definition 2.1. Let X be a non-empty set and $S : X^3 \rightarrow \mathbb{R}^+$, a function delighting the ensuring properties:

- (i) $\bar{S} : [0, \infty) \rightarrow [0, \infty)$ such that $\bar{S}(0) = 1$;
- (ii) $\bar{S}(u, v, w) \leq [\bar{S}(u, u, a) \times \bar{S}(v, v, a) \times \bar{S}(w, w, a)]^b, \forall a, b \geq 1$.

Then, (X, S) is called a b -S-multiplicative metric space.

Example 2.2.

$$\bar{S}(u, v, w) = \begin{cases} 1 & ; u = v = w = 0 \\ e^{u+v+w} & ; \text{otherwise} \end{cases}$$

Here

$$\begin{aligned} \bar{S}(u, v, w) &= e^{u+v+w} \\ &\leq e^{2u+a+2v+a+2w+a} \\ &\leq [e^{2u+2v+2w+3a}]^b \\ &\leq [\bar{S}(u, u, a) \times \bar{S}(v, v, a) \times \bar{S}(w, w, a)]^b \end{aligned}$$

Thus, $(X, \bar{S})$ is a b -S-multiplicative metric space.

Lemma 2.3. Let (X, S) be a b -S-multiplicative metric space and $\{w_n\}$ a sequence in X . Then, $\{w_n\}$ converges to w if and only if $\bar{S}(w_n, w, w) \rightarrow 1$ as $n \rightarrow \infty$.

Lemma 2.4. Let (X, S) be a b - S -multiplicative metric space and $\{w_n\}$ a sequence in X . Then, $\{w_n\}$ is said to be a Cauchy sequence if and only if $\bar{S}(w_n, w_m, w_l) \rightarrow 1$ as $n, m, l \rightarrow \infty$.

Theorem 2.5. Let X be a complete b - S -multiplicative metric space and $T: X \rightarrow X$ a map for which there exist the real number p with $p > 1$ such that for each pair $u, v, w \in X$

$$(1) \quad \bar{S}(Tu, Tv, Tw) \leq p(\bar{S}(u, v, w))^b$$

Then, T has a unique fixed point.

Proof. Suppose T satisfies condition (1) and $w_0 \in X$ is an arbitrary point and define a sequence $\{w_n\}$ by $w_{n+1} = Tw_n$, that is $w_1 = Tw_0$, $w_2 = Tw_1 = T^2w_0$, $w_3 = Tw_2 = T^3w_0$, ... $w_n = T^n w_0$, then

$$\begin{aligned} \bar{S}(w_n, w_n, w_{n+1}) &= \bar{S}(Tw_{n-1}, Tw_{n-1}, Tw_n)^b \\ &\leq p(\bar{S}(w_{n-1}, w_{n-1}, w_n))^b \\ &\leq p(\bar{S}(w_{n-2}, w_{n-2}, w_{n-1}))^{b^2} \end{aligned}$$

Applying this again, we get

$$\bar{S}(w_n, w_n, w_{n+1}) \leq p(\bar{S}(w_0, w_0, w_1))^{b^n}.$$

By applying (ii) in Definition 2.1, we have

$$(2) \quad \bar{S}(w_n, w_m, w_m) \leq p \left[\bar{S}(w_n, w_n, w_{n+1}) \times (\bar{S}(w_m, w_m, w_{n+1}))^{b^2} \right]$$

Applying this again with $m > n$, we get

$$\begin{aligned} (3) \quad \bar{S}(w_n, w_m, w_m) &\leq p \left[\bar{S}(w_n, w_n, w_{n+1}) \times (\bar{S}(w_{n+1}, w_{n+1}, w_{n+2}))^{b^2} \right] \\ &\quad \times p(\bar{S}(w_{n+2}, w_{n+2}, w_{n+3}))^{b^4} \times \dots \\ &\quad \times p(\bar{S}(w_{m-1}, w_{m-1}, w_m))^{b^{2n}}. \end{aligned}$$

From (2) and (3), we have

$$\begin{aligned} (4) \quad \bar{S}(w_n, w_m, w_m) &\leq p(\bar{S}(w_0, w_0, w_1))^{b^{2n}} \times (\bar{S}(w_0, w_0, w_1))^{b^{2n}} \\ &\quad \times p(\bar{S}(w_0, w_0, w_1))^{b^{4n}} \times \dots \\ &\quad \times p(\bar{S}(w_0, w_0, w_1))^{b^{2n^2}} \end{aligned}$$

Taking the limit of $\bar{S}(w_n, w_m, w_m)$ as $n, m \rightarrow \infty$, we have

$$\lim_{n, m \rightarrow \infty} \bar{S}(w_n, w_m, w_m) = 1$$

For $n, m, l \in \mathbb{N}$ with $n > m > l$,

$$\bar{S}(w_n, w_m, w_l) \leq \bar{S}(w_n, w_n, w_{n-1}) + \bar{S}(w_m, w_m, w_{n-1}) + \bar{S}(w_l, w_l, w_{n-1})$$

Taking the limit of $\bar{S}(w_n, w_m, w_l)$ as $n, m, l \rightarrow \infty$, we have

$$\lim_{n,m,l \rightarrow \infty} \bar{S}(w_n, w_m, w_l) = 1$$

Hence, $\{w_n\}$ is a b-S-Cauchy sequence. By the completeness of $(X, \bar{S})$, there exist $u \in X$ such that $\{w_n\}$ is b-S-convergent to u . Suppose $Tu \neq u$, we have

$$(5) \quad \bar{S}(w_n, Tu, Tu) \leq p(\bar{S}(w_{n-1}, u, u))^b$$

Taking the limit as $n \rightarrow \infty$ and applying the fact that function is b-S-continuous in its variables, we get

$$(6) \quad \bar{S}(u, Tu, Tu) \leq p(\bar{S}(u, u, u))^b.$$

Therefore,

$$\bar{S}(u, Tu, Tu) \leq 1.$$

This is a contradiction. Therefore, $Tu = u$.

To show the uniqueness, suppose $v \neq u$ is such that $Tv = v$, then

$$\bar{S}(Tu, Tv, Tv) \leq p(\bar{S}(u, v, v))^b.$$

Since $Tu = u$ and $Tv = v$, we have

$$\bar{S}(u, v, v) \leq 1$$

which indicates $v = u$.

Remark 2.6. Suppose $(X, \bar{S})$ be a b -S-multiplicative metric space and $d: X \times X \rightarrow [0, \infty)$ is a function defined by $d(u, v) = \bar{S}(u, v, v)^b$, then Theorem 2.5 reduces to the Banach contraction principle in S-multiplicative metric space (similar to the Banach contraction principle in S-multiplicative metric space).

Lemma 2.7. Suppose $(X, \bar{S})$ be a b-S-multiplicative metric space and $\{w_n\}$ be a sequence in X . Then,

$$\bar{S}(w_n, w_{n+1}, w_{n+1})^b \leq \bar{S}(w_n, w_n, w_{n+1})^b.$$

Proof. By (ii) of Definition 2.1,

$$\begin{aligned} \bar{S}(w_n, w_{n+1}, w_{n+1}) &\leq [\bar{S}(w_n, w_n, w_{n+1}) \times \bar{S}(w_{n+1}, w_{n+1}, w_{n+1})^2]^b \\ &\leq [\bar{S}(w_n, w_n, w_{n+1}) \times (1)^2]^b \\ &\leq [\bar{S}(w_n, w_n, w_{n+1})]^b. \end{aligned}$$

Lemma 2.8. Suppose $(X, \bar{S})$ be a b-S-multiplicative metric space and $\{w_n\}$ a sequence in X . Then,

$$\bar{S}(w_n, w_{n+1}, w_{n+1})^b \leq \bar{S}(w_n, w_n, w_{n+1})^b.$$

Proof. By (ii) of Definition 2.1,

$$\begin{aligned} \bar{S}(w_n, w_{n+1}, w_{n+1}) &\leq [\bar{S}(w_n, w_n, w_{n+1}) \times \bar{S}(w_{n+1}, w_{n+1}, w_{n+1})^2]^b \\ &\leq [\bar{S}(w_n, w_n, w_{n+1}) \times (1)^2]^b \\ &\leq [\bar{S}(w_n, w_n, w_{n+1})]^b. \end{aligned}$$

Theorem 2.9. Suppose X be a complete b - S -multiplicative metric space and $T: X \times X$ a map for which there occur a real number p with $0 < p < 1$ such that for each pair $u, v, w \in X$,

$$\bar{S}(Tu, Tv, Tw) \leq p[\bar{S}(u, Tu, Tu) \times \bar{S}(v, Tv, Tv) \times \bar{S}(w, Tw, Tw)]^b.$$

Then, T has a unique fixed point.

Proof. Suppose T satisfies condition (i) and $w_0 \in X$ is an arbitrary point and define a sequence $\{w_n\}$ by $w_{n+1} = Tw_n$, that is $w_1 = Tw_0, w_2 = Tw_1 = T^2w_0, w_3 = Tw_2 = T^3w_0, \dots, w_n = T^n w_0$, then

$$\begin{aligned} (7) \quad \bar{S}(w_n, w_n, w_{n+1}) &= \bar{S}(Tw_{n-1}, Tw_{n-1}, Tw_n) \\ &\leq p[(\bar{S}(w_{n-1}, w_n, w_n))^2 \times \bar{S}(w_n, w_{n+1}, w_{n+1})]^b \\ &\leq p[\bar{S}(w_{n-1}, w_n, w_n)^2 \times \bar{S}(w_n, w_n, w_{n+1})]^b \\ &\leq p[\bar{S}(w_{n-1}, w_n, w_n)]^b \end{aligned}$$

From (7), we have

$$\bar{S}(w_n, w_n, w_{n+1}) \leq p(\bar{S}(w_0, w_0, w_1))^{b^n}$$

By applying (ii) in Definition 2.1, we get

$$\bar{S}(w_n, w_m, w_m) \leq p[\bar{S}(w_n, w_n, w_{n+1}) \times \bar{S}(w_m, w_m, w_{n+1})]^{b^2}.$$

Applying this repeatedly with $m > n$, we have

$$\begin{aligned} \bar{S}(w_n, w_m, w_m) &\leq p[\bar{S}(w_n, w_n, w_{n+1}) \times \bar{S}(w_{n+1}, w_{n+1}, w_{n+2})]^{b^2} \\ &\quad \times p[\bar{S}(w_{n+2}, w_{n+2}, w_{n+3})]^{b^4} \times \dots \\ &\quad \times p[\bar{S}(w_{m-1}, w_{m-1}, w_m)]^{b^{2n}} \end{aligned}$$

From (4) and (6), we get

$$\begin{aligned} \bar{S}(w_n, w_m, w_m) &\leq p \left[\bar{S}(w_0, w_0, w_1)^{b^n} \times (\bar{S}(w_0, w_0, w_1))^{b^{2n}} \right] \\ &\quad \times p[\bar{S}(w_0, w_0, w_1)]^{b^{2n}} \times \dots \\ &\quad \times p[\bar{S}(w_0, w_0, w_1)]^{b^{(2n)^2}} \end{aligned}$$

Taking the limit of $\bar{S}(w_n, w_m, w_m)$ as $n, m \rightarrow \infty$, we have

$$\lim_{n, m \rightarrow \infty} \bar{S}(w_n, w_m, w_m) = 1$$

For $n, m, l \in \mathbb{N}$ with $n > m > l$,

$$\begin{aligned} \bar{S}(w_n, w_m, w_l) &\leq p[\bar{S}(w_n, w_n, w_{n-1}) + \bar{S}(w_m, w_m, w_{n-1}) \\ &\quad + \bar{S}(w_l, w_l, w_{n-1})]. \end{aligned}$$

Taking the limit of $\bar{S}(w_n, w_m, w_l)$ as $n, m, l \rightarrow \infty$, we have

$$\lim_{n,m,l \rightarrow \infty} \bar{S}(w_n, w_m, w_l) = 1.$$

Therefore, $\{w_n\}$ is a b-S-Cauchy sequence. By the completeness of $(X, \bar{S})$, there occurs $u \in X$ such that $\{w_n\}$ is b-S-convergent to u . Suppose $Tu \neq u$, we have

$$\bar{S}(w_n, Tu, Tu) \leq p[(\bar{S}(w_{n-1}, w_n, w_n) \times \bar{S}(u, Tu, Tu)^2]^b.$$

Taking the limit as $n \rightarrow \infty$ and using the fact that function is S-continuous in its variables, we get

$$\bar{S}(u, Tu, Tu) \leq p[\bar{S}(u, u, u) \times \bar{S}(u, Tu, Tu)^2]^b.$$

This is a contradiction. Hence, $Tu = u$. To show the uniqueness, suppose $v \neq u$ is such that $Tv = v$, then

$$\bar{S}(Tu, Tv, Tv) \leq p[\bar{S}(u, Tu, Tu) \times \bar{S}(v, Tv, Tv)^2]^b$$

Since $Tu = u$ and $Tv = v$, we have

$$\bar{S}(u, v, v) \leq 1,$$

which indicates $v = u$.

Remark 2.10. Let $(X, \bar{S})$ be a b-S-multiplicative metric space and $d: X \times X \rightarrow [0, \infty)$ a mapping defined by $d(u, v) = \bar{S}(u, v, v)$, then Theorem 2.9 reduces to the Banach type contraction principle in b-S-multiplicative metric space.

For $b = 1$ we have:

Corollary 2.11. Suppose $(X, \bar{S})$ is a sequentially compact b-S-multiplicative metric space and let $T: X \rightarrow X$ assumed the ensured condition.

$$\bar{S}(Tu, Tv, Tw) \leq \bar{S}(u, v, w),$$

whenever $u, v, w \in X$ and $u \neq v \neq w$. Then, T has a unique fixed point.

Example 2.12. Suppose

$$\begin{aligned} \bar{S}(u, v, w) &= \log[S(u, v, w)] \\ &= \bar{S}(Tu, Tv, Tw) \\ &\leq \bar{S}(u, v, w) \\ &\leq \log[S(u, v, w)] \\ &\leq \bar{S}(u, v, w), \end{aligned}$$

which indicates $S(Tu, Tv, Tw) \leq S(u, v, w)$.

3. Conclusion

A new metric space called b-S multiplicative metric space is introduced and some fixed-point theorems are declared and evidence in these spaces. Some examples are presented to verify the inventiveness and implication of our main results.

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